ON SOME EQUATIONS THAT CAN BE BROUGHT TO THE EQUATIONS OF DIFFUSION TYPE.

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Abstract. For the system of second order quasilinear parabolic equations the problem of reducing them to the equations of diffusion type is considered. In non-degenerate case an effective algorithm for solving this problem is suggested.

1. Introduction.

Let's consider the system of differential equations defined by some matrix $A = A(y^1,\dots,y^n)$ with non-zero determinant:

$$\frac{\partial y^i}{\partial t}=\sum_{j=1}^{n}\frac{\partial}{\partial x}\left(A^i_j\,\frac{\partial y^j}{\partial x}\right),\quad\text{where}\quad i=1,\dots,n. \tag{1.1}$$

The equations (1.1) do not form the set invariant under the point transformations given by the following changes of variables:

$$\begin{aligned}\tilde y^1&=\tilde y^1(y^1,\dots,y^n),\\ &\dots\dots\dots\dots\dots\\ \tilde y^n&=\tilde y^1(y^1,\dots,y^n).\end{aligned} \tag{1.2}$$

However, one can expand this set up to the point-invariant class of equations:

$$\frac{\partial y^i}{\partial t}=\sum_{j=1}^{n}A^i_j\left(\frac{\partial^2 y^j}{\partial x^2}+\sum_{r=1}^{n}\sum_{s=1}^{n}\Gamma^j_{rs}\,\frac{\partial y^r}{\partial x}\,\frac{\partial y^s}{\partial x}\right). \tag{1.3}$$

Here coefficients A^i_j and Γ^j_{rs} depend on $y^1,\dots,y^n$ so that $\det A\neq 0$. Denote by T the Jacoby matrix for the change of variables (1.2) and denote by S the Jacoby matrix for inverse change of variables:

$$S^i_j=\frac{\partial y^i}{\partial\tilde y^j},\qquad\qquad T^i_j=\frac{\partial\tilde y^i}{\partial y^j}.$$

In geometry matrices S and T are called direct and inverse transition matrices respectively (this implies that transition from $y^1,\dots,y^n$ to $\tilde y^1,\dots,\tilde y^n$ is considered as direct transition).

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Theorem 1.1. *Under the point transformation* (1.2) *the equations* (1.3) *transfer to the equations of the same form*

$$\frac{\partial\tilde{y}^i}{\partial t}=\sum_{j=1}^{n}\tilde{A}^i_j\left(\frac{\partial^2\tilde{y}^j}{\partial x^2}+\sum_{r=1}^{n}\sum_{s=1}^{n}\tilde{\Gamma}^j_{rs}\,\frac{\partial\tilde{y}^r}{\partial x}\,\frac{\partial\tilde{y}^s}{\partial x}\right).\tag{1.4}$$

Parameters A^i_j, Γ^j_{rs}, $\tilde{A}^i_j$ *and* $\tilde{\Gamma}^j_{rs}$ *in* (1.3) *and* (1.4) *are related as follows*

$$A^k_i=\sum_{m=1}^{n}\sum_{p=1}^{n}S^k_m\,T^p_i\,\tilde{A}^m_p,\tag{1.5}$$

$$\Gamma^k_{ij}=\sum_{m=1}^{n}\sum_{p=1}^{n}\sum_{q=1}^{n}S^k_m\,T^p_i\,T^q_j\,\tilde{\Gamma}^m_{pq}+\sum_{m=1}^{n}S^k_m\,\frac{\partial T^m_i}{\partial y^j}.\tag{1.6}$$

If we suppose $y^1,\ldots,y^n$ to be local coordinates on some manifold M, then due to theorem 1.1 parameters A^k_i define a tensor field $\mathbf{A}$ of the type $(1,1)$ on M. Parameters Γ^k_{ij} in turn can be interpreted as components of some affine connection Γ on M. This enables us to apply various geometrical methods to the investigation of the equations (1.3). Here we consider only one problem due to equations (1.3), which appears to have graceful geometrical solution. Similar problems for other classes of differential equations were considered in [1–6].

2. Reduction to the equations of diffusion type.

Equations (1.1) arise in various models describing the diffusion phenomena in multicomponent mixtures (see [7–24]) and in theory of integrability (see [25–30]). They are called the equations *of diffusion type*. As for the equations (1.3) those of them, that can be brought to the form (1.1) by means of point transformation (1.2), should obviously be involved to the class of equations of diffusion type. This gives rise to the problem of finding effective description for this class of equations.

Problem. *What condition for tensor field* $\mathbf{A}$ *and affine connection* Γ *should be fulfilled on* M *in order to guarantee the existence of point transformation* (1.2) *that brings equations* (1.2) *to the diffusion form* (1.1)*?*

Suppose that one could bring (1.3) to the form (1.1) in local variables $\tilde{y}^1,\ \ldots,\ \tilde{y}^n$. Then in such variables for components of affine connection Γ we have

$$\tilde{\Gamma}^m_{pq}=\frac{1}{2}\sum_{s=1}^{n}\tilde{B}^m_s\left(\frac{\partial\tilde{A}^s_p}{\partial\tilde{y}^q}+\frac{\partial\tilde{A}^s_q}{\partial\tilde{y}^p}\right).\tag{2.1}$$

Here $\tilde{B}^m_s$ are the components of the matrix $\tilde{B}$ which is inverse to the matrix $\tilde{A}$ formed by components of operator field $\mathbf{A}$ in local coordinates $\tilde{y}^1,\ \ldots,\ \tilde{y}^n$. Now let's apply formulas (1.5) in order to transform the components of the field $\mathbf{A}$ in (2.1):

$$\tilde{A}^s_p=\sum_{r=1}^{n}\sum_{i=1}^{n}T^s_r\,S^i_p\,A^r_i,$$

Further differentiate this relationship with respect to $\tilde{y}^q$:

$$\frac{\partial \tilde{A}^s_p}{\partial \tilde{y}^q}=\sum_{r=1}^{n}\sum_{i=1}^{n}\left(\frac{\partial T^s_r}{\partial \tilde{y}^q}\,S^i_p\,A^r_i+T^s_r\,\frac{\partial S^i_p}{\partial \tilde{y}^q}\,A^r_i+T^s_r\,S^i_p\,\frac{\partial A^r_i}{\partial \tilde{y}^q}\right). \tag{2.2}$$

Let's express the derivatives $\partial A^r_i/\partial\tilde{y}^q$, $\partial S^i_p/\partial\tilde{y}^q$, and $\partial T^s_r/\partial\tilde{y}^q$ through partial derivatives in initial variables $y^1, \ldots, y^n$. We shall do this by means of formulas

$$\frac{\partial A^r_i}{\partial \tilde{y}^q}=\sum_{j=1}^{n}S^j_q\,\frac{\partial A^r_i}{\partial y^j},\qquad \frac{\partial T^s_r}{\partial \tilde{y}^q}=\sum_{j=1}^{n}S^j_q\,\frac{\partial T^s_r}{\partial y^j},\qquad \frac{\partial S^i_p}{\partial \tilde{y}^q}=-\sum_{j=1}^{n}\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n}S^i_\beta\,S^j_q\,\frac{\partial T^\beta_\alpha}{\partial y^j}\,S^\alpha_p.$$

Substitute them into (2.2), then substitute the resulting expression into the formula (2.1) for the components of Γ. Missing expression for the derivative $\partial\tilde{A}^s_q/\partial\tilde{y}^p$ we get by transposing indices p and q:

$$\begin{aligned}\tilde{\Gamma}^m_{pq}&=\frac{1}{2}\sum_{r=1}^{n}\sum_{s=1}^{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\tilde{B}^m_s\,\frac{\partial T^s_r}{\partial y^j}\,A^r_i\,(S^i_p\,S^j_q+S^i_q\,S^j_p)\,+\\&+\frac{1}{2}\sum_{r=1}^{n}\sum_{s=1}^{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\tilde{B}^m_s\,T^s_r\,\frac{\partial A^r_i}{\partial y^j}\,(S^i_p\,S^j_q+S^i_q\,S^j_p)\,-\\&-\frac{1}{2}\sum_{s=1}^{n}\sum_{j=1}^{n}\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n}\tilde{B}^m_s\,\tilde{A}^s_\beta\,\frac{\partial T^\beta_\alpha}{\partial y^j}\,(S^\alpha_p\,S^j_q+S^\alpha_q\,S^j_p).\end{aligned}$$

Now substitute the above expression for $\tilde{\Gamma}^m_{pq}$ into the formula (1.6). As a result we obtain the following formula for Γ^k_{ij}:

$$\Gamma^k_{ij}=\sum_{r=1}^{n}\frac{B^k_r}{2}\left(\frac{\partial A^r_i}{\partial y^j}+\frac{\partial A^r_j}{\partial y^i}\right)+\sum_{r=1}^{n}\sum_{p=1}^{n}\sum_{q=1}^{n}B^k_r\,\theta^r_{pq}\,\frac{A^p_i\,\delta^q_j+A^p_j\,\delta^q_i}{2}. \tag{2.3}$$

Here δ^q_i is Kronecker's delta-symbol (representing components of unit matrix):

$$\delta^q_i=\begin{cases}1 & \text{for}\quad q=i,\\ 0 & \text{for}\quad q\neq i.\end{cases}$$

By B^k_r we denote the components of operator field $\mathbf{B}=\mathbf{A}^{-1}$. The entrance of matrices S and T into (2.3) is completely defined by the quantities θ^r_{pq}:

$$\theta^r_{pq}=\sum_{m=1}^{n}S^r_m\,\frac{\partial T^m_p}{\partial y^q}=\sum_{m=1}^{n}S^r_m\,\frac{\partial^2\tilde{y}^m}{\partial y^p\,\partial y^q}. \tag{2.4}$$

From (2.4) we see that $\theta^r_{pq}=\theta^r_{qp}$. If we know that system of equations (1.3) admits the coordinates $\tilde{y}^1, \ldots, \tilde{y}^n$, where it takes diffusion form (1.1), and if these coordinates are given, then by means of formula (2.4) we derive θ^r_{pq} for further substitution into the relationship (2.3).

Otherwise, when coordinates $\tilde{y}^1, \ldots, \tilde{y}^n$ are not given, one should determine θ^r_{pq} from (2.3) and should consider (2.4) as the equations that determine required change of variables (1.2). We write these equations as

$$\frac{\partial T^m_p}{\partial y^q}=\sum_{r=1}^{n}\theta^r_{pq}\,T^m_r. \tag{2.5}$$

Equations (2.5) form complete system of linear Pfaff equations respective to the components of the matrix T. They are well known in differential geometry. Compatibility condition for (2.5) is derived from permutability of partial derivatives:

$$\frac{\partial^2 T^m_p}{\partial y^q\,\partial y^k}=\frac{\partial^2 T^m_p}{\partial y^k\,\partial y^q}. \tag{2.6}$$

Let's calculate both sides of (2.6) according to equations (2.5). As a result we get the relationship, which is known as the condition of "zero curvature"[1]:

$$\frac{\partial\theta^m_{kp}}{\partial y^q}-\frac{\partial\theta^m_{qp}}{\partial y^k}+\sum_{r=1}^{n}\theta^m_{qr}\,\theta^r_{kp}-\sum_{r=1}^{n}\theta^m_{kr}\,\theta^r_{qp}=0. \tag{2.7}$$

On the base of theory of Pfaff equations (see, for instance, appendix A in [4]) one can prove the following theorem.

Theorem 2.1. *Let v be some point of the manifold M. Matrix Pfaff equations* (2.5) *have solution with non-zero determinant* $\det T\neq 0$ *in some neighborhood of the point v on M if and only if the condition* (2.7) *is fulfilled.*

Quantities θ^r_{pq} are symmetric in lower pair of indices p and q. Hence for arbitrary solution of the equations (2.5) we have

$$\frac{\partial T^m_p}{\partial y^q}=\frac{\partial T^m_q}{\partial y^p}. \tag{2.8}$$

This condition (2.8) in turn is the compatibility condition for the equations

$$\frac{\partial\tilde{y}^m}{\partial y^p}=T^m_p(y^1,\ldots,y^n)$$

which determine new variables $\tilde{y}^1, \ldots, \tilde{y}^n$. We shall formulate the result of the above speculations in form of two theorems.

Theorem 2.2. *Quantities $\theta^r_{pq}=\theta^r_{qp}$ are defined by some change of local coordinates in some neighborhood of the point v on M according to the formulas* (2.4) *if and only if the conditions of "zero curvature"* (2.7) *are fulfilled.*

[1]Note that "zero curvature" condition (2.7) does not mean that the curvature tensor of the affine connection Γ is zero.

Theorem 2.3. *System of equations* (1.3) *can be brought to the diffusion form* (1.1) *by some change of local coordinates in some neighborhood of the point v on M if and only if one can find quantities $\theta^r_{pq} = \theta^r_{qp}$ satisfying both conditions* (2.3) *and* (2.7) *simultaneously.*

Theorem 2.3 distinguishes two cases in the theory of quasilinear equations of the form (1.3): first is **the case of general position** and second is **the degenerate case**. In the case of general position quantities θ^r_{pq} are uniquely determined by the relationships (2.3). Substituting them into (2.7) further we can get required equations for $\mathbf{A}$ and Γ which guarantee that the equations (1.3) are reducible to the form (1.1).

In degenerate case relationships (2.3) do not determine the quantities θ^r_{pq} at all or determine with some extent of uncertainty (not uniquely). In this case we should analyze the relationships (2.3) for their solvability with respect to θ^r_{pq} and for their compatibility with (2.7).

3. Non-degeneracy condition.

The non-degeneracy condition distinguishing the case of general position, as formulated above, is the condition of unique solvability of the equations (2.3) with respect to θ^r_{pq}. These equations can be written as follows:

$$\sum_{p=1}^{n}\sum_{q=1}^{n}\theta^r_{pq}\,\frac{A^p_i\,\delta^q_j + A^p_j\,\delta^q_i}{2} = \sum_{k=1}^{n} A^r_k\,\Gamma^k_{ij} - \frac{1}{2}\left(\frac{\partial A^r_i}{\partial y^j} + \frac{\partial A^r_j}{\partial y^i}\right). \tag{3.1}$$

The equations (3.1) are linear equations respective to θ^r_{pq}. Note that they breaks into n separate parts. For each $r = 1, \ldots, n$ we have closed system of n^2 linear equations. Matrices of all these systems are the same, they are defined by the components of tensor field $\boldsymbol{\Lambda}$ in the following form:

$$\Lambda^{pq}_{ij} = \frac{A^p_i\,\delta^q_j + A^p_j\,\delta^q_i + A^q_i\,\delta^p_j + A^q_j\,\delta^p_i}{4}. \tag{3.2}$$

Denote by w^r_{ij} right hand sides of (3.1). Then these equations can be written as

$$\sum_{p=1}^{n}\sum_{q=1}^{n}\Lambda^{pq}_{ij}\,\theta^r_{pq} = w^r_{ij}. \tag{3.3}$$

Let v be some fixed point on the manifold M. Denote by V the tangent space to M at this point: $V = T_v(M)$. By V^* denote the dual space for V. Then let's consider a tensor product of two samples of dual space:

$$W = V^* \otimes V^* = T^0_2(v, M).$$

According to this definition W is a space of bilinear forms on the space V. It is represented as a direct sum of two subspaces:

$$W = W_{\text{sym}} \oplus W_{\text{skew}}. \tag{3.4}$$

First is the subspace of symmetric bilinear forms and second is the subspace of skew-symmetric forms. Tensor $\boldsymbol{\Lambda}$ can be treated as a linear operator in the space W. Both subspaces in the sum (3.4) are invariant with respect to the action of the operator $\mathbf{\Lambda}\colon W \to W$, restriction of $\mathbf{\Lambda}$ to W_{skew} being identically zero (since Λ^{pq}_{ij} is symmetric in both pairs of indices upper and lower).

Right hand side of (3.3) is symmetric in i and j, quantities θ^r_{pq}, that should be determined from the equations (3.1), are also symmetric in their lower indices p and q. Therefore the above condition determining the case of general position can be formulated as the condition of non-degeneracy for the restriction of the operator $\mathbf{\Lambda}$ to the subspace of symmetric bilinear forms:

$$\mathbf{\Lambda}_{\text{sym}}\colon W_{\text{sym}} \to W_{\text{sym}}.$$

In other words this is written as

$$\det \mathbf{\Lambda}_{\text{sym}} \neq 0. \tag{3.5}$$

Components of the operator $\mathbf{\Lambda}$ are uniquely determined by the components of $\mathbf{A}$ according to the formula (3.2). But nevertheless (3.5) doesn't follow immediately from $\det \mathbf{A} \neq 0$. One should check it up separately.

4. Effectivization of the non-degeneracy condition.

Let's study the question of how to check up the condition (3.5), if the matrix of operator field $\mathbf{A}$ is given. We shall start with the case when operator $\mathbf{A}$ has purely simple spectrum, i. e. when its eigenvalues $\lambda_1, \ldots, \lambda_n$ are distinct and its characteristic polynomial has no multiple roots:

$$f(\lambda) = \det(\mathbf{A} - \lambda\,\mathbf{I}) = (-\lambda)^n + \sum_{s=1}^{n} \sigma_s(n)\,(-\lambda)^{n-s} = \prod_{s=1}^{n} (\lambda_s - \lambda). \tag{4.1}$$

Here $\sigma_1(n), \ldots, \sigma_n(n)$ are basic symmetric polynomials defined by formula

$$\sigma_s(n) = \sum\ldots\sum_{1 \leqslant i_1 < \ldots < i_s \leqslant n} \lambda_{i_1} \cdot \ldots \cdot \lambda_{i_s}. \tag{4.2}$$

Last of these polynomials is equal to the determinant of the operator $\mathbf{A}$ (see more details in [31]).

Operator $\mathbf{A}$ with simple spectrum in finite dimensional space $V = T_v(M)$ is diagonalizable, it has a base of eigenvectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$. However, the construction of such base may require complexification of the space V, since roots or characteristic polynomial (4.1) are complex numbers in general. Denote by $\mathbf{h}^1, \ldots, \mathbf{h}^n$ the dual base in dual space V^*. We have the following relationships:

$$\mathbf{A}\mathbf{e}_i = \lambda_i\,\mathbf{e}_i, \qquad \mathbf{A}^*\mathbf{h}^i = \lambda_i\,\mathbf{h}^i, \qquad \mathbf{h}^i(\mathbf{e}_k) = \delta^i_k. \tag{4.3}$$

Here $\mathbf{A}^*\colon V^* \to V^*$ is a conjugate operator for $\mathbf{A}$. By means of covectors $\mathbf{h}^1, \ldots, \mathbf{h}^n$ we shall build the base in the space of symmetric bilinear forms W_{sym}:

$$\mathbf{w}^{ij} = \mathbf{h}^i \otimes \mathbf{h}^j + \mathbf{h}^j \otimes \mathbf{h}^i, \text{ where } 1 \leqslant i \leqslant j \leqslant n. \tag{4.4}$$

Denote by N the dimension of the space W_{sym}. On counting the number of vectors in the base (4.4) we find

$$N = \dim W_{\text{sym}} = \frac{n(n+1)}{2}. \tag{4.5}$$

Due to (3.2) and (4.3) the result of applying $\mathbf{\Lambda}$ to the vector $\mathbf{h}^p \otimes \mathbf{h}^q$ is given by

$$\mathbf{\Lambda}(\mathbf{h}^p \otimes \mathbf{h}^q) = \frac{\lambda_p + \lambda_q}{2}\,\mathbf{h}^p \otimes \mathbf{h}^q. \tag{4.6}$$

Now let's apply the operator (4.6) to the vectors of the base (4.4). By means of direct calculations we get

$$\mathbf{\Lambda}_{\text{sym}}(\mathbf{w}^{ij}) = \mathbf{\Lambda}(\mathbf{w}^{ij}) = \frac{\lambda_i + \lambda_j}{2}\,\mathbf{w}^{ij}. \tag{4.7}$$

The relationship (4.7) means that operator $\mathbf{\Lambda}_{\text{sym}}$ is diagonalizable, base (4.4) being the base of eigenvectors of this operator. Eigenvalues μ_{ij} corresponding to the eigenvectors $\mathbf{w}^{ij}$ are given by the formula

$$\mu_{ij} = \frac{\lambda_i + \lambda_j}{2}, \text{ where } 1 \leqslant i \leqslant j \leqslant n. \tag{4.8}$$

Using (4.8) we can calculate the determinant of the operator $\mathbf{\Lambda}_{\text{sym}}$:

$$\det \mathbf{\Lambda}_{\text{sym}} = \sigma_N(N) = \prod_{i \leqslant j} \left(\frac{\lambda_i + \lambda_j}{2}\right). \tag{4.9}$$

Let's introduce lexicographic ordering in the set of pairs of indices (i, j), where $i \leqslant j$. Say that $(i_1, j_1) < (i_2, j_2)$ if $j_1 < j_2$ or if $i_1 < i_2$ in the case of coincidence $j_1 = j_2$. Then consider the sum

$$\sigma_s(N) = \sum\ldots\sum_{(i_1,j_1)<\ldots<(i_s,j_s)} \frac{(\lambda_{i_1} + \lambda_{j_1}) \cdot \ldots \cdot (\lambda_{i_s} + \lambda_{j_s})}{2^s}. \tag{4.10}$$

Formula (4.10) determines basic symmetric polynomials for the set of numbers μ_{ij} from (4.8). For $s = N$ it is equivalent to (4.9). Its easy to note, that right hand sides of (4.10) do not change under any transposition of numbers $\lambda_1, \ldots, \lambda_n$. They are symmetric polynomials in $\lambda_1, \ldots, \lambda_n$. Hence according the well known theorem (see [31], chapter 6, § 2) one can express them trough basic symmetric polynomials:

$$\sigma_s(N) = P_s(\sigma_1(n), \ldots, \sigma_n(n)), \text{ where } s = 1, \ldots, N. \tag{4.11}$$

For any particular n we can derive explicit expressions for polynomials $P_s(\sigma_1, \ldots, \sigma_n)$. For instance, if $n = 2$, we have

$$P_1 = \frac{3}{2}\,\sigma_1, \qquad P_2 = \frac{1}{2}\,\sigma_1^2 + \sigma_2, \qquad P_3 = \frac{1}{2}\,\sigma_2\,\sigma_1, \tag{4.12}$$

From (4.5) here we get $N = 3$. Since $\sigma_2(2) = \det A$ is non-zero, for $n = 2$ the non-degeneracy condition (3.5) can be written as

$$\sigma_1(2) = \operatorname{tr}\mathbf{A} \neq 0. \tag{4.13}$$

Theorem 4.1. *For $n = 2$ the system of equations* (1.3) *belongs to the non-degenerate class if and only if* $\det\mathbf{A} \neq 0$ *and* $\operatorname{tr}\mathbf{A} \neq 0$.

The relationship (4.13) means that operator $\mathbf{A}$ shouldn't have two eigenvalues which differ only in sign. In this form the non-degeneracy condition can be stated in multidimensional case as well. For arbitrary n it follows from (4.9). Indeed, we have

$$\prod_{i\leqslant j}\left(\frac{\lambda_i+\lambda_j}{2}\right) = \det\mathbf{A}\cdot\prod_{i<j}\left(\frac{\lambda_i+\lambda_j}{2}\right) \neq 0, \tag{4.14}$$

which is equivalent to $\det\mathbf{A} \neq 0$ and $\lambda_i \neq -\lambda_j$ for $i \neq j$.

In order to check up the condition (4.14) one need not find eigenvalues of the operator $\mathbf{A}$. This condition can be written in terms of coefficients of characteristic polynomial (4.1), provided we have explicit expressions for polynomials in (4.11). The latter is algorithmically solvable problem.

Another effective way of checking up the condition (4.14) is based on the theory of resultants. Let $f(\lambda)$ be a characteristic polynomial for the operator $\mathbf{A}$. Its roots are non-zero since $\det\mathbf{A} \neq 0$, therefore the condition $\lambda_i \neq -\lambda_j$ for $i \neq j$ means that polynomials $f(\lambda)$ and $f(-\lambda)$ have no common roots. Then resultant of these two polynomials is non-zero (see [31] or [32]):

$$\operatorname*{Res}_{\lambda}\bigl[\det(\mathbf{A}-\lambda\,\mathbf{I}), \det(\mathbf{A}+\lambda\,\mathbf{I})\bigr] \neq 0. \tag{4.15}$$

Theorem 4.2. *System of equations* (1.3) *belongs to the non-degenerate class if and only if* $\det\mathbf{A} \neq 0$ *and condition* (4.15) *is fulfilled.*

5. Finding inverse operator for $\mathbf{\Lambda}_{\text{sym}}$.

Suppose that non-degeneracy condition stated in theorem 4.2 is fulfilled. Let's find regular way for inverting the operator $\mathbf{\Lambda}_{\text{sym}}\colon W_{\text{sym}} \to W_{\text{sym}}$. One need to do it for to find θ^r_{pq} from the equations (3.3). Let $f(\lambda)$ be a characteristic polynomial for the operator $\mathbf{A}$. Lets consider pair of polynomials $f(\lambda+\mu)$ and $f(\mu-\lambda)$, where μ is a parameter. These polynomials are factored as follows:

$$\begin{aligned}
&f(\lambda+\mu) = (-1)^n\prod_{i=1}^{n}\bigl(\lambda-(\lambda_i-\mu)\bigr)\\
&f(\mu-\lambda) = \prod_{i=1}^{n}\bigl(\lambda-(\mu-\lambda_i)\bigr).
\end{aligned} \tag{5.1}$$

For to calculate the resultant of polynomials (5.1) let's apply the following theorem of algebra (see [31] or [32] for proof).

Theorem 5.1. *Suppose that two polynomials $g(\lambda)$ and $h(\lambda)$ with leading coefficients a and b are factored to a product of linear terms*

$$\begin{aligned} g(\lambda) &= a\cdot(\lambda-\alpha_1)\cdot\ldots\cdot(\lambda-\alpha_n),\\ h(\lambda) &= b\cdot(\lambda-\beta_1)\cdot\ldots\cdot(\lambda-\beta_m). \end{aligned} \tag{5.2}$$

Then resultant of these polynomials is given by the formula

$$\operatorname*{Res}_{\lambda}\bigl[g(\lambda),h(\lambda)\bigr]=a^n\,b^m\prod_{i=1}^{n}\prod_{j=1}^{m}(\alpha_i-\beta_j). \tag{5.3}$$

Comparing (5.1) with (5.2) from (5.3) we obtain the resultant of polynomials (5.1):

$$\begin{aligned} \operatorname*{Res}_{\lambda}\bigl[f(\mu-\lambda),f(\lambda+\mu)\bigr] &= (-1)^n\prod_{i=1}^{n}\prod_{j=1}^{n}(2\,\mu-\lambda_i-\lambda_j)=\\ &= 2^{n^2}\prod_{i<j}\left(\frac{\lambda_i+\lambda_j}{2}-\mu\right)\prod_{i\leqslant j}\left(\frac{\lambda_i+\lambda_j}{2}-\mu\right). \end{aligned} \tag{5.4}$$

Let's compare (5.4) with formula (4.8) for eigenvalues of the operator $\mathbf{\Lambda}_{\mathrm{sym}}$. This comparison gives the formula

$$\varphi(\mu)^2=2^{-n^2}\,f(\mu)\,\operatorname*{Res}_{\lambda}\bigl[f(\mu-\lambda),f(\lambda+\mu)\bigr]. \tag{5.5}$$

Here $\varphi(\mu)=\det(\mathbf{\Lambda}_{\mathrm{sym}}-\mu\,\mathbf{I})$ is a characteristic polynomial for the operator $\mathbf{\Lambda}_{\mathrm{sym}}$. Its remarkable that formula (5.5) express the square of characteristic polynomial of operator $\mathbf{\Lambda}_{\mathrm{sym}}$ in terms of characteristic polynomial of operator $\mathbf{A}$.

Denote by $\varepsilon(\mu)$ the polynomial in right hand side of (5.5). According to the well known theorem of Hamilton and Cayley (see [32] or [33]) when substituting the operator $\mathbf{\Lambda}_{\mathrm{sym}}$ into its characteristic polynomial $\varphi(\mu)$ we should get zero operator. This is true for the polynomial $\varepsilon(\mu)=\varphi(\mu)^2$ as well:

$$\varepsilon(\mathbf{\Lambda}_{\mathrm{sym}})=\sum_{i=0}^{M}\varepsilon_i\,\mathbf{\Lambda}_{\mathrm{sym}}^{i}=0. \tag{5.6}$$

Here $M=2\,N=n(n+1)$ is the degree of polynomial $\varepsilon(\mu)=\varphi(\mu)^2$. Now we can rewrite (5.6) as follows

$$\left(\sum_{i=1}^{M}\varepsilon_i\,\mathbf{\Lambda}_{\mathrm{sym}}^{i-1}\right)\cdot\mathbf{\Lambda}_{\mathrm{sym}}=-\varepsilon_0\,\mathbf{I}. \tag{5.7}$$

Quantity ε_0 in right hand side of (5.7) is the value of polynomial $\varepsilon(\lambda)$ at $\lambda=0$:

$$\varepsilon_0=\varepsilon(0)=\varphi(0)^2=(\det\mathbf{\Lambda}_{\mathrm{sym}})^2.$$

Due to non-degeneracy condition (3.5) the quantity ε_0 is non-zero. Therefore we can use (5.7) in order to calculate the operator $\mathbf{\Lambda}^{-1}_{\text{sym}}$:

$$\mathbf{\Lambda}^{-1}_{\text{sym}} = -\sum_{i=1}^{M} \frac{\varepsilon_i}{\varepsilon_0}\,\mathbf{\Lambda}^{i-1}_{\text{sym}}.$$

We can expand operator $\mathbf{\Lambda}^{-1}_{\text{sym}}\colon W_{\text{sym}} \to W_{\text{sym}}$ to be identically zero in the subspace of skew-symmetric bilinear forms W_{skew}. Though being degenerate, expanded operator $\mathbf{\Lambda}^{-1}_{\text{sym}}$ is defined in the whole space of bilinear forms W. One can define this operator by the following formula

$$\mathbf{\Lambda}^{-1}_{\text{sym}} = -\sum_{i=1}^{M} \frac{\varepsilon_i}{\varepsilon_0}\,\mathbf{\Lambda}^{i-1}. \tag{5.8}$$

Right hand side of (5.8) is interpreted as a tensor field $\mathbf{D}$ of the type $(2,2)$ whose components D^{ij}_{pq} are symmetric in upper and lower indices.

Theorem 5.2. *If* $\det A \neq 0$ *and if non-degeneracy condition* (3.5) *written in the form* (4.15) *is fulfilled, then equations* (3.3) *can be solved with respect to* θ^r_{pq}*:*

$$\theta^r_{pq} = \sum_{i=1}^{n}\sum_{j=1}^{n} D^{ij}_{pq}\,w^r_{ij}. \tag{5.9}$$

Here D^{ij}_{pq} *are components of tensor field* $\mathbf{D} = \mathbf{\Lambda}^{-1}_{\text{sym}}$ *defined by the formula* (5.8).

Naturally we can choose more straightforward way of calculating components of tensor field $\mathbf{D}$ other than formula (5.8). In order to do it we should only remember that $\mathbf{D}$ defines an operator inverse to the operator $\mathbf{\Lambda}_{\text{sym}}$ in subspace W_{sym}:

$$\mathbf{D} \cdot \mathbf{\Lambda}_{\text{sym}} = \mathbf{I}. \tag{5.10}$$

Written in coordinates the relationship (5.10) looks like

$$\sum_{i=1}^{n}\sum_{j=1}^{n} D^{ij}_{rs}\,\Lambda^{pq}_{ij} = \frac{\delta^p_r\,\delta^q_s + \delta^p_s\,\delta^q_r}{2}.$$

This is the system of N^2 linear equations with respect to D^{ij}_{rs}, where N is defined by (4.5). Non-degeneracy condition (4.15) guarantees the compatibility of these equations and the uniqueness of their common solution.

6. Criterion for being in diffusion class.

On solving the equations (3.3) respective to θ^r_{pq} in the form (5.9) we can substitute this expression into (2.7) and we can apply theorem 2.3.

Theorem 6.1. *In non-degenerate case equations* (1.3) *can be brought to the diffusion form* (1.1) *by means of some point transformation* (1.2) *if and only if the quantities*

$$\theta^r_{pq} = \sum_{i=1}^{n}\sum_{j=1}^{n} D^{ij}_{rs}\left(\sum_{k=1}^{n} A^r_k\,\Gamma^k_{ij} - \frac{1}{2}\left(\frac{\partial A^r_i}{\partial y^j} + \frac{\partial A^r_j}{\partial y^i}\right)\right). \tag{6.1}$$

do satisfy "zero curvature" condition (2.7).

Due to formulas derived in sections 3, 4 and 5 we can check up non-degeneracy condition and we can explicitly calculate D^{ij}_{rs} in those variables $y^1, \ldots, y^n$, where the components of operator field $\mathbf{A}$ are initially given. Therefore theorem 2.3 is an effective criterion for testing if the equation belong to the diffusion class in non-degenerate case. The algorithm given by this theorem can be easily realized by means of any of now existing program packages for symbolic calculations.

Note. *All results in sections 3, 4 and 5 were obtained under the assumption that operator* $\mathbf{A}$ *has purely simple spectrum. However, they remain true for arbitrary operators, since any operator with degenerate spectrum can be obtained as a limit of some sequence of operators with purely simple spectrum.*

7. Explicit formulas for $n = 2$.

As it was shown above, in two-dimensional case the non-degeneracy condition reduces to the following relationships

$$\det \mathbf{A} \neq 0, \qquad\qquad \operatorname{tr} \mathbf{A} \neq 0$$

(see theorem 4.1). Using (4.12) we can find explicit expression for the characteristic polynomial $\varphi(\mu)$ of the operator $\mathbf{\Lambda}_{\text{sym}}$. Its coefficients are expressed in terms of parameters $\sigma_1 = \operatorname{tr} \mathbf{A}$ and $\sigma_2 = \det \mathbf{A}$:

$$\varphi(\mu) = -\mu^3 + \frac{3\,\sigma_1}{2}\,\mu^2 - \frac{\sigma_1^2 + 2\,\sigma_2}{2}\,\mu + \frac{\sigma_1\,\sigma_2}{2}. \tag{7.1}$$

From operator equality $\varphi(\mathbf{\Lambda}_{\text{sym}}) = 0$ due to (7.1) we get the following expression:

$$\mathbf{\Lambda}^{-1}_{\text{sym}} = \frac{2\,\mathbf{\Lambda}^2 - 3\,\sigma_1\,\mathbf{\Lambda} + (\sigma_1^2 + 2\,\sigma_2)\,\mathbf{I}}{\sigma_1\,\sigma_2}. \tag{7.2}$$

Substituting (3.2) into (7.2) we obtain the components of the tensor field $\mathbf{D} = \mathbf{\Lambda}^{-1}_{\text{sym}}$:

$$D^{11}_{11} = \frac{\sigma_2 + (A^2_2)^2}{\sigma_1\,\sigma_2}, \qquad D^{22}_{22} = \frac{\sigma_2 + (A^1_1)^2}{\sigma_1\,\sigma_2},$$

$$D^{11}_{22} = \frac{(A^1_2)^2}{\sigma_1\,\sigma_2}, \qquad D^{22}_{11} = \frac{(A^2_1)^2}{\sigma_1\,\sigma_2},$$

$$D^{12}_{11} = D^{21}_{11} = -\frac{A^2_1\,A^2_2}{\sigma_1\,\sigma_2}, \qquad D^{12}_{22} = D^{21}_{22} = -\frac{A^1_2\,A^1_1}{\sigma_1\,\sigma_2},$$

$$D^{11}_{12} = D^{11}_{21} = -\frac{A^1_2\,A^2_2}{\sigma_1\,\sigma_2}, \qquad D^{22}_{12} = D^{22}_{21} = -\frac{A^2_1\,A^1_1}{\sigma_1\,\sigma_2},$$

$$D^{12}_{12} = D^{21}_{12} = \frac{A^1_1\,A^2_2}{\sigma_1\,\sigma_2}, \qquad\qquad D^{12}_{21} = D^{21}_{21} = \frac{A^1_1\,A^2_2}{\sigma_1\,\sigma_2}.$$

Now we should substitute the above expressions into (6.1) for to find parameters θ^r_{pq} and substitute them into (2.7). As a result we obtain four differential equations binding A^i_j and Γ^k_{ij}. If they hold, then the equations (1.3) can be brought to the form (1.1) in two-dimensional case $n = 2$.

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