PROBLEM OF METRIZABILITY FOR THE DYNAMICAL SYSTEMS ACCEPTING THE NORMAL SHIFT.

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Abstract. The problem of metrizability for the dynamical systems accepting the normal shift is formulated and solved. The explicit formula for the force field of metrizable Newtonian dynamical system $\ddot{\mathbf{r}} = \mathbf{F}(\mathbf{r}, \dot{\mathbf{r}})$ is found.

1. Introduction.

The class of Newtonian dynamical systems accepting the normal shift was first defined in [1] (see also [2] and [3]). It's the class of dynamical systems in $\mathbb{R}^n$ given by the differential equations of the form

$$\ddot{\mathbf{r}} = \mathbf{F}(\mathbf{r}, \dot{\mathbf{r}}) \tag{1.1}$$

and possessing some additional geometrical property: the property of conserving the orthogonality of trajectories and the hypersurfaces shifted along these trajectories. The idea for considering such systems was found as a result of generalizing the classical construction of normal shift which is known also as a Bonnet transformation. Let S be the hypersurface in $\mathbb{R}^n$. From each point M on S we draw the segment MM' with the fixed length l along the normal vector to S. The points M' then form another hypersurface S', the segment MM' being perpendicular to S'. The transformation $f : S \longrightarrow S'$ just described is the classical Bonnet transformation. It has the generalization for non-Euclidean situation: for the Riemannian metric g_{ij} one should replace the segment MM' of a straight line by the segment of geodesic line with the length l. The transformation $f : S \longrightarrow S'$ in this case is the metrical Bonnet transformation or the normal shift with respect to the metric g_{ij}. Trajectories of this shift are defined by the equation of geodesic lines

$$\ddot{r}^k = -\Gamma^k_{ij}\dot{r}^i\dot{r}^j \tag{1.2}$$

This equation can be treated as a particular case of the Newtonian dynamical system (same indices on different levels in (1.2) and everywhere below imply summation).

In [1] (see also [2] and [3]) the classical Bonnet transformation was generalized for the case of dynamical systems (1.1) in $\mathbb{R}^n$. Other generalization is a metrical Bonnet transformation. So quite natural question is: how do these two generalizations relate each other? In order to treat (1.2) as a dynamical system accepting the normal shift in standard Euclidean metric in $\mathbb{R}^n$ we should have the coincidence of the concept of orthogonality with respect to both Euclidean and non-Euclidean metrics δ_{ij} and g_{ij} in $\mathbb{R}^n$. That is $g_{ij} = e^{-2f}\delta_{ij}$ should be conformally Euclidean metric. Problem of metrizability then can be stated as follows.

Problem of metrizability. *Under which circumstances the dynamical system* (1.1) *accepting the normal shift in the sense of* [1] *and* [4] *is equivalent to the metrical normal shift for some conformally Euclidean metric in* $\mathbb{R}^n$.

Studying this problem and solving it is the goal of present paper. As we shall see below its solution is explicit and constructive.

2. Geodesic lines of conformally Euclidean metric in $\mathbb{R}^n$

Let δ_{ij} be the metric tensor for the standard Euclidean metric in $\mathbb{R}^n$. Let's consider conformally Euclidean metric

$$g_{ij} = e^{-2f}\delta_{ij} \tag{2.1}$$

where $f = f(\mathbf{r}) = f(r^1, \dots, r^n)$ is some scalar function in $\mathbb{R}^n$. Metrical connection for the metric (2.1) is given by

$$\Gamma^k_{ij} = \delta_{ij}\delta^{ks}\partial_s f - (\delta^k_i \partial_j f + \delta^k_j \partial_i f) \tag{2.2}$$

Here by ∂_s we denote the partial derivative with respect to r^s. The equation of geodesic lines for (2.2) has the following form

$$\ddot{r}^k = -\delta^{ks}\partial_s f \dot{r}^i \delta_{ij} \dot{r}^j + 2\partial_i f \dot{r}^i \dot{r}^k \tag{2.3}$$

In a vectorial form like (1.1) the force field for the dynamical system corresponding to (2.3) is written as follows

$$\mathbf{F}(\mathbf{r}, \mathbf{v}) = -\nabla f |\mathbf{v}|^2 + 2\,\langle \nabla f, \mathbf{v}\rangle\, \mathbf{v} \tag{2.4}$$

Here ∇f is a gradient of the function f considered as a vector in $\mathbb{R}^n$ and $\langle \nabla f, \mathbf{v}\rangle$ is a standard Euclidean scalar product of ∇f with the vector of velocity $\mathbf{v}$. The modulus of the velocity vector in (2.4) is also calculated in a standard metric δ_{ij}.

Because of its geometrical origin the dynamical system (1.1) with force field (2.4) accepts the normal shift. However it is curious to check this fact directly. Following the receipt of [4] and [5] we choose the coordinates $u^1, \dots, u^{n-1}$ on a unit sphere $|\mathbf{v}| = 1$ in the velocity space. Let's denote $v = |\mathbf{v}|$ and let $\mathbf{N}$ be the unit vector directed along the vector $\mathbf{v}$. Then

$$\mathbf{N} = \mathbf{N}(u^1, \dots, u^{n-1}) \qquad\qquad \mathbf{M}_i = \frac{\partial \mathbf{N}}{\partial u^i} \tag{2.5}$$

For the derivatives of the vectors $\mathbf{M}_i$ defined by (2.5) one has a Weingarten derivation formula

$$\frac{\partial \mathbf{M}_i}{\partial u^j} = \vartheta^k_{ij}\mathbf{M}_k - G_{ij}\mathbf{N} \tag{2.6}$$

where $G_{ij} = \langle \mathbf{M}_i, \mathbf{M}_j\rangle$ is an induced metric on the unit sphere. Let's consider the expansion of the force field (2.4) in the basis formed by $\mathbf{N}$ and the vectors $\mathbf{M}_i$

$$\mathbf{F} = A\mathbf{N} + B^i\mathbf{M}_i \tag{2.7}$$

Let's find the spatial gradients of the coefficients of the expansion (2.7) and then let's expand them in the same basis. As a result we get

$$\frac{\partial A}{\partial r^k} = aN_k + \alpha^p M_{pk} \qquad \frac{\partial B^i}{\partial r^k} = b^i N_k + \beta^{ip} M_{pk} \tag{2.8}$$

According to the results of [4] and [5] the weak normality condition for the dynamical system with the force field (2.7) is given by the following equations

$$B^i = -G^{ik}\frac{\partial A}{\partial u^k} \tag{2.9}$$

$$\begin{aligned} &\alpha^i + \frac{B^q B^k}{v^2}\vartheta^i_{qk} - \frac{B^i A}{v^2} + b^i + \\ &\quad + \frac{A}{v}\frac{\partial B^i}{\partial v} + \frac{\partial B^i}{\partial u^k}\frac{B^k}{v^2} - \frac{b^i}{v}\frac{\partial A}{\partial v} = 0 \end{aligned} \tag{2.10}$$

In order to check the equations (2.9) and (2.10) in the case of force field (2.4) we calculate the coefficients of expansion (2.7) explicitly. Because of the orthogonality $\langle \mathbf{N}, \mathbf{M}_i \rangle = 0$ we have

$$\begin{aligned} A &= \langle \mathbf{F}, \mathbf{N} \rangle = \langle \nabla f, \mathbf{N} \rangle v^2 \\ B^i &= G^{ik} \langle \mathbf{F}, \mathbf{M}_k \rangle = -G^{ik} \langle \nabla f, \mathbf{M}_k \rangle v^2 \end{aligned} \tag{2.11}$$

From (2.11) we can see that the relationship (2.9) becomes the identity due to (2.5). For the coefficients α^i and b^i in (2.8) we find

$$\alpha^i = G^{ik} \frac{\partial A}{\partial r^q} \delta^{qs} M_{ks} = G^{ik} \frac{\partial^2 f}{\partial r^q \partial r^m} \delta^{mp} N_p \delta^{qs} M_{ks} v^2$$
$$b^i = \frac{\partial B^i}{\partial r^q} \delta^{qs} N_s = -G^{ik} \frac{\partial^2 f}{\partial r^q \partial r^m} \delta^{mp} M_{kp} \delta^{qs} N_s v^2$$

From these two equalities we see that α^i and b^i differ only by sign. When substituting them into (2.10) they vanish. Let's calculate the sixth term in (2.10) using the formula (2.6)

$$\frac{\partial B^i}{\partial u^k} \frac{B^k}{v^2} = \frac{B^i A}{v^2} + \left(\frac{\partial G^{iq}}{\partial u^k} + G^{is} \vartheta^q_{sk} \right) G_{qp} \frac{B^p B^k}{v^2}$$

Taking into account the concordance of the metric G_{ij} and the metrical connection ϑ^k_{ij} we can bring this equation into the following form

$$\frac{\partial B^i}{\partial u^k} \frac{B^k}{v^2} = \frac{B^i A}{v^2} - G^{sq} \vartheta^i_{sk} G_{qp} \frac{B^p B^k}{v^2} = \frac{B^i A}{v^2} - \frac{B^s B^k}{v^2} \vartheta^i_{sk}$$

When substituting this form of sixth term into (2.10) it cancels the second and the third terms in (2.10). Because of quadratic dependence of A and B^i in (2.11) upon v fifth and seventh terms in (2.10) cancel each other. Resuming all above we conclude that the equations (2.9) and (2.10) hold identically for the components of the expansion (2.7). Weak normality condition for (2.4) is fulfilled.

Next step consists in substituting the geodesic flows of the form (2.4) by some dynamical systems similar to them. Let's start with the Euclidean metric $g_{ij} = \delta_{ij}$. Force field (2.4) then is zero $\mathbf{F}(\mathbf{r}, \mathbf{v}) = 0$, trajectories are straight lines. For the dynamical system with the force field $\mathbf{F}(\mathbf{r}, \mathbf{v}) = \mathbf{v}$ they are also straight lines. From geometrical point of view these two dynamical systems realize the same normal shift. This example shows that for to solve the problem of metrizability one should find all dynamical systems accepting the normal shift in $\mathbb{R}^n$ for which the trajectories are the geodesic lines of conformally Euclidean metrics. The problem of geometrical coincidence of trajectories for two different dynamical systems was first stated in [6]. There the following pairs of dynamical systems were considered

$$\begin{aligned} \ddot{r}^k + \Gamma^k_{ij} \dot{r}^i \dot{r}^j &= F^k(\mathbf{r}, \dot{\mathbf{r}}) \\ \ddot{r}^k + \tilde{\Gamma}^k_{ij} \dot{r}^i \dot{r}^j &= \tilde{F}^k(\mathbf{r}, \dot{\mathbf{r}}) \end{aligned} \tag{2.12}$$

In the case of $F^k = \tilde{F}^k = 0$ the condition of coincidence of trajectories for the dynamical systems (2.12) is known as a condition of geodesical equivalence for the affine connections Γ^k_{ij} and $\tilde{\Gamma}^k_{ij}$. The detailed discussion of the questions connected with geodesical equivalence and geodesical maps can be found in the monograph [7] (see also [8] and [9]).

For nonzero F^k and $\tilde{F}^k$ if the trajectories of the dynamical systems (2.12) coincide then one say that one of these systems is a modeling system for another. This case was considered in [10-13]. The terms *inheriting the trajectories* and *trajectory equivalence* below seem to be more preferable than the term *modeling* from purely linguistical point of view.

3. Inheriting the trajectories and trajectory equivalence of dynamical systems.

Let's consider the pair of Newtonian dynamical systems (1.1) of the second order

$$\partial_{tt} \mathbf{r} = \mathbf{F}_1(\mathbf{r}, \partial_t \mathbf{r}) \tag{3.1}$$

$$\partial_{\tau\tau} \mathbf{r} = \mathbf{F}_2(\mathbf{r}, \partial_\tau \mathbf{r}) \tag{3.2}$$

Trajectories of these systems are defined by the initial position and initial velocity

$$\mathbf{r}\big|_{t=0}=\mathbf{r}_0 \qquad \partial_t\mathbf{r}\big|_{t=0}=\mathbf{v}_0 \tag{3.3}$$

$$\mathbf{r}\big|_{\tau=0}=\mathbf{r}_0 \qquad \partial_\tau\mathbf{r}\big|_{\tau=0}=\mathbf{w}_0 \tag{3.4}$$

Let $\mathbf{r}=\mathbf{R}_1(t,\mathbf{r}_0,\mathbf{v}_0)$ and $\mathbf{r}=\mathbf{R}_2(\tau,\mathbf{r}_0,\mathbf{w}_0)$ are the solutions of the equations (3.1) and (3.2) with the initial conditions (3.3) and (3.4).

Definition 1. Say that the dynamical system (3.2) inherits the trajectories of the system (3.1) if for any pair of vectors $\mathbf{r}_0$ and $\mathbf{w}_0\neq 0$ one can find the vector $\mathbf{v}_0$ and the function $T(\tau)$ such that $T(0)=0$ and the following equality

$$\mathbf{R}_1(T(\tau),\mathbf{r}_0,\mathbf{v}_0)=\mathbf{R}_2(\tau,\mathbf{r}_0,\mathbf{w}_0) \tag{3.5}$$

holds identically by τ in some neighborhood of zero $\tau=0$.

Definition 2. Two dynamical systems are called trajectory equivalent if each of them inherits the trajectories of the other.

Differentiating (3.5) by τ for $\tau=0$ we get the relation between the vectors $\mathbf{w}_0$ and $\mathbf{v}_0$ in the following form

$$\mathbf{v}_0\,\partial_\tau T(0)=\mathbf{w}_0 \tag{3.6}$$

From (3.6) we see that $\mathbf{v}_0\neq 0$ and $\partial_\tau T(0)\neq 0$. For nonzero τ we have

$$\partial_t\mathbf{R}_1(T(\tau),\mathbf{r}_0,\mathbf{v}_0)\,\partial_\tau T=\partial_\tau\mathbf{R}_2(\tau,\mathbf{r}_0,\mathbf{w}_0) \tag{3.7}$$

Let's differentiate the relationship (3.7) by τ. Then for $\tau=0$ we get

$$\mathbf{F}_1(\mathbf{r}_0,\mathbf{v}_0)\,\partial_\tau T(0)^2+\mathbf{v}_0\,\partial_{\tau\tau}T(0)=\mathbf{F}_2(\mathbf{r}_0,\mathbf{w}_0) \tag{3.8}$$

The relationship (3.8) bind the force field of the dynamical system (3.2) with the force field of the system (3.1) trajectories of which are inherited according to the definition 1. Because of $\mathbf{v}_0\neq\mathbf{w}_0$ this relationship is nonlocal. However, if $\mathbf{F}_1(\mathbf{r},\mathbf{v})$ is homogeneous function by $\mathbf{v}$ this relationship becomes local. It's the very case we shall consider below.

Let γ be the degree of homogeneity for the function $\mathbf{F}_1(\mathbf{r},\mathbf{v})$ with respect to its vectorial argument $\mathbf{v}$. The relationship (3.8) then has the following form

$$\mathbf{F}_1(\mathbf{r},\mathbf{w})\,\partial_\tau T(0)^{2-\gamma}+\mathbf{w}\,\partial_{\tau\tau}T(0)\,\partial_\tau T(0)^{-1}=\mathbf{F}_2(\mathbf{r},\mathbf{w}) \tag{3.9}$$

Since the initial data $\mathbf{r}_0$ and $\mathbf{w}_0$ in (3.4) are arbitrary we omit the index 0 everywhere in (3.9). Because of (3.9) the vectors $\mathbf{w}$, $\mathbf{F}_1(\mathbf{r},\mathbf{w})$ and $\mathbf{F}_2(\mathbf{r},\mathbf{w})$ are linearly dependent. Two cases are possible:

(1) special case when the vectors $\mathbf{F}_1(\mathbf{r},\mathbf{w})$ and $\mathbf{w}$ are linearly dependent,
(2) generic case when vectors $\mathbf{F}_1(\mathbf{r},\mathbf{w})$ and $\mathbf{w}$ are linearly independent.

Let's start with the first case. Here for the force field of the dynamical system (3.1) we have

$$\mathbf{F}_1(\mathbf{r},\mathbf{w})=\mathbf{w}\frac{H_1(\mathbf{r},\mathbf{w})}{|\mathbf{w}|} \tag{3.10}$$

where $H_1(\mathbf{r},\mathbf{w})$ is a scalar function homogeneous by $\mathbf{w}$ with order of homogeneity γ. Because of (3.9) the force field $\mathbf{F}_2(\mathbf{r},\mathbf{w})$ has the similar form

$$\mathbf{F}_2(\mathbf{r},\mathbf{w})=\mathbf{w}\frac{H_2(\mathbf{r},\mathbf{w})}{|\mathbf{w}|} \tag{3.11}$$

However the function $H_2(\mathbf{r},\mathbf{w})$ in (3.11) shouldn't be homogeneous by $\mathbf{w}$. Trajectories of the dynamical systems (3.1) and (3.2) with the force fields (3.10) and (3.11) are straight lines, therefore any two such systems are inheriting the trajectories of each other even when the function $H_1(\mathbf{r},\mathbf{w})$ is not homogeneous by $\mathbf{w}$.

The second case is more complicated. Here vector $\mathbf{F}_2(\mathbf{r},\mathbf{w})$ can be decomposed by the vectors $\mathbf{F}_1(\mathbf{r},\mathbf{w})$ and $\mathbf{w}$

$$\mathbf{F}_2(\mathbf{r},\mathbf{w}) = C(\mathbf{r},\mathbf{w})\mathbf{F}_1(\mathbf{r},\mathbf{w}) + \mathbf{w}\frac{H(\mathbf{r},\mathbf{w})}{|\mathbf{w}|} \tag{3.12}$$

Differentiating (3.7) by τ for $\tau \neq 0$ we get the equation for the function $T(\tau)$

$$\mathbf{F}_1(\mathbf{r},\mathbf{w})\,\partial_\tau T^{2-\gamma} + \mathbf{w}\,\partial_{\tau\tau}T\,\partial_\tau T^{-1} = \mathbf{F}_2(\mathbf{r},\mathbf{w}) \tag{3.13}$$

where $\mathbf{r}$ and $\mathbf{w}$ are the functions of τ defined by (3.2) and (3.4)

$$\mathbf{r} = \mathbf{r}(\tau) = \mathbf{R}_2(\tau,\mathbf{r}_0,\mathbf{w}_0) \qquad\qquad \mathbf{w} = \mathbf{w}(\tau) = \partial_\tau \mathbf{r}$$

Since the expansion of $\mathbf{F}_2(\mathbf{r},\mathbf{w})$ by two linearly independent vectors $\mathbf{F}_1(\mathbf{r},\mathbf{w})$ and $\mathbf{w}$ is unique from (3.12) and (3.13) we get

$$\begin{aligned} &(\partial_\tau T)^{2-\gamma} = C(\mathbf{r},\mathbf{w}) \\ &\partial_{\tau\tau}T = \frac{H(\mathbf{r},\mathbf{w})}{|\mathbf{w}|}\partial_\tau T \end{aligned} \tag{3.14}$$

Because of (3.14) we should consider two different subcases depending on the value of γ: $\gamma \neq 2$ and $\gamma = 2$. Let's consider the first subcase. Here the function $T(\tau)$ is defined by the equation of the first order $\partial_\tau T = C(\mathbf{r},\mathbf{w})^{\gamma-2}$. Therefore the second equation (3.14) should be the consequence of the first one. Differentiating the first equation (3.14) with respect to τ we get

$$\partial_\tau \ln(\partial_\tau T) = (\gamma - 2)\left(w^i\frac{\partial \ln(C)}{\partial r^i} + F_2^i\frac{\partial \ln(C)}{\partial w^i}\right)$$

Comparing this equation with the second equation (3.14) and substituting F_2^i in it by (3.13) we get the relationship

$$\frac{H}{|\mathbf{w}|} = \left(w^i\frac{\partial \ln(C)}{\partial r^i} + F_1^i\frac{\partial C}{\partial w^i}\right)\left(\frac{1}{\gamma-2} - w^i\frac{\partial ln(C)}{\partial w^i}\right)^{-1}$$

which express $H(\mathbf{r},\mathbf{w})$ through $C(\mathbf{r},\mathbf{w})$. Force field $\mathbf{F}_2(\mathbf{r},\mathbf{w})$ of the dynamical system (3.2) inheriting the trajectories of the system (3.1) in this case is defined by one arbitrary scalar function $C(\mathbf{r},\mathbf{w})$.

Now let's consider the second subcase $\gamma = 2$ Here the first equation (3.14) is trivial and $C(\mathbf{r},\mathbf{w}) = 1$. The function $T(\tau)$ is defined only by the second equation (3.14). The force field of the system (3.2) defined by (3.12) contains one arbitrary function $H(\mathbf{r},\mathbf{w})$. This case $\gamma = 2$ is the most interesting since the force fields of the form (2.4) are covered by this case. Dynamical systems inheriting their trajectories are defined by the following force fields

$$\mathbf{F}(\mathbf{r},\mathbf{v}) = -\nabla f|\mathbf{v}|^2 + 2\,\langle\nabla f,\mathbf{v}\rangle\,\mathbf{v} + \frac{\mathbf{v}}{|\mathbf{v}|}H(\mathbf{r},\mathbf{v}) \tag{3.15}$$

where $f = f(\mathbf{r})$. However not all the dynamical systems with the force field (3.15) are accepting the normal shift.

4. Problem of metrizability.

Let the dynamical system (1.1) with the force field (3.15) be accepting the normal shift. The force field (3.15) differs from (2.4) by the term $\mathbf{N}H(\mathbf{r},\mathbf{v})$. Therefore the components of the expansion (2.7) for (3.15) are slightly different from that of (2.4)

$$\tilde{A} = A + H \qquad\qquad \tilde{B}^i = B^i \tag{4.1}$$

But both they are satisfying the equations (2.9) and (2.10). Substituting (4.1) into (2.9) we get the vanishing of the derivatives

$$\frac{\partial H}{\partial u^k} = 0 \qquad\qquad H = H(\mathbf{r}, v) = H(\mathbf{r}, |\mathbf{v}|)$$

Therefore the function H in (3.15) doesn't depend on the direction of the vector of velocity. It depends only on the modulus of velocity and on the position of the mass point in the space. For the components of the expansion (2.8) which are used in (2.10) we have

$$\tilde{\alpha}^i = \alpha^i + h^i \qquad\qquad \tilde{b}^i = b^i \tag{4.2}$$

The parameters h^i are defined by the function $H(\mathbf{r}, v)$ according to the formula

$$h^i = G^{ik} \langle \nabla H, \mathbf{M}_k \rangle = G^{ik} \frac{\partial H}{\partial r^q} \delta^{qs} M_{ks} \tag{4.3}$$

Let's substitute (4.1) and (4.2) into the equation (2.10). As a result we have

$$h^i - \frac{B^i H}{v^2} + \frac{H}{v}\frac{\partial B^i}{\partial v} - \frac{B^i}{v}\frac{\partial H}{\partial v} = 0 \tag{4.4}$$

The relationships (4.4) form the system of $n - 1$ equations for the function $H(\mathbf{r}, v)$. Taking into account (2.11) and (4.3) we can transform them to the following ones

$$\frac{\partial H}{\partial r^k} + \left(v \frac{\partial f}{\partial r^k}\right) \frac{\partial H}{\partial v} = \frac{\partial f}{\partial r^k} H \tag{4.5}$$

Let's consider the vector fields which are defined by the differential equations (4.5)

$$X^k = \frac{\partial}{\partial r^k} + \left(v \frac{\partial f}{\partial r^k}\right) \frac{\partial}{\partial v} \tag{4.6}$$

It's easy to check that the vector fields (4.6) are commuting, therefore the equations (4.5) are compatible. It's wonderful that the common solution for the system of equations (4.5) can be written explicitly

$$H(\mathbf{r}, v) = H(ve^{-f})e^f \tag{4.7}$$

Using (4.7) the final result can be formulated as a following theorem giving the solution for the problem of metrizability.

Theorem 1. *The dynamical system* (1.1) *in* $\mathbb{R}^n$ *is accepting the normal shift if its force field has the following form*

$$\mathbf{F}(\mathbf{r}, \mathbf{v}) = -\nabla f |\mathbf{v}|^2 + 2 \langle \nabla f, \mathbf{v} \rangle \mathbf{v} + \frac{\mathbf{v}}{|\mathbf{v}|} H(|\mathbf{v}|e^{-f})e^f$$

where $f = f(\mathbf{r}) = f(r^1, \ldots, r^n)$ *and* $H = H(v)$ *are two arbitrary functions. When it is metrizable the dynamical system* (1.1) *realizes the metrical normal shift for some conformally Euclidean metric in* $\mathbb{R}^n$*.*

The comparison of the force field from the theorem 1 with the examples of [1] shows that even in $\mathbb{R}^2$ one can find the non-metrizable dynamical systems accepting the normal shift. So the concept of the dynamical systems accepting the normal shift is more wide, it cannot be reduced to the metrical normal shift.

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