ON THE SOLUTION OF NORMALITY EQUATIONS FOR THE DIMENSION $n \geq 3$.

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Abstract. The normality equations for the Newtonian dynamical systems on an arbitrary Riemannian manifold of the dimension $n \geq 3$ are considered. Locally the solution of such equations reduces to three possible cases: in two of them the solution is written out explicitly, and in the third case the equations of normality are reduced to an ordinary differential equation of the second order. Some new examples of explicit solutions of normality equations are constructed.

1. Introduction.

The concept of dynamical systems that admit the normal shift was introduced in [1,2]. These are the systems which describe the motion of a mass point according to the Newton's second law $m\,\ddot{\mathbf{r}} = \mathbf{F}(\mathbf{r}, \dot{\mathbf{r}})$ and which satisfy some specific geometrical condition that permits them to implement the normal shift of any hypersurface along their trajectories. Such geometrical condition was called *the condition of normality*. In [1,2] this geometrical condition of normality was brought to the form of a system of partial differential equations for the force field $\mathbf{F}(\mathbf{r}, \dot{\mathbf{r}})$ of the dynamical system. These equations were called *the equations of normality*.

In [3,4] the equations of normality were generalized for the case of Newtonian dynamical systems on an arbitrary Riemannian manifold. Note that in this case the normality equations were written in covariant form. Therefore one can apply the methods of differential geometry to these equations.

The geodesic flows of the metrics that is conformally equivalent to an original metric of the Riemannian manifold form a wide subclass of the dynamical systems accepting the normal shift. The problem of coincidence of trajectories of a dynamical system with the trajectories of such geodesic flow was called *the problem of metrizability*. It was considered in [5,6] where an explicit form of the force field for all metrizable dynamical system that admit the normal shift was obtained.

The symmetry analysis of the equation of normality was started in [7] where all classical symmetries of these equations in two-dimensional spatially homogeneous case were found and the corresponding self-similar (invariant) solutions were constructed. As a result of the symmetry analysis some examples of non-trivial (non-metrizable) dynamical systems accepting the normal shift in two-dimensional case were found. The importance of the search for such dynamical systems especially for the dimension $n \geq 3$ was pointed out to authors by academician A. T. Fomenko.

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Some simplest examples of non-metrizable systems were constructed in [8], however the systematical search for such systems has yet to be undertaken.

The main goal of the present article is the complete local analysis of normality equations for the dimension $n \geq 3$ and the construction of new non-metrizable dynamical systems that admit the normal shift.

2. The equations of normality.

Let M be a n-dimensional Riemannian manifold with the metric tensor g_{ij}. Let's denote by $x^1, \ldots, x^n$ the local coordinates of some point on M and by $v^1, \ldots, v^n$ the coordinates of a tangent vector at this point. The equations of Newtonian dynamical system describing the motion of the point of unit mass $m = 1$ have the following form:

$$\dot{x}^i = v^i, \qquad \nabla_t v^i = F^i(\mathbf{x}, \mathbf{v}). \tag{2.1}$$

The normality equations for the force field $\mathbf{F}$ of the dynamical system (2.1) are written as two systems of differential equations:

$$\begin{cases} \bigl(v^{-1} F_i + \tilde{\nabla}_i (F^k N_k)\bigr) P^i_q = 0 \\ (\nabla_i F_k + \nabla_k F_i - 2v^{-2} F_i F_k) N^k P^i_q + \\ \quad + v^{-1}(\tilde{\nabla}_k F_i F^k - \tilde{\nabla}_k F^r N^k N_r F_i) P^i_q = 0 \end{cases} \tag{2.2}$$

$$\begin{cases} (P^k_i P^q_j - P^q_i P^k_j) \left(N^r \dfrac{\tilde{\nabla}_r F_k}{v} F_q - \nabla_q F_k \right) = 0 \\ P^k_i \tilde{\nabla}_k F^q P^j_q = \dfrac{P^k_r \tilde{\nabla}_k F^q P^r_q}{n-1} P^j_i \end{cases} \tag{2.3}$$

The systems of equations (2.2) and (2.3) were derived separately in [3] and [4]. The first system was called *the equations of weak normality*, and the second one was called *the additional normality equations*. The equations of normality (2.2) and (2.3) are differential equations in the covariant derivatives ∇ and $\tilde{\nabla}$ in the expanded algebra of tensor fields. A tensor field from the expanded algebra is distinguished from an ordinary tensor field on M by the doubled set of arguments: it depends on a point $\mathbf{x} \in M$ as well as a tangent vector $\mathbf{v}$ at this point. Therefore there are two types of covariant derivatives in the expanded algebra: *the spatial gradient* ∇ and *the velocity gradient* $\tilde{\nabla}$. The calculation of components of the covariant derivative $\tilde{\nabla}$ for a tensor field $\mathbf{U}$ of (r, s)–type is simply the differentiation with respect to the components of velocity:

$$\tilde{\nabla}_i U^{i_1 \ldots i_r}_{j_1 \ldots j_s} = \frac{\partial U^{i_1 \ldots i_r}_{j_1 \ldots j_s}}{\partial v^i}. \tag{2.4}$$

There are no connection components Γ^k_{ij} in (2.4). Therefore the covariant derivative $\tilde{\nabla}$ can be defined on a manifold without any metric. The components of the covariant derivative ∇ are calculated in more complicated way:

$$\begin{aligned} \nabla_i U^{i_1 \ldots i_r}_{j_1 \ldots j_s} &= \frac{\partial U^{i_1 \ldots i_r}_{j_1 \ldots j_s}}{\partial x^i} + \sum_{m=1}^{r} \sum_{q_m=1}^{n} U^{i_1 \ldots q_m \ldots i_r}_{j_1 \ldots j_s} \, \Gamma^{i_m}_{i\, q_m} - \\ &- \sum_{m=1}^{s} \sum_{q_m=1}^{n} U^{i_1 \ldots i_r}_{j_1 \ldots q_m \ldots j_s} \, \Gamma^{q_m}_{i\, j_m} - \sum_{p=1}^{n} \sum_{q=1}^{n} \frac{\partial U^{i_1 \ldots i_r}_{j_1 \ldots j_s}}{\partial v^p} \, \Gamma^p_{i\, q} \, v^q. \end{aligned} \tag{2.5}$$

The equations (2.2) and (2.3) are the equations with respect to the force vector $\mathbf{F}$. But apart from the components of the force vector, in (2.2) and (2.3) there are also the components of tensor fields $\mathbf{N}$ and $\mathbf{P}$, and a scalar field v. These are given parameters in the normality equations: v is the field of velocity modulus $v = |\mathbf{v}|$; $\mathbf{N}$ is the unit vector directed along $\mathbf{v}$ and $\mathbf{P}$ is the orthogonal projector on the hyperplane that is perpendicular to the vector $\mathbf{v}$:

$$N^i = v^i/v, \qquad P^i_j = \delta^i_j - N^i\,N_j. \tag{2.6}$$

Both covariant derivatives ∇ and $\tilde{\nabla}$ of a metric tensor are equal to zero. This makes calculations in the expanded algebra similar to calculations in the ordinary algebra of tensor fields on M. The covariant derivatives of given fields v, $\mathbf{N}$ and $\mathbf{P}$ are calculated from the formulae (2.4) and (2.5).

3. A scalar substitution.

Consider the first equation from (2.2). Let us denote $A = F^k N_k$. The scalar field A from the expanded algebra has the sense of the orthogonal projection of the force $\mathbf{F}$ on the direction of the velocity vector. So, the force field F can be written in the form: $\mathbf{F} = A\,\mathbf{N} + \tilde{\mathbf{F}}$. Substituting this expansion into the first equation from (2.2), we find:

$$v^{-1}\,\tilde{F}_i\,P^i_q + \tilde{\nabla}_i A\,P^i_q = 0. \tag{3.1}$$

Here we've used the obvious equality $P^i_q\,N_i = 0$, which follows from (2.6), and the fact that $\mathbf{N}$ is a unit vector. From the orthogonality of the vectors $\tilde{\mathbf{F}}$ and $\mathbf{N}$ we have $\tilde{F}_i\,P^i_q = \tilde{F}_q$. So, taking into account everything told above, from (3.1) we obtain:

$$F_q = A\,N_q - v\,P^i_q\,\tilde{\nabla}_i A. \tag{3.2}$$

From (3.2) we see that all components of the force field $\mathbf{F}$ are defined by the scalar field A.

Let's substitute (3.2) into the second equation of weak normality (2.2). It leads to quite cumbersome, but not complicated computations. As a result of these computations we obtain:

$$\begin{gathered} P^i_q\,(\nabla_i A + v\,P^{ks}\,\tilde{\nabla}_s A\,\tilde{\nabla}_k\tilde{\nabla}_i A - \\ -\,v\,N^k\,\nabla_k\tilde{\nabla}_i A - N^k\,A\,\tilde{\nabla}_k\tilde{\nabla}_i A) = 0. \end{gathered} \tag{3.3}$$

In order to derive (3.3) we've used the following relationships:

$$\nabla_i v = 0, \qquad \tilde{\nabla}_i v = N_i, \tag{3.4}$$

$$\nabla_i N^k = 0, \qquad \tilde{\nabla}_i N^k = v^{-1}\,P^k_i, \tag{3.5}$$

$$\nabla_i P^k_q = 0, \qquad \tilde{\nabla}_i P^k_q = -v^{-1}\,(P_{iq}\,N^k + P^k_i\,N_q), \tag{3.6}$$

that are obtained by direct computations taking into account the relationships (2.4), (2.5) and (2.6).

Now let us substitute (3.2) into the additional equations of normality (2.3) and use once more the relationships (3.4), (3.5) and (3.6) in combination with $P^i_q\,N_i=0$ and $P^i_q\,P^q_k=P^i_k$. From the first equation (2.3) we derive

$$\begin{aligned}P^k_i\,P^q_j\,(N^r\,\tilde\nabla_r\tilde\nabla_k A\,\tilde\nabla_q A+\nabla_q\tilde\nabla_k A-\\ -N^r\,\tilde\nabla_r\tilde\nabla_q A\,\tilde\nabla_k A-\nabla_k\tilde\nabla_q A)=0.\end{aligned}\tag{3.7}$$

After the substitution (3.2) into the second equation (2.3) we obtain:

$$P^k_i\,\tilde\nabla_k\tilde\nabla_q A\,P^{qj}=\frac{P^{kq}\,\tilde\nabla_k\tilde\nabla_q A}{n-1}\,P^j_i,\tag{3.8}$$

where the number n is a dimension of the manifold M. It's convenient to rewrite the last equation in the following form:

$$P^k_i\,\tilde\nabla_k\tilde\nabla_q A\,P^{qj}=\lambda\,P^j_i,\tag{3.9}$$

where $\lambda=\lambda(\mathbf{x},\mathbf{v})$ is a function on the tangent bundle TM. It is the very form this equation was initially derived in [4].

4. Fiber spherical coordinates.

The equations (3.3), (3.7) and (3.9) form the complete list of the equations of normality after the scalar substitution (3.2). The last equation from this list has an important difference from the other ones. If we write out the covariant derivatives $\tilde\nabla_k\tilde\nabla_q A$, in explicit form, from (3.9) we obtain:

$$P^k_i\,P^q_j\,\frac{\partial^2 A}{\partial v^k\,\partial v^q}=\lambda\,P_{ij}.\tag{4.1}$$

The equations (4.1) contain only the derivatives of A with respect to the coordinates $v^1,\dots,v^n$ in the fiber of the tangent bundle TM. The fiber is a n-dimensional linear vector space with the metric g_{ij} which is constant within the fiber.

Let's fix some point P on the manifold M. The conditions $v=|\mathbf{v}|=\mathrm{const}$ decompose the fiber of TM over P into the union of non-intersecting spheres S^{n-1} with the radii $r=v$. Let $u^1,\dots,u^{n-1}$ be local coordinates on the unit sphere S^{n-1} and let $\mathbf{v}=\mathbf{N}(u^1,\dots,u^{n-1})$ be the parametrical equation of the unit sphere in the fiber of TM over the point P. Then the relationships

$$\begin{aligned}&v^1=u^n\cdot N^1(u^1,\dots,u^{n-1}),\\ &\dots\dots\dots\dots\dots\dots\dots\dots\\ &v^n=u^n\cdot N^n(u^1,\dots,u^{n-1})\end{aligned}\tag{4.2}$$

define the change of Cartesian coordinates $v^1,\dots,v^n$ for the spherical coordinates $u^1,\dots,u^n$, where $u^n=|\mathbf{v}|$. The choice of coordinates (4.2) in each fiber of TM defines *the fiber spherical coordinates* on the tangent bundle TM. The dependence of the choice of such coordinates upon the point P can be made smooth, however for analysis of the equations (4.1) this does not matter. Therefore we'll consider the relationships (4.2) taking the point P to be fixed.

Let's transform the equations (4.1) by making the change of variables (4.2). The metric g_{ij} is constant within the fiber and the derivatives $\tilde{\nabla}_i = \partial/\partial v^i$ are covariant derivatives in the metric g_{ij}. The equations (4.1) are tensorial equations, therefore it's sufficient to recalculate the components of tensors P^k_i, P^q_j, g_{ij} and the derivatives $\tilde{\nabla}_i$ in the new coordinates.

Note that according to (4.2) we have $\partial v^i/\partial u^n = N^i$. Therefore vector $\mathbf{N}$ is the n-th coordinate vector in the spherical coordinates: $\mathbf{N} = (0, \dots, 0, 1)$. So, for the components of the projector $\mathbf{P}$ in spherical coordinates we have

$$P^k_i = \begin{Vmatrix} 1 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 1 & 0 \\ 0 & \dots & 0 & 0 \end{Vmatrix}, \qquad P_{ij} = \begin{Vmatrix} g_{11} & \dots & g_{1\,n-1} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ g_{n-1\,1} & \dots & g_{n-1\,n-1} & 0 \\ 0 & \dots & 0 & 0 \end{Vmatrix}. \tag{4.3}$$

In view of what we said above, in the spherical coordinates the equation (4.1) has the initial form (3.9). But the components of the connection ϑ^k_{ij} in spherical coordinates $u^1, \dots, u^n$ are nonzero. For i, j, k in the range $1 \leq i, j, k \leq n-1$ the quantities ϑ^k_{ij} do coincide with the components of the metric connection on the spheres $|\mathbf{v}| = \text{const}$. The other components are calculated explicitly:

$$\begin{aligned} &\vartheta^n_{nn} = 0, && \vartheta^n_{in} = \vartheta^n_{ni} = 0, && \vartheta^i_{nn} = 0, \\ &\vartheta^n_{ij} = -\frac{g_{ij}}{v}, && \vartheta^i_{jn} = \vartheta^i_{nj} = \frac{\delta^i_j}{v}. \end{aligned} \tag{4.4}$$

By virtue of the structure of the matrices (4.3) for $i = n$ or $j = n$ the equations (3.9) are fulfilled identically. Taking into account (4.4) we can write the rest non-trivial equations as the equations in the covariant derivatives $\bar{\nabla}$ on the spheres $|\mathbf{v}| = \text{const}$:

$$\bar{\nabla}_i \bar{\nabla}_j A = \left(\lambda + \frac{1}{v}\,\frac{\partial A}{\partial u^n}\right) g_{ij}. \tag{4.5}$$

Since the function λ is an indeterminate parameter in (4.5), it's convenient to introduce another parameter μ and to write (4.5) more simply:

$$\bar{\nabla}_i \bar{\nabla}_j A = \mu\, g_{ij}. \tag{4.6}$$

Consider a differential consequence from the equation (4.6). Let us write (4.6) in the form $\bar{\nabla}_j \bar{\nabla}_k A = \mu\, g_{jk}$ then apply the operator $\bar{\nabla}_i$ to both sides of this equation and alternate the obtained equation with respect to the indices i and j:

$$\begin{aligned} &(\bar{\nabla}_i \bar{\nabla}_j - \bar{\nabla}_j \bar{\nabla}_i)\bar{\nabla}_k A = \bar{\nabla}_i \mu\, g_{jk} - \bar{\nabla}_j \mu\, g_{ik}. \\ &-\bar{R}^s_{kij}\, \bar{\nabla}_s A = \bar{\nabla}_i \mu\, g_{jk} - \bar{\nabla}_j \mu\, g_{ik}. \end{aligned} \tag{4.7}$$

Here $R^s_{kij} = v^{-2}\,(\delta^s_i\, g_{jk} - \delta^s_j\, g_{ik})$ is the curvature tensor of the sphere S^{n-1} given by the equation $v = |\mathbf{v}| = \text{const}$.

Let's contract the above equation (4.7) with the metric g^{jk}. It leads to the following relationship:

$$\frac{n-2}{v^2}\,\bar{\nabla}_i A = (n-2)\,\bar{\nabla}_i \mu.$$

Note that the additional equations of normality (2.3) arise in the dimension $n \geq 3$ only, therefore in the obtained equations $n-2 \neq 0$. Hence

$$-\nabla_i A = v^2\,\nabla_i\mu. \tag{4.8}$$

Let's substitute the last expression for $\nabla_i A$ into the equations (4.6). It leads to the following differential equations for the functional parameter μ:

$$\bar\nabla_i\bar\nabla_j\mu = -v^{-2}\,g_{ij}\,\mu. \tag{4.9}$$

Theorem 4.1. *The general solution of the system of equations* (4.9) *on the sphere* $|\mathbf{v}| = \mathrm{const}$ *has the form:*

$$\mu = \sum_{i=1}^{n} m_i\,N^i(u^1,\ldots,u^n), \tag{4.10}$$

where $m_1,\ldots,m_n$ *are some arbitrary constants and* $N^1,\ldots,N^n$ *are the components of the vector* $\mathbf{N}$.

Proof. Because the equations (4.9) and the expression (4.10) are covariant tensorial equalities it's convenient to check the statement of the theorem for some particular choice of coordinates $u^1,\ldots,u^n$ on the sphere $|\mathbf{v}| = \mathrm{const}$. Let's render concrete the change of variables (4.2), taking the metric g_{ij} to be brought to the unitary form $g_{ij} = \delta_{ij}$ in the coordinates $v^1,\ldots,v^n$ in the fiber of TM over the considered fixed point P:

$$\begin{aligned}
v^1 &= v\,\sin(u_{n-1})\,\sin(u_{n-2})\cdot\ldots\cdot\sin(u_3)\sin(u_2)\,\sin(u_1),\\
v^2 &= v\,\sin(u_{n-1})\,\sin(u_{n-2})\cdot\ldots\cdot\sin(u_3)\sin(u_2)\,\cos(u_1),\\
v^3 &= v\,\sin(u_{n-1})\,\sin(u_{n-2})\cdot\ldots\cdot\sin(u_3)\cos(u_2),\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
v^{n-1} &= v\,\sin(u_{n-1})\,\cos(u_{n-2}),\\
v^n &= v\,\cos(u_{n-1}).
\end{aligned} \tag{4.11}$$

In spherical coordinates $u^1,\ldots,u^n$ defined by the relationships (4.11) the metric g_{ij} is diagonal, moreover $g_{ii} > 0$ and the quantities $H_i = \sqrt{g_{ii}}$ are called *the Lame coefficients*:

$$\begin{aligned}
H_1 &= v\,\sin(u_{n-1})\,\sin(u_{n-2})\cdot\ldots\cdot\sin(u_3)\sin(u_2),\\
H_2 &= v\,\sin(u_{n-1})\,\sin(u_{n-2})\cdot\ldots\cdot\sin(u_3),\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
H_{n-2} &= v\,\sin(u_{n-1}),\\
H_{n-1} &= v,\\
H_n &= 1.
\end{aligned} \tag{4.12}$$

The components of the metrical connection ϑ^k_{ij} are defined by the Lame coefficients (4.12) in the following way:

$$\vartheta^k_{ij} = \frac{1}{H_k}\,\frac{\partial H_k}{\partial u^j}\,\delta_{ik} + \frac{1}{H_k}\,\frac{\partial H_k}{\partial u^i}\,\delta_{jk} - \frac{H_i}{(H_k)^2}\,\frac{\partial H_i}{\partial u^k}\,\delta_{ij}.$$

Most of the components of the connection ϑ^k_{ij} are zero. The calculation of the other ones makes possible to write (4.9) explicitly. As a result we obtain the equations of two types. The equations of the first type correspond to the case $i < j \leq n-1$ in (4.9):

$$\mu_{ij} = \frac{\cos(u_j)}{\sin(u_j)}\,\mu_i. \tag{4.13}$$

Here μ_i and μ_{ij} are the partial derivatives of μ of the first and the second order with respect to the corresponding variables u^i and u^j. The equations of the second type correspond to the case $i = j \leq n-1$:

$$\mu_{ii} + \sum_{k=i+1}^{n-1} \frac{\cos(u^k)}{\sin(u^k)} \left(\prod_{p=i+1}^{k} \sin^2(u^p) \right) \mu_k = -\left(\prod_{p=i+1}^{n-1} \sin^2(u^p) \right) \mu. \tag{4.14}$$

Note that, when $i = n-1$, the equation (4.14) is reduced to the form $\mu_{ii} = -\mu$.

Let us make the further proof of the theorem by induction on n. We take, as the base of induction, the case $n = 2$. In the initial geometrical problem the additional equations of normality arise only for the dimension $n > 2$. Therefore the equations (4.6) are also absent. The relationship (4.8), which lets us bring (4.6) to (4.9), was also derived for $n > 2$. But once the equations (4.9) are already derived, their extrapolation for the case $n = 2$ has no contradiction.

So, when $n = 2$, the systems (4.13) and (4.14) are reduced to the equation: $\mu_{11} = -\mu$. The general solution of this equation is:

$$\mu(u^1) = m_1\,\sin(u^1) + m_2\,\cos(u^1), \tag{4.15}$$

where m_1 and m_2 are constants. In the case $n = 2$ the relationships (4.11) have the form:

$$v^1 = v\,N^1 = v\,\sin(u_1), \qquad\qquad v^2 = v\,N^2 = v\,\cos(u_1). \tag{4.16}$$

Comparison of (4.15) with (4.10) and taking into account (4.16) show that the statement of theorem 4.1 is fulfilled for the case $n = 2$.

Now let $n > 2$. Consider the last equation from (4.14) with $i = n-1$. It has the form $\mu_{ii} = -\mu$. Hence

$$\mu = \tilde{\mu}\,\sin(u^{n-1}) + m_n\,\cos(u^{n-1}), \tag{4.17}$$

where $\tilde{\mu}$ and m_n are parameters that in general case depend on $u^1, \ldots, u^{n-2}$. Let's substitute (4.17) into the equations (4.13), setting $j = n-1$. As a result the contribution of the first summand from (4.17) is canceled and we obtain the following relationships for m_n:

$$\frac{\partial m_n}{\partial u^i} = 0 \text{ for all } i = 1, \ldots, n-2.$$

Hence $m_n = \text{const}$. Therefore the substitution of (4.17) into the rest equations (4.13) and (4.14) leads to the equations for the parameter $\tilde{\mu}$, which have the same form as (4.13) and (4.14), but with $n-1$ in place of n. It means that one can apply the assumption of induction. As a result we obtain:

$$\tilde{\mu} = m_1 \left(\prod_{p=1}^{n-2} \sin(u^p)\right) + \sum_{k=2}^{n-1} m_k \left(\prod_{p=k}^{n-2} \sin(u^p)\right) \cos(u^{p-1}), \tag{4.18}$$

where $m_1, \dots, m_{n-1}$ are some constants. Now it remains to substitute (4.18) into (4.17) and to compare each of the summands in the obtained formula with (4.11). From this comparison we see that (4.18) and (4.10) are the same. So, the theorem is proved. □

Now let's return to the relationship (4.8) and write it in the form $\nabla_i(A + v^2\,\mu) = 0$, from where we obtain:

$$A + v^2\,\mu = \text{const}\,. \tag{4.19}$$

Note that (4.19) and parameters m_i from (4.10) are constants within the separate spheres $|\mathbf{v}| = \text{const}$ only. Therefore, denoting $b_i = |\mathbf{v}|, m_i$, we can formulate the following statement.

Theorem 4.2. *The scalar field $A(\mathbf{x}, \mathbf{v})$ from the extended algebra of tensor fields satisfying the equations of normality* (3.9) *has the form:*

$$A = a(\mathbf{x}, |\mathbf{v}|) + |\mathbf{v}|\, b_i(\mathbf{x}, |\mathbf{v}|)\, N^i, \tag{4.20}$$

where the scalar field a and the covector field $\mathbf{b}$ from the expanded algebra depend on the velocity modulus $v = |\mathbf{v}|$ in the fibers of the tangent bundle TM.

5. The refinement of the scalar substitution.

The obtained formula (4.20) completely defines the dependence of the function A upon velocity $\mathbf{v}$. The dependence of A upon coordinates should be refined by the substitution of (4.20) into the equations (3.3) and (3.7). Introduce the following notations $a' = \partial a/\partial v$ and $a'' = \partial^2 a/\partial v^2$. It is not difficult to check that a' and a'' are scalar fields from the expanded algebra of tensor fields and they depend on velocity modulus only. Moreover

$$\tilde{\nabla}_i a = a'\,N_i, \qquad\qquad \tilde{\nabla}_i a' = a''\,N_i. \tag{5.1}$$

The derivatives $b'_k = \partial b_k/\partial v$ and $b''_k = \partial b'_k/\partial v$ define two covector fields $\mathbf{b}'$ and $\mathbf{b}''$ from the expanded algebra of tensor fields. These fields also depend on velocity modulus only and there are following relationships analogous to (5.1):

$$\tilde{\nabla}_i b_k = b'_k\,N_i, \qquad\qquad \tilde{\nabla}_i b'_k = b''_k\,N_i. \tag{5.2}$$

Substituting (4.20) into the equations of normality (3.3) and taking into account (5.1) and (5.2), we obtain:

$$P^i_q\left(\nabla_i a + a'\,b_i - b'_i\,a + v\,N^r\,\nabla_i b_r - v\,N^r\,\nabla_r b_i\right) = 0 \tag{5.3}$$

The equation (5.3) contains the covariant derivatives $\nabla_i a$ and $\nabla_i b_r$. The formula (2.5) is written in the variables $x^1, \dots, x^n, v^1, \dots, v^n$. But for the fields a and $\mathbf{b}$ the natural variables are $x^1 \dots, x^n, v$, where $v = |\mathbf{v}| = \sqrt{g_{kq}\, v^k\, v^q}$. While in the variables $x^1, \dots, x^n, v^1, \dots, v^n$ the velocity modulus v depends on $x^1, \dots, x^n$ because g_{kq} depends on $x^1, \dots, x^n$. Therefore

$$\nabla_i a = \frac{\partial a}{\partial x^i} + \frac{a'}{2v}\,\frac{\partial g_{kq}}{\partial x^i}\, v^k\, v^q - \Gamma^k_{iq}\,\frac{a'}{v}\, v_k\, v^q.$$

In view of concordance of the metric and connection two last summands in this expression are canceled. An analogous cancellation happens in the computation of $\nabla_i b_r$. For this reason the covariant derivatives of the fields a and $\mathbf{b}$ are calculated as if they contain no dependence on $\mathbf{v}$:

$$\nabla_i a = \partial a/\partial x^i, \qquad \nabla_i b_r = \partial b_r/\partial x^i - \Gamma^k_{ir}\, b_k. \tag{5.4}$$

By virtue of (5.4) the covariant derivatives $\nabla_i a$ and $\nabla_i b_r$ depend on the velocity modulus only.

Let's replace the projector P^i_q in (5.4) by the explicit expression $P^i_q = \delta^i_q - N^i\, N_q$ from the formula (2.6)

$$\begin{aligned} &\nabla_q a + a'\, b_q - b'_q\, a + v\,(\nabla_q b_r - \nabla_r b_q)\, N^r \\ &\qquad - N_q\,(\nabla_r a + a'\, b_r - b'_r\, a)\, N^r = 0. \end{aligned} \tag{5.5}$$

There are three groups of summands in the equation (5.5). The first three summands depend on the velocity modulus only. The other ones depend on the modulus of the velocity vector and on its direction as well. The direction of $\mathbf{v}$ is given by $\mathbf{N}$. One of these summands is linear in $\mathbf{N}$, while another is quadratic with respect to $\mathbf{N}$. The substitution of $\mathbf{N}$ by $-\mathbf{N}$ does not change the modulus of the velocity vector. This substitution changes a sign of the summand that is linear in $\mathbf{N}$ and the rest summands in (5.4) remain unaltered. Therefore the summand linear with respect to $\mathbf{N}$ vanishes identically. Hence

$$\nabla_q b_r - \nabla_r b_q = 0. \tag{5.6}$$

The summand quadratic with respect to $\mathbf{N}$ contains the contraction of the vector $\mathbf{N}$ and the covector with the components $\nabla_r a + a'\, b_r - b'_r\, a$ which depend on $|\mathbf{v}|$ only. While the modulus of the velocity vector is unchanged, the vector $\mathbf{N}$ can be made orthogonal to this covector. So, the summand quadratic in $\mathbf{N}$ is also zero:

$$\nabla_q a + a'\, b_q - b'_q\, a = 0. \tag{5.7}$$

Conclusion: the equation of normality (3.3) for the field A of the form (4.20) is reduced to the equations (5.6) and (5.7) for the fields a and $\mathbf{b}$.

The substitution of (4.20) into (3.7) add nothing new. It leads to the equation that is the consequence of (5.6). For this reason (5.6) and (5.7) remain as unique basic equations of normality for the fields a and $\mathbf{b}$. Taking into account the relationships (5.4) and symmetry of the connection components Γ^k_{ij} enables us to bring (5.6) to the form:

$$\frac{\partial b_r}{\partial x^i} - \frac{\partial b_i}{\partial x^r} = 0. \tag{5.8}$$

The covector field $\mathbf{b}(\mathbf{x}, v)$ from the expanded algebra of tensor fields depending only on the modulus of velocity can be treated as the one-parameter family of 1-forms on the manifold: $\mathbf{b} = b_i\, dx^i$, where v is a parameter. Then the equation (5.8) is a closedness condition for each form from this family.

Example 1. Let $a \equiv 0$. Then the equation (5.7) is fulfilled for any choice of $\mathbf{b}$. So, each one- parameter family of closed 1-forms corresponds to some Newtonian dynamical system accepting the normal shift. The force field of such system has the following form:

$$F_q = |\mathbf{v}|\, b_i(\mathbf{x}, |\mathbf{v}|)\,(2\, N^i\, N_q - \delta^i_q). \tag{5.9}$$

Note that the simplest examples of the dynamical systems accepting the normal shift from the papers [1], [2] and [8] are systems of the form (5.9).

Example 2. Let $\mathbf{b} \equiv 0$. Then the equation (5.7) has the form: $\nabla_q a = 0$. It means that the function a does not depend on a point of the manifold M, i.e. $a = a(v)$. For the force field of the corresponding dynamical system we have:

$$F_q = a'(|\mathbf{v}|)\, N_q. \tag{5.10}$$

This is a trivial class of the dynamical systems that admit the normal shift. The trajectories of the systems (5.10) coincide with geodesics of the manifold M.

Now consider the case $a \not\equiv 0$ and $\mathbf{b} \not\equiv 0$. In this case the equation (5.7) is not trivial. Let's write it in the following form:

$$\left(\frac{\partial}{\partial x^q} + b_q\,\frac{\partial}{\partial v}\right)\ln|a| = b'_q. \tag{5.11}$$

Let us set $\hat{X}_q = \partial/\partial x^q + b_q\,\partial/\partial v$ and compute the commutator of two such differential operators, taking into account the relationships (5.8):

$$[\hat{X}_k,\, \hat{X}_q] = (b_k\, b'_q - b_q\, b'_k)\,\partial/\partial v. \tag{5.12}$$

The equations (5.11) are overdetermined. Using (5.8) and (5.12), from (5.11) we derived the following differential consequence:

$$b_k\,\left(b''_q - \frac{a'}{a}\, b'_q\right) = b_q\,\left(b''_k - \frac{a'}{a}\, b'_k\right). \tag{5.13}$$

Lemma 5.1. *The components of two covectors* $\mathbf{b}$ *and* $\mathbf{c}$ *satisfy the relationships* $b_k\, c_q = b_q\, c_k$ *if and only if they are linear dependent.*

Proof. If one of the covectors is equal to zero the lemma statement is trivial. Let $\mathbf{b} \neq 0$. Then, at least one component of the covector $\mathbf{b}$ is not zero. Suppose $b_1 \neq 0$. Then from $b_k\, c_q = b_q\, c_k$ we derive $c_k = \lambda\, b_q$, where $\lambda = c_1/b_1$. It means $\mathbf{c} = \lambda\,\mathbf{b}$. So, the lemma is proved. □

Let us apply Lemma 5.1 to the equation (5.13) where $\mathbf{b} \neq 0$. So, as a consequence of (5.13) we obtain the following equation:

$$b''_q - \frac{a'}{a}\, b'_q = \lambda\, b_q, \tag{5.14}$$

where λ is some scalar. Let's differentiate the equation (5.11) with respect to v and write the obtained result in the form:

$$b''_q - \frac{a'}{a}\, b'_q = \frac{\partial^2 \ln|a|}{\partial v^2}\, b_q + \frac{\partial^2 \ln|a|}{\partial v\,\partial x^q}. \tag{5.15}$$

The comparison of (5.14) with (5.15) leads to the equation that is also to be considered as a differential consequence of the equations (5.7) and (5.9):

$$\frac{\partial}{\partial x^q}\left(\frac{a'}{a}\right) = \mu\, b_q, \tag{5.15}$$

where the scalar μ is obtained from λ by subtracting the second logarithmic derivative of the function a with respect to v.

For further analysis of the obtained equations (5.15) note that the equations (5.8) are locally solvable. Any field $\mathbf{b}$ satisfying the equations (5.8) is defined by some scalar field β from the expanded algebra which depends on the modulus of velocity only:

$$b_r = \nabla_r \beta = \partial\beta/\partial x^r. \tag{5.16}$$

The substitution (5.16) into the equation (5.15) transforms this equation to the form:

$$\frac{\partial}{\partial x^q}\left(\frac{a'}{a}\right) = \mu\, \frac{\partial\beta}{\partial x^q}. \tag{5.17}$$

Lemma 5.2. *If spatial gradients of two functions $\alpha(\mathbf{x}, v)$ and $\beta(\mathbf{x}, v)$ are proportional: $\partial\alpha/\partial x^q = \mu\, \partial\beta/\partial x^q$ then in some neighborhood of points at which the gradient of the function β is not zero, there is the representation $\alpha = F(\beta, v)$ where F is some function of two variables.*

Proof. In Lemma 5.2 the quantity v plays the role of a parameter. Therefore to prove the lemma it is convenient to assume v by fixed $v = v_0 = \text{const}$. Then, while $\operatorname{grad}\beta \neq 0$, the local coordinates $x^1, \ldots, x^n$ on M can be taken so that $x^1 = \beta(\mathbf{x}, v_0)$. In this case from the proportionality of the gradients α and β we have:

$$\frac{\partial\alpha}{\partial x^2} = \ldots = \frac{\partial\alpha}{\partial x^2} = 0.$$

It means $\alpha = \alpha(x^1, v_0)$. And the functional dependence of α on x^1 and v_0 in these specially chosen coordinates defines the function $F(\beta, v)$. The choice of such coordinates can be made smoothly depending on the parameter v_0. Therefore $F(\beta, v)$ is the smooth function of two variables that defines a relation between α and β in the form of following relationship: $\alpha = F(\beta, v)$. □

Let's apply Lemma 5.2 to the equations (5.17). As a result we obtain the differential equation binding a and β:

$$\frac{\partial \ln|a|}{\partial v} = F(\beta, v). \tag{5.18}$$

Let's substitute (5.18) into the equation (5.11). Then

$$\frac{\partial \ln|a|}{\partial x^q} + F(\beta, v)\,\frac{\partial\beta}{\partial x^q} = \frac{\partial^2\beta}{\partial x^q\,\partial v}. \tag{5.19}$$

Let $\Phi(\beta, v)$ be the antiderivative of the function $F(\beta, v)$ with respect to β for fixed v. In other words, the function Φ is connected with F by the relationship:

$$F(\beta, v) = \frac{\partial\Phi(\beta, v)}{\partial\beta}, \tag{5.20}$$

that defines $\Phi(\beta, v)$ up to the summand depending on v only:

$$\Phi(\beta, v) \to \Phi(\beta, v) + \Psi(v). \tag{5.21}$$

After the substitution (5.20) into (5.19) taking into account that $\beta = \beta(\mathbf{x}, v)$ lets us to rewrite the equation (5.19) in the following form:

$$\frac{\partial}{\partial x^q}\left(\ln|a| + \Phi(\beta, v) - \frac{\partial\beta}{\partial v}\right) = 0. \tag{5.22}$$

By virtue of (5.22) the expression in parentheses is the quantity depending on v only. Therefore because of the arbitrariness (5.21) in the choice of function $\Phi(\beta, v)$ we have:

$$\ln|a| = \frac{\partial\beta}{\partial v} - \Phi(\beta, v). \tag{5.23}$$

The substitution of (5.23) into the equation (5.18) leads to the differential equation that for the given $\Phi(\beta, v)$ defines the dependence of β on v:

$$\beta'' = \frac{\partial\Phi(\beta, v)}{\partial\beta}\,(\beta' + 1) + \frac{\partial\Phi(\beta, v)}{\partial v}. \tag{5.24}$$

The relation of the quantities β and $\Phi(\beta, v)$ and the force field of the dynamical system contains an element of arbitrariness (see (5.21)). This arbitrariness defines the following gauge transformations that do not change the form of the equation (5.24):

$$\begin{aligned}&\beta \to \tilde\beta(\mathbf{x}, v) = \beta(\mathbf{x}, v) + \psi(v),\\ &\Phi \to \tilde\Phi(\tilde\beta, v) = \Phi(\tilde\beta - \psi(v), v) + \psi'(v).\end{aligned} \tag{5.25}$$

The transformations (5.25) do not change the quantities $\alpha = \ln|a|$ in (5.23). The equation (5.24) can be rewritten in the form of a system of two equations of the first order

$$\begin{cases}\beta' = \alpha + \Phi(\beta, v),\\ \alpha' = \dfrac{\partial\Phi(\beta, v)}{\partial\beta}.\end{cases} \tag{5.26}$$

Note that the system (5.26) also admits the gauge transformations (5.25) complemented by the relationship $\alpha \to \tilde\alpha(v) = \alpha(v)$.

In general case the equation (5.24) as well as the system (5.26) is not possible to solve explicitly. However this equation enables to characterize exactly the extent of arbitrariness in the definition of force field of the dynamical system accepting the normal shift. For fixed choice of the function $\Phi(\beta, v)$ the general solution of the equation (5.24) contains two integration constants f and h:

$$\beta(v) = B_\Phi(v, f, h). \tag{5.27}$$

The parameters f and h in (5.27) can depend on the coordinates $x^1, \dots, x^n$. So, f and h are two scalar fields on the manifold M: $f = f(\mathbf{x})$ and $h = h(\mathbf{x})$. The corresponding force field of the dynamical system for (5.27) has the following components:

$$F_q = \frac{\exp(\partial B_\Phi/\partial v)}{\exp(\Phi(B_\Phi, v))}\,N_q + |\mathbf{v}|\left(\frac{\partial B_\Phi}{\partial f}\,\nabla_i f + \frac{\partial B_\Phi}{\partial h}\,\nabla_i h\right)(2\,N^i\,N_q - \delta^i_q). \tag{5.28}$$

The formula (5.28) is considerably less effective than the formulae (5.9) and (5.10) for the cases considered earlier. It contains the function B_Φ which is not arbitrary. Some special cases when this formula can be made more effective we'll consider in following section.

Theorem 5.1. *The force field* $\mathbf{F}$ *of the dynamical system that admit the normal shift is locally defined by one of three formulae* (5.9), (5.10) *or* (5.28). *In the last case it contains three arbitrary parameters: two scalar fields* $f(\mathbf{x})$ *and* $h(\mathbf{x})$ *and the function of two parameters* $\Phi(\beta, v)$.

Example 3. In [5] and [6] the class of metrizable dynamical systems that admit the normal shift was studied. Consider these systems in the context of the above construction. Let us choose the function $\Phi(\beta, v)$ of the special form:

$$\Phi(\beta, v) = -\ln H(v\, e^{-\beta/v}), \tag{5.29}$$

where $H = H(\xi)$ is some smooth function of one variable. The substitution of (5.29) into (5.24) leads to the equation:

$$\beta'' = \frac{H'}{H}\, e^{-\beta/v} \left(\beta' - \frac{\beta}{v}\right). \tag{5.30}$$

Finding the general solution of the equation (5.30) in explicit form is too problematic. However one particular solution is easily guessed:

$$\beta(\mathbf{x}, v) = v\, f(\mathbf{x}). \tag{5.30}$$

Here f is some scalar field on the manifold M. The substitution of (5.29) and (5.30) into (5.23) defines the field a. The field $\mathbf{b}$ is also easily calculated:

$$a = H(v\, e^{-f})\, e^{f}, \qquad\qquad b_q = v\, \nabla_q f. \tag{5.32}$$

The fields (5.32) define dynamical systems with the force field of the form:

$$F_q = -|\mathbf{v}|^2\, \nabla_q f + 2\, \nabla_k f\, v^k\, v_i + N_q\, H(|\mathbf{v}|\, e^{-f})\, e^{f}.$$

This is the force field of the metrizable dynamical systems from [6]. It contains one arbitrary scalar field f and an arbitrary function H of one variable. It's clear that this case does not exhaust the functional arbitrariness declared by theorem 5.1. This fact confirms the existence of non-trivial non-metrizable dynamical systems accepting the normal shift in the dimension $n \geq 3$.

6. Some new examples.

Example 4. Consider the case, when the equation (5.24) becomes linear. So, let us choose the function $\Phi(\beta, v)$ being linear with respect to β and set

$$\Phi(\beta, v) = -\frac{\phi''(v)}{\phi'(v)}\, \beta + \ln \phi'(v). \tag{6.1}$$

The expression (6.1) is not a general form of the function linear in β, however the general case can be reduced to (6.1) by the gauge transformation (5.25). The system of equations (5.6) equivalent to the equation (5.24) for the function $\Phi(\beta, v)$ of the form (6.1) is written as:

$$\begin{cases} \beta' = \alpha - \frac{\phi''(v)}{\phi'(v)}\, \beta + \ln \phi'(v), \\ \alpha' = -\frac{\phi''(v)}{\phi'(v)} \end{cases} \tag{6.2}$$

The system (6.2) is integrated easily. As constants of integration two scalar fields $f(\mathbf{x})$ and $h(\mathbf{x})$ arise:

$$\alpha = -\ln \phi'(v) + \ln h, \qquad\qquad \beta = \frac{\ln h\,\phi(v) + f}{\phi'(v)}. \tag{6.3}$$

Hence for the force field $\mathbf{F}$ of the dynamical system by virtue of (4.20) and the scalar substitution (3.2) we obtain the expression:

$$F_q = \frac{h}{\phi'(|\mathbf{v}|)}\,N_q + |\mathbf{v}|\left(\frac{\nabla_i h}{h}\,\frac{\phi(|\mathbf{v}|)}{\phi'(|\mathbf{v}|)} + \frac{\nabla_i f}{\phi'(|\mathbf{v}|)}\right)(2\,N^i\,N_q - \delta^i_q). \tag{6.4}$$

Consider the equation (5.24) once again. The general solution of this equation depends on two integration constants: $\beta = B_\Phi(v, f, h)$. Let the parameter h be functionally expressed via the parameter f, i.e. $h = h(f)$. This diminishes the number of arbitrary parameters and leads to the function

$$\beta = \beta(v, f) = B_\Phi(v, f, h(f)), \tag{6.5}$$

that defines some one-parameter subfamily in two-parameter family of solutions of the equation (5.24). Let's consider (6.5) as a function of two variables and differentiate it with respect to v. As a result we obtain one more function of two variables: $\beta' = \beta'(v, f)$. Without loss of generality, one can suppose that the dependence on f in (6.5) is locally reversible: $f = f(\beta, v)$. Let's substitute this into the function $\beta' = \beta'(v, f)$. As a result we obtain the following differential equation of the first order:

$$\beta' = U(\beta, v), \tag{6.6}$$

where $U(\beta, v) = \beta'(v, f(\beta, v))$. The equation (6.6) is compatible with (5.24). One-parameter family of its solutions is exactly the subfamily (6.5) of solutions of the equation (5.24). Let us differentiate (6.6) with respect to v and substitute the result into (5.24). It leads to the relationship:

$$\frac{\partial U(\beta, v)}{\partial \beta}\,U(\beta, v) + \frac{\partial U(\beta, v)}{\partial v} = \frac{\partial \Phi(\beta, v)}{\partial \beta}\,(U(\beta, v) + 1) + \frac{\partial \Phi(\beta, v)}{\partial v}, \tag{6.7}$$

that binds the functions $U(\beta, v)$ and $\Phi(\beta, v)$. The relationship (6.7) is the compatibility condition of two ordinary differential equations (6.6) and (5.24).

Note that (6.7) is one equation for two functions. To solve (6.7) let's introduce a new function $W(\beta, v) = U(\beta, v) - \Phi(\beta, v)$. Then the equation (6.7) can be written in the form:

$$\frac{\partial U(\beta, v)}{\partial \beta} - \frac{\partial W(\beta, v)}{\partial \beta}\,U(\beta, v) = \frac{\partial W(\beta, v)}{\partial v} + \frac{\partial W(\beta, v)}{\partial \beta}. \tag{6.8}$$

Let's make the following substitution into the differential equation (6.8):

$$W(\beta, v) = -\ln w(\beta, v), \qquad\qquad U(\beta, v) = \frac{u(\beta, v)}{w(\beta, v)}. \tag{6.9}$$

After this substitution the equation (6.8) becomes quite simple:

$$\frac{\partial u}{\partial \beta} + \frac{\partial w}{\partial \beta} + \frac{\partial w}{\partial v} = 0. \tag{6.10}$$

The general solution of the equation (6.10) is defined locally by one arbitrary function of two variables $\phi(\beta, v)$:

$$u = -\frac{\partial \phi}{\partial \beta} - \frac{\partial \phi}{\partial v}, \qquad\qquad w = \frac{\partial \phi}{\partial \beta}.$$

Hence for the function $U(\beta, v)$ and for the function $W(\beta, v)$ introduced above we obtain:

$$U = -\frac{\partial \phi / \partial v}{\partial \phi / \partial \beta} - 1, \qquad\qquad W = -\ln\left(\frac{\partial \phi}{\partial \beta}\right). \tag{6.11}$$

Let us write the first relationship from (6.11) in the form of an equation for the function $\phi(\beta, v)$:

$$\frac{\partial \phi(\beta, v)}{\partial \beta}\,(U(\beta, v) + 1) + \frac{\partial \phi(\beta, v)}{\partial v} = 0. \tag{6.12}$$

Such equation is solved by the methods of characteristics. The characteristics of the equation (6.12) are defined by the following ordinary differential equation

$$\beta' = U(\beta, v) + 1, \tag{6.13}$$

and the function $\phi(\beta, v)$ by virtue of (6.12) is the first integral (conservation law) of the equation (6.13). Hence a value of the function ϕ can be used to parameterize the family of solutions of the equation (6.13). Let's write it as:

$$\beta = B(v, \phi). \tag{6.14}$$

The function (6.14) differs from (6.5). It inverts the functional dependence ϕ on β, i.e. $B(v, \phi(\beta, v)) \equiv \beta$. Differentiating this relationship we obtain:

$$\frac{\partial B}{\partial \phi}\,\frac{\partial \phi}{\partial \beta} = 1, \qquad\qquad \frac{\partial B}{\partial v} + \frac{\partial B}{\partial \phi}\,\frac{\partial \phi}{\partial v} = 0. \tag{6.15}$$

Let us express the derivatives $\partial\phi/\partial\beta$ and $\partial\phi/\partial v$ from (6.15) via $\partial B/\partial\phi$ and $\partial B/\partial v$ and substitute the result into (6.11). It gives:

$$U = \frac{\partial B}{\partial v} - 1, \qquad\qquad W = \ln\left(\frac{\partial B}{\partial \phi}\right). \tag{6.16}$$

Form (6.16) it is easy to define the function Φ but in the variables v and ϕ:

$$\Phi = U - W = B_v - \ln(B_\phi) - 1. \tag{6.17}$$

In the same variables v and ϕ it is naturally to write the equation (6.6) too. So, we obtain the equation defining the dependence of ϕ on v:

$$\phi' = -\frac{1}{B_\phi(v, \phi)}. \tag{6.18}$$

It is derived from $\phi' = \partial\phi/\partial\beta \cdot U + \partial\phi/\partial v$ and from the relationships (6.15) defining partial derivatives ϕ with respect to v and β.

Summarizing the computations just done, note that the initial equation (5.24) contains an arbitrary function $\Phi(\beta, v)$, we've expressed it via $\phi(\beta, v)$ and then via the function $B(v, \phi)$. Hence it is convenient to consider $B(v, \phi)$ as a primary function and to express everything in terms of this function.

Example 5. Let us render concrete the choice of the function $B(v,\phi)$ to solve the equation (6.18) explicitly. Let

$$B(v,\phi)=-\frac{1}{3\,v}\,\ln\left(\frac{\phi}{\phi+3}\right).$$

Then the equation (6.18) for the function $\phi(v)$ gets the form of the Bernoulli equation:

$$\phi'=\phi^2\,v+3\,v\,\phi.$$

The general solution of this equation contains one integration constant f that depends on the space variables $x^1,\dots,x^n$ and defines a scalar field on the manifold M:

$$\phi=\phi(v,f)=\frac{3}{\exp(-3v^2/2-2\,f)-1}.$$

Now it remains to compute the fields a and $\mathbf{b}$ in (4.20) to define the field A of the scalar substitution (3.2)

$$\begin{aligned}
a&=e^W=B_\phi=-\frac{4}{9\,v}\,\sinh^2\left(-\frac{3\,v^2}{4}-f\right),\\
b_i&=\frac{\partial\beta}{\partial x^i}=\frac{\partial B}{\partial\phi}\,\frac{\partial\phi}{\partial f}\,\nabla_i f=-\frac{\nabla_i f}{3\,v}.
\end{aligned}$$

Now the expression for the components of force field of the dynamical system accepting the normal shift is easily derived:

$$F_q=-\frac{4}{9\,|\mathbf{v}|}\,\sinh^2\left(-\frac{3\,|\mathbf{v}|^2}{4}-f\right)\,N_q-\frac{\nabla_i f\,(2\,N^i\,N_q-\delta^i_q)}{3}. \tag{6.19}$$

The force field (6.19) contains one scalar field f only and this fact distinguishes this force field from the general case (5.28) and from the case (6.4). It happens because we've replaced the equation of the second order (5.24) by the equation of the first order (6.6).

8. Final remarks.

Three cases defined by the formulae (5.9), (5.10) and (5.28) characterize completely the local construction of the dynamical systems accepting the normal shift on Riemannian manifolds of the dimension $n\geq 3$. Two dimensional case is distinguished by a larger extent of diversity. In multidimensional case the system of normality equations is strongly overdetermined (however it is compatible). Hence the set of its solutions is more restricted.

Note that the existence and number of local solutions of normality equations are not connected with the features of metric: the constantness of curvature, the number of isometries etc. Hence the existence and number of smooth global solutions of these equations depend exclusively on the topology of the manifold M.

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