MULTIDIMENSIONAL DYNAMICAL SYSTEMS ACCEPTING THE NORMAL SHIFT.

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Abstract. The dynamical systems of the form $\ddot{\mathbf{r}} = \mathbf{F}(\mathbf{r}, \dot{\mathbf{r}})$ in $\mathbb{R}^n$ accepting the normal shift are considered. The concept of weak normality for them is introduced. The partial differential equations for the force field $\mathbf{F}(\mathbf{r}, \dot{\mathbf{r}})$ of the dynamical systems with weak and complete normality are derived.

1. Introduction

The concept of dynamical system accepting the normal shift was introduced in [1] as a result of generalization of the classical geometrical Bonnet transformation (or normal shift) for the case of dynamical systems. In [1] (see also [2] and [3]) the dynamical systems in $\mathbb{R}^2$ are studied (see [1], [2] and [3] for detailed reference list). In present paper we generalize the results of [1] for the multidimensional case and considser some peculiarities absent in 2-dimensional case. These results are declared in [4]. They also form the section 5 in preprint [3].

Let's consider the dynamical system describing the trajectories $\mathbf{r} = \mathbf{r}(t)$ in $\mathbb{R}^n$ particles with unit mass in the force field $\mathbf{F}(\mathbf{r}, \dot{\mathbf{r}})$

$$\ddot{\mathbf{r}} = \mathbf{F}(\mathbf{r}, \dot{\mathbf{r}}) \tag{1.1}$$

We shall use the trajectories of (1.1) for transforming the submanifolds of $\mathbb{R}^n$. Let's consider the set of particles each of which is starting at $t = 0$ from some point P on some hypersurface $S \subset \mathbb{R}^n$ in the direction of normal vector $\mathbf{n}(P)$ with some initial velocity $v(P)$. In the end of time interval t these particles form another hypersurface $S_t \subset \mathbb{R}^n$. This defines the one-parameter family of hypersurfaces and the family of diffeomorphisms

$$f_t : S \longrightarrow S_t \tag{1.2}$$

Now let's recall the following two definitions from [1] and [2].

Definition 1. Each transformation $f = f_t$ of the family (2.1) is called the normal shift along the dynamical system (1.1) if each trajectory of (1.1) crosses each submanifold S_t along its normal vector $\mathbf{n}$.

Definition 2. Dynamical system (1.1) is called the dynamical system accepting the normal shift of submanifolds of codimension 1 if for any submanifold S of codimension 1 there is the function $v = v(P)$ on S such that the transformation (2.1) defined by the system (1.1) and the initial velocity function $|\mathbf{v}(P)| = v(P)$ is the transformation of normal shift.

2. Normality conditions for the dynamical systems.

Phase space for dynamical system (1.1) is defined by the pairs of vectors $\mathbf{r}$ and $\mathbf{v}$. In all points of phase space where $\mathbf{v} \neq 0$ we introduce the spherical coordinates in $\mathbf{v}$-space. Let $u^1, \dots, u^{n-1}$ be the coordinates on the unit sphere $|\mathbf{v}| = 1$ and let $u^n = v = |\mathbf{v}|$. We define also the unit vector $\mathbf{N}$ along the vector $\mathbf{v}$ and the derivatives of $\mathbf{N}$

$$\mathbf{N} = \mathbf{N}(u^1, \dots, u^{n-1}) \qquad \mathbf{M}_i = \frac{\partial \mathbf{N}}{\partial u^i} \tag{2.1}$$

For the derivatives of the introduced vectors $\mathbf{M}_i$ one may use the standard Weingarten formulae with the metric connection ϑ^k_{ij} on the unit sphere $|\mathbf{v}| = 1$

$$\frac{\partial \mathbf{M}_i}{\partial u^j} = \vartheta^k_{ij} \mathbf{M}_k - G_{ij} \mathbf{N} \tag{2.2}$$

here $G_{ij} = G_{ij}(u^1, \dots, u^{n-1})$ is the metric tensor on unit sphere defined by scalar products $G_{ij} = \langle \mathbf{M}_i, \mathbf{M}_j \mathbf{M}_j \rangle$. In (2.2) and in what follows coinciding upper and lower indices imply summation. Force field for the dynamical system (1.1) may be represented by the formula similar to that of [1]

$$\mathbf{F} = A\mathbf{N} + B^i \mathbf{M}_i \tag{2.3}$$

The equation (1.1) itself then is rewritten as the following system of differential equations with respect to $\mathbf{r}$, v and u^i

$$\dot{\mathbf{r}} = v\mathbf{N} \qquad \dot{v} = A \qquad \dot{u}_i = v^{-1} B^i \tag{2.4}$$

Let's consider the solution of (2.4) depending on some extra parameter s and introduce the following notations for the derivatives of the coordinates in the phase space

$$\partial_s \mathbf{r} = \tau \qquad \partial_s v = w \qquad \partial_s u^i = z^i \tag{2.5}$$

Differentiating (2.4) and keeping in mind (2.1) and (2.5) we obtain the time derivatives for $\boldsymbol{\tau}$, w and z^i

$$\begin{aligned} \dot{\boldsymbol{\tau}} &= w\mathbf{N} + v\mathbf{M}_i z^i \\ \dot{w} &= \frac{\partial A}{\partial r^k} \tau^k + \frac{\partial A}{\partial v} w + \frac{\partial A}{\partial u^k} z^k \\ \dot{z}^i &= -\frac{B^i w}{v^2} + \frac{1}{v} \left(\frac{\partial B^i}{\partial r^k} \tau^k + \frac{\partial B^i}{\partial v} w + \frac{\partial B^i}{\partial u^k} z^k \right) \end{aligned} \tag{2.6}$$

The equations (2.6) here are the analogs of (3.7) from [1]. In addition to (2.5) let's introduce the following notations

$$\varphi = \langle \tau, \mathbf{N} \rangle \qquad \psi_i = \langle \tau, \mathbf{M}_i \rangle \tag{2.7}$$

Differentiating (2.7) and taking into account (2.1), (2.2), (2.6) and (2.5) we get the following equations

$$\begin{aligned}\dot\varphi &= w + \frac{B^i\psi_i}{v}\\ \dot\psi_i &= vG_{ik}z^k + \frac{B^k}{v}\left(\vartheta^p_{ik}\psi_p - G_{ik}\varphi\right)\end{aligned}\tag{2.8}$$

For the space gradients of A and B^i we may define the expansions

$$\frac{\partial A}{\partial r^k} = aN_k + \alpha^p M_{pk} \qquad\qquad \frac{\partial B^i}{\partial r^k} = b^i N_k + \beta^{ip} M_{pk}\tag{2.9}$$

Substituting (2.9) into (2.6) we derive the following equations

$$\begin{aligned}\dot w &= a\varphi + \alpha^k\psi_k + \frac{\partial A}{\partial v}w + \frac{\partial A}{\partial u^k}z^k\\ \dot z^i &= -\frac{B^i w}{v^2} + \frac{1}{v}\left(vb^i\varphi + \beta^{ik}\psi_k + \frac{\partial B^i}{\partial v}w + \frac{\partial B^i}{\partial u^k}z^k\right)\end{aligned}\tag{2.10}$$

The equations (2.8) and (2.10) form the complete system of linear differential equations with respect to φ, ψ_i, w, z_i. Let's differentiate first of the equations (2.8) by t keeping in mind all above expressions for the time derivatives of the quantities involved. As a result we obtain

$$\begin{aligned}\ddot\varphi = &\left(a - \frac{B^qB^k}{v^2}G_{qk}\right)\varphi + \left(\frac{\partial A}{\partial v}\right)w + \left(\frac{\partial A}{\partial u^i} + G_{ik}B^k\right)z^i +\\ &+\left(\alpha^i + \frac{B^qB^k}{v^2}\vartheta^i_{qk} - \frac{B^iA}{v^2} + b^i + \frac{1}{v}\frac{\partial B^i}{\partial v}A + \frac{\partial B^i}{\partial u^k}\frac{B^k}{v^2}\right)\psi_i\end{aligned}$$

Let's consider the following expression denoted by L

$$\ddot\varphi - P\dot\varphi - Q\varphi = L\tag{2.11}$$

with P and Q being the coefficients enclosed in brackets in the above expression for $\ddot\varphi$

$$P = \left(\frac{\partial A}{\partial v}\right) \qquad\qquad Q = \left(a - \frac{B^qB^k}{v^2}G_{qk}\right)$$

For L in (2.11) then we derive the following expression also being the linear combination of the expressions “in brackets”

$$L = \left(\frac{\partial A}{\partial u^i} + G_{ik}B^k\right)z^i + \left(\alpha^i + \frac{B^qB^k}{v^2}\vartheta^i_{qk} - \frac{B^iA}{v^2} + b^i + \frac{A}{v}\frac{\partial B^i}{\partial v} - \frac{B^i}{v}\frac{\partial A}{\partial v}\right)\psi_i$$

Since z^i and ψ_i form the linearly independent set of functions L can vanish if and only if these “brackets” vanish. This gives us the following equations for A and B^i

$$B^i = -G^{ik}\frac{\partial A}{\partial u^k}\tag{2.12}$$

$$\begin{aligned}&\alpha^i + \frac{B^qB^k}{v^2}\vartheta^i_{qk} - \frac{B^iA}{v^2} + b^i +\\ &\quad+\frac{A}{v}\frac{\partial B^i}{\partial v} + \frac{\partial B^i}{\partial u^k}\frac{B^k}{v^2} - \frac{B^i}{v}\frac{\partial A}{\partial v} = 0\end{aligned}\tag{2.13}$$

being the generalizations of (3.27) and (3.28) from [1] for the multidimensional case.

Definition 3. The dynamical system (1.1) with force field of the form (2.3) is called the system with the weak normality condition if the equations (2.12) and (2.13) hold.

For the systems with the weak normality function φ satisfies the second order ordinary differential equation derived from (2.11)

$$\ddot{\varphi} - P\dot{\varphi} - Q\varphi = 0 \tag{2.14}$$

For the dynamical system of definition 3 to be the system accepting the normal shift in the sense of definition 2 we should be able to obtain the initial conditions

$$\varphi|_{t=0} = 0 \qquad \dot{\varphi}|_{t=0} = 0$$

for any submanifold of codimension 1 by the choice of the modulus of initial velocity $v = |\mathbf{v}|$. Let S be some arbitrary manifold of codimension 1 in $\mathbb{R}^n$. Defining the unit normal vector for each point on S we define the spherical map $S \longrightarrow S^{n-1}$ from S to unit sphere S^{n-1} in $\mathbb{R}^n$. For the submanifolds of general position this map is the local diffeomorphism. The latter fact let us transfer the coordinates $u^1, \dots, u^{n-1}$ (see above) from the unit sphere S^{n-1} to S vector $\mathbf{N}(u^1, \dots, u^{n-1})$ from (2.1) being the common unit normal vector for both. Tangent vectors to S are defined like $\mathbf{M}_i$ in (2.1)

$$\mathbf{E}_i = \frac{\partial \mathbf{r}}{\partial u^i} \qquad \frac{\partial \mathbf{E}_i}{\partial u^j} = \Gamma^k_{ij}\mathbf{E}_k + b_{ij}\mathbf{N} \tag{2.15}$$

Tensor b_{ij} in (2.15) is the second quadratic form for S and Γ^k_{ij} are the components of metric connection on S while ϑ^k_{ij} form the metric connection on S^{n-1}. From (2.1) and (2.15) we also derive

$$\mathbf{M}_i = -b^k_i\mathbf{E}_k \qquad G_{ij} = b^k_i b^q_j g_{kq} \tag{2.16}$$

For S of general position the matrix b^k_i is nondegenerate. Let $d = b^{-1}$ be the inverse matrix. Then from (2.16) we have

$$\mathbf{E}_i = -d^k_i\mathbf{M}_k \qquad g_{ij} = d^k_i d^q_j G_{kq} \tag{2.17}$$

Components of matrix d^k_i in (2.17) form the G-symmetric tensor

$$d^k_i G_{kj} = G_{ik} d^k_j = b_{ij} \qquad \nabla_i b_{jk} = \nabla_j b_{ik} \tag{2.18}$$

Second of the equations (2.18) known as the Peterson-Coddazy equation holds with respect to both connections Γ^k_{ij} and ϑ^k_{ij}. It is known that the difference of two connections is the tensor

$$Y^k_{ij} = \Gamma^k_{ij} - \vartheta^k_{ij} = b^k_q \nabla_i d^q_j \tag{2.19}$$

Covariant derivatives in (2.18), (2.19) and everywhere below are defined by the spherical connection ϑ^k_{ij} on the unit sphere S^{n-1}.

According to the definition 2 we are to determine the scalar function $v = f(u^1, \dots, u^{n-1})$ such that

$$\dot{\mathbf{r}}|_{t=0} = f(u^1, \dots, u^{n-1})\mathbf{N}(u^1, \dots, u^{n-1}) \tag{2.20}$$

(compare with (3.23) from [1]). Variables $u^1 = u^1(0), \dots, u^{n-1} = u^{n-1}(0)$ here play the same role as s in (3.23) from [1]. Denoting $s = u^i(0)$ for a while from (2.5), (2.7) and (2.20) we obtain

$$\varphi|_{t=0} = \langle \mathbf{N}, \partial_s \mathbf{r}\rangle|_{t=0} = \langle \mathbf{N}, \mathbf{E}_i\rangle \equiv 0$$

So first initial condition for the equation (2.14) is identically zero since all particles are starting from S along the normal vector to this submanifold. For the second we have

$$\dot\varphi|_{t=0} = \langle \partial_t \mathbf{N}, \partial_s \mathbf{r}\rangle|_{t=0} + \langle \mathbf{N}, \partial_{st} \mathbf{r}\rangle|_{t=0}$$

Using (2.20) and the latter expression we find the following one

$$\dot\varphi|_{t=0} = -\frac{b_{ik}}{f} B^k + \frac{\partial f}{\partial u^i} \tag{2.21}$$

To make zero the initial condition (2.21) we need to choose the function f satisfying the following equations

$$\frac{\partial f}{\partial u^i} = \frac{b_{ik}(u^1, \dots, u^{n-1}) B^k(\mathbf{r}, f, u^1, \dots, u^{n-1})}{f} \tag{2.22}$$

where $\mathbf{r} = \mathbf{r}(u^1, \dots, u^{n-1})$ is the vector of cartesian coordinates of a point on S. Equations (2.22) are analogs of (3.25) from [1]. Since the equations (5.22) form the overdetermined system of differential equations one can derive from them some other equations being the compatibility conditions for (2.22). The way of obtaining them is standard: one should differentiate (2.22) by u_j and then use the equality $\partial_{ij} f = \partial_{ji} f$ by changing the order of derivatives. As a result of such calculations we get

$$\begin{aligned} &\frac{1}{v}\frac{\partial B^k}{\partial v} B^q - \beta^{kq} = \frac{1}{v}\frac{\partial B^q}{\partial v} B^k - \beta^{qk} \\ &\nabla_k B^q = \frac{\nabla_p B^p}{n-1} \delta^q_k \end{aligned} \tag{2.23}$$

Functions B^k in (2.23) are considered as the functions of $2n$ independant variables $r^1, \dots, r^n$, v and $u^1, \dots, u^{n-1}$ as in (2.3). Covariant derivatives by $u^1, \dots, u^{n-1}$ are respective to the spherical connection ϑ^k_{ij}.

Theorem 2. *The equations* (2.12), (2.13) *and* (2.23) *form the enough condition for the dynamical system* (1.1) *with force field* (2.3) *to be accepting the normal shift as described by the definition 2.*

Equations (2.12), (5.13) and (5.23) are compatible in some sense since they have common solution for A and B^k (at least trivial one with $A = A(v)$ and $B^k \equiv 0$ corresponding to the geometrical situation from [5] and [6]). Detailed analysis of these equations and nontrivial examples of the multidimensional dynamical systems associated with their solutions are the subject of separate paper.

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