

**SPECIAL COURSE IN MATHEMATICS**

SHARIPOV R. A.

**REPRESENTATIONS OF FINITE GROUPS**

Part 1

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Sharipov R. A. **Representations of finite groups. Part 1.**

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This book is an introduction to a fast developing branch of mathematics — the theory of representations of groups. It presents classical results of this theory concerning finite groups. This book is written on the base of the special course which I gave in Mathematics Department of Bashkir State University.

In preparing Russian edition of this book I used computer typesetting on the base of  $\mathcal{A}\mathcal{M}\mathcal{S}\text{-}\mathcal{T}\mathcal{E}\mathcal{X}$  package and I used cyrillic fonts of Lh-family distributed by CyrTUG association of Cyrillic  $\mathcal{T}\mathcal{E}\mathcal{X}$  users. English edition is also typeset by  $\mathcal{A}\mathcal{M}\mathcal{S}\text{-}\mathcal{T}\mathcal{E}\mathcal{X}$ .

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## PREFACE.

The theory of group representations is a wide branch of mathematics. In this book I explain very small part of this theory concerned with representations of finite groups. The material of the book is approximately equal to a one semester course.

In explaining the material of the book I tried to make it maximally detailed, complete and self-consistent. The reader need not refer to other literature. He is only required to know the linear algebra and the group theory on the level of a standard university course. The linear algebra references are given to my book which now is available online:

- [1] Sharipov R. A. «Course of linear algebra and multidimensional geometry», Bashkir State University, Ufa 1996.

The primary material of the book is prepared on the base of the following two brilliant monographs:

- [2] Neimark M. A. «Theory of representations of groups», Nauka publishers, Moscow 1976;
- [3] Kirillov A. A. «Elements of the theory of representations», Nauka publishers, Moscow 1978.

Some problems concerning the character algebra for finite groups representations are not considered in this book. They will be given in Part 2, which is planned as a separate book.

September, 1995;  
December, 2006.

R. A. Sharipov.

## CHAPTER I

# REPRESENTATIONS OF GROUPS

### § 1. Representations of groups and their homomorphisms.

It is clear that matrix groups are in some sense simpler than abstract groups. Multiplication rule in them is explicitly specific. Dealing with matrix groups one can use methods of the linear algebra and calculus. The theory of group representations grows from the aim to reproduce an abstract group in a matrix form.

Let  $V$  be a linear vector space over the field of complex numbers  $\mathbb{C}$ . By  $\text{End}(V)$  we denote the set of linear operators mapping  $V$  into  $V$ . The subset of non-degenerate operators within  $\text{End}(V)$  is denoted by  $\text{Aut}(V)$ . It is easy to see that  $\text{Aut}(V)$  is a group. The operation of composition, i. e. applying two operators successively, is the group multiplication in  $\text{Aut}(V)$ .

**DEFINITION 1.1.** A representation  $f$  of a group  $G$  in a linear vector space  $V$  is a group homomorphism  $f: G \rightarrow \text{Aut}(V)$ .

If  $f$  is a representation of a group  $G$  in  $V$ , this fact is briefly written as  $(f, G, V)$ . Let  $g \in G$  be an element of the group  $G$ , then  $f(g)$  is a non-degenerate operator acting within the space  $V$ . It is called the *representation operator* corresponding to the element  $g \in G$ . By  $f(g)\mathbf{x}$  we denote the result of applying this operator to a vector  $\mathbf{x} \in V$ . The notation with two pairs of braces  $f(g)(\mathbf{x})$  will also be used provided it is more clear

in a given context. For instance,  $f(g)(\mathbf{x} + \mathbf{y})$ . Representation operators satisfy the following evident relationships:

- (1)  $f(g_1 g_2) = f(g_1) f(g_2)$ ;
- (2)  $f(1) = 1$ ;
- (3)  $f(g^{-1}) = f(g)^{-1}$ ;

DEFINITION 1.2. Let  $(f, G, V)$  and  $(h, G, W)$  be two representations of the same group  $G$ . A linear mapping  $A: V \rightarrow W$  is called a *homomorphism* sending the representation  $(f, G, V)$  to the representation  $(h, G, W)$  if the following condition is fulfilled:

$$A \circ f(g) = h(g) \circ A \quad \text{for all } g \in G. \quad (1.1)$$

The mapping  $A$  in (1.1), which performs a homomorphism of two representations, sometimes is called an *interlacing map*.

DEFINITION 1.3. A homomorphism  $A$  interlacing two representations  $(f, G, V)$  and  $(h, G, W)$  is called an *isomorphism* if it is bijective as a linear mapping  $A: V \rightarrow W$ .

It is easy to verify that the relation of being isomorphic is an equivalence relation for representations. Two isomorphic representations are also called *equivalent representations*. In the theory of representations two isomorphic representations are treated as identical representations because all essential properties of such two representations do coincide.

THEOREM 1.1. *If  $(f, G, V)$  is a representation of a group  $G$  in a space  $V$  and if  $A: V \rightarrow W$  is a bijective linear mapping, then  $A$  induces a unique representation of the group  $G$  in  $W$  which is equivalent to  $(f, G, V)$  and for which  $A$  is an interlacing map.*

PROOF. The proof of the theorem is trivial. Let's define the operators of a representation  $h$  in  $W$  as follows:

$$h(g) = A \circ f(g) \circ A^{-1}. \quad (1.2)$$

It is easy to verify that the formula (1.2) does actually define a representation of the group  $G$  in  $W$ . Multiplying (1.2) on

the right by  $A$ , we get (1.1). Hence  $A$  is an isomorphism of  $f$  and  $h$ . Moreover, any representation of  $G$  in  $W$  for which  $A$  is an interlacing isomorphism should coincide with  $h$ . This fact is proved by multiplying (1.1) on the right by  $A^{-1}$ .  $\square$

From the theorem proved just above we conclude that if a representation  $f$  in  $V$  is given, in order to construct an equivalent representation in  $W$  it is sufficient to have a linear bijection from  $V$  to  $W$ . However, in practice the problem is stated somewhat differently. Two representations  $f$  and  $h$  in  $V$  and  $W$  are already given. The problem is to figure out if they are equivalent and if so to find an interlacing operator. In this statement this is one of the basic problems of the theory of representations.

## § 2. Finite dimensional representations.

DEFINITION 2.1. A representation  $(f, G, V)$  is called finite-dimensional if its space  $V$  is a finite-dimensional linear vector space, i. e.  $\dim V < \infty$ .

Below in this book we consider only finite-dimensional representations, though many facts proved for this case can then be transferred or generalized for the case of infinite-dimensional representations.

Note that any finite-dimensional linear vector space over the field of complex numbers  $\mathbb{C}$  can be bijectively mapped onto the standard arithmetic coordinate vector space  $\mathbb{C}^n$ , where  $n = \dim V$ . And  $\text{Aut}(\mathbb{C}^n) = \text{GL}(n, \mathbb{C})$ . Therefore each finite-dimensional representation is equivalent to some matrix representation  $f: G \rightarrow \text{GL}(n, \mathbb{C})$ . This fact follows from the theorem 1.1. In spite of this fact we shall consider finite-dimensional representations in abstract vector spaces because all statements in this case are more elegant and their proofs are sometimes even more simple than the proofs of corresponding matrix statements.

### § 3. Invariant subspaces. Restriction and factorization of representations.

DEFINITION 3.1. Let  $(f, G, V)$  be a representation of a group  $G$  in a linear vector space  $V$ . A subspace  $W \subseteq V$  is called an *invariant subspace* if for any  $g \in G$  and for any  $\mathbf{x} \in W$  the result of applying  $f(g)$  to  $\mathbf{x}$  belongs to  $W$ , i. e.  $f(g)\mathbf{x} \in W$ .

The concept of *irreducibility* is introduced in terms of invariant subspaces. This is the central concept in the theory of representations.

DEFINITION 3.2. A representation  $(f, G, V)$  of the group  $G$  is called *irreducible* if it has no invariant subspaces other than  $W = \{0\}$  and  $W = V$ . Otherwise the representation  $(f, G, V)$  is called *reducible*.

Assume that  $(f, G, V)$  is an irreducible representation. Let's choose some vector  $\mathbf{x} \neq 0$  of  $V$  and consider its orbit:

$$\text{Orb}_f(\mathbf{x}) = \{\mathbf{y} \in V : \mathbf{y} = f(g)\mathbf{x} \text{ for some } g \in G\}.$$

The orbit  $\text{Orb}_f(\mathbf{x})$  is a subset of the space  $V$  invariant under the action of the representation operators. However, in general case, it is not a linear subspace. Let's consider its linear span

$$W = \langle \text{Orb}_f(\mathbf{x}) \rangle.$$

The subspace  $W$  is invariant and  $W \neq \{0\}$  since it possesses the non-zero vector  $\mathbf{x}$ . Then due to the irreducibility of  $f$  we get  $W = V$ . As a result one can formulate the following criterion of irreducibility.

**THEOREM 3.1 (irreducibility criterion).** *A representation  $(f, G, V)$  is irreducible if and only if the orbit of an arbitrary non-zero vector  $\mathbf{x} \in V$  spans the whole space  $V$ .*

The necessity of this condition was proved above. Let's prove its sufficiency. Let  $W \subseteq V$  be an invariant subspace such that



$W \neq \{0\}$ . Let's choose a nonzero vector  $\mathbf{x} \in W$ . Due to the invariance of  $W$  we have  $\text{Orb}_f(\mathbf{x}) \subseteq W$ . Hence,  $\langle \text{Orb}_f(\mathbf{x}) \rangle \subseteq W$ . But  $\langle \text{Orb}_f(\mathbf{x}) \rangle = V$ , therefore  $W = V$ . The criterion is proved.

Irreducible representations are similar to chemical elements. One cannot extract other more simple representations of a given group from them. Any reducible representation in some sense splits into irreducible ones. Therefore in the theory of representations the following two problems are solved:

- (1) to find and describe all irreducible representations of a given group;
- (2) to suggest a method for splitting an arbitrary representation into its irreducible components.

The first problem is analogous to building the Mendeleev's table in chemistry, the second problem is analogous to chemical analysis of substances.

Let's consider some reducible representation  $(f, G, V)$  of a group  $G$ . Assume that  $W$  is an invariant subspace such that  $\{0\} \subsetneq W \subsetneq V$ . Let's denote by  $\varphi(g)$  the restriction of the operator  $f(g)$  to the subspace  $W$ :

$$\varphi(g) = f(g) \Big|_W. \quad (3.1)$$

For the operators  $\varphi(g)$  we have the following relationships:

$$\varphi(g) \varphi(g^{-1}) = (f(g) f(g^{-1})) \Big|_W = 1; \quad (3.2)$$

$$\varphi(g_1) \varphi(g_2) = (f(g_1) f(g_2)) \Big|_W = \varphi(g_1 g_2). \quad (3.3)$$

From the relationship (3.2) we conclude that the operator  $\varphi(g)$  is invertible and  $\varphi(g)^{-1} = \varphi(g^{-1})$ . Hence,  $\varphi(g) \in \text{Aut}(W)$ . The relationship (3.3) in its turn shows that the mapping  $\varphi: G \rightarrow \text{Aut}(W)$  is a group homomorphism defining a representation.

**DEFINITION 3.3.** The representation  $(\varphi, G, W)$  of a group  $G$  obtained by restricting the operators of a representation  $(f, G, V)$  to its invariant subspace  $W \subseteq V$  according to the formula (3.1) is called the *restriction* of  $f$  to  $W$ .

The presence of an invariant subspace  $W$  let's us define factoroperators in the factorspace  $V/W$ :

$$\psi(g) = f(g) \Big|_{V/W}. \quad (3.4)$$

Let's recall that the action of the operator  $\psi(g)$  upon a coset  $\text{Cl}_W(\mathbf{x})$  in  $V/W$  is defined as follows:

$$\psi(g) \text{Cl}_W(\mathbf{x}) = \text{Cl}_W(f(g)\mathbf{x}). \quad (3.5)$$

The correctness of the definition (3.5) is verified by direct calculations (see [1]). The factoroperators (3.5) obey the following relationships:

$$\begin{aligned} \psi(g) \psi(g^{-1}) \text{Cl}_W(\mathbf{x}) &= \text{Cl}_W(f(g)f(g^{-1})\mathbf{x}) = \\ &= \text{Cl}_W(f(gg^{-1})\mathbf{x}) = \text{Cl}_W(\mathbf{x}), \end{aligned} \quad (3.6)$$

$$\begin{aligned} \psi(g_1) \psi(g_2) \text{Cl}_W(\mathbf{x}) &= \text{Cl}_W(f(g_1)f(g_2)\mathbf{x}) = \\ &= \text{Cl}_W(f(g_1g_2)\mathbf{x}) = \psi(g_1g_2) \text{Cl}_W(\mathbf{x}). \end{aligned} \quad (3.7)$$

From (3.6) and (3.7) we conclude that the factoroperators (3.4) satisfy the relationships similar to (3.2) and (3.3). They define a representation  $(\psi, G, V/W)$  which is usually called a *factorrepresentation*.

The representations  $(\varphi, G, W)$  and  $(\psi, G, V/W)$  are generated by the representation  $f$ . Each of them inherits a part of the information contained in the representation  $f$ . In order to understand which part of the information is kept in  $\varphi$  and  $\psi$  let's study the representation operators  $f(g)$  in some special basis. Let's choose a basis  $\mathbf{e}_1, \dots, \mathbf{e}_s$  within the invariant subspace  $W$ .

Then we complete this basis up to a basis in the space  $V$ . This construction is based on the theorem on completing a basis of a subspace (see [1]). The matrix of the operator  $f(g)$  in such a composite basis is a block-triangular matrix:

$$F(g) = \left\| \begin{array}{c|c} \varphi_j^i & u_j^i \\ \hline 0 & \psi_j^i \end{array} \right\|. \quad (3.8)$$

The upper left diagonal block coincides with the matrix of the operator  $\varphi(g)$  in the basis  $\mathbf{e}_1, \dots, \mathbf{e}_s$ . The lower right diagonal block coincides with the matrix of the factoroperator  $\psi(g)$  in the basis  $\mathbf{E}_1, \dots, \mathbf{E}_{n-s}$ , where

$$\mathbf{E}_1 = \text{Cl}_W(\mathbf{e}_{s+1}), \quad \mathbf{E}_2 = \text{Cl}_W(\mathbf{e}_{s+2}), \quad \dots, \quad \mathbf{E}_{n-s} = \text{Cl}_W(\mathbf{e}_n).$$

From (3.8) we conclude that when passing from  $f$  to its restriction  $\varphi$  and to its factorrepresentation  $\psi$  the amount of the lost information is determined by the upper right non-diagonal block  $u_j^i$  in the matrix (3.8).

Despite to the loss of information, the passage from  $f$  to the pair of representations  $\varphi$  and  $\psi$  can be treated as splitting  $f$  into more simple components. If  $\varphi$  and  $\psi$  are also reducible, they can be split further. However, this process of splitting is finite since in each step we have the reduction of the dimension:  $\dim(W) < \dim(V)$  and  $\dim(V/W) < \dim(V)$ . The process will terminate when we reach irreducible representations.

#### § 4. Completely reducible representations.

As we have seen above, the process of fragmentation of a reducible representation leads to the loss of information. However, there is a special class of representations for which the loss of information is absent. This is the class of *completely reducible representations*.

DEFINITION 4.1. A representation  $(f, G, V)$  of a group  $G$  is called *completely reducible* if each its invariant subspace  $W$  has an invariant direct complement  $U$ , i.e.  $V$  is a direct sum of two invariant subspaces  $V = W \oplus U$ .

Note that an irreducible representation is a trivial example of a completely reducible one. Here we have  $V = V \oplus \{0\}$ .

Let  $(f, G, V)$  be a reducible and completely reducible representation. Let  $W$  be an invariant subspace for  $f$  and  $U$  be its invariant direct complement. Then we have the following isomorphisms of the restrictions and factors:

$$f|_U \cong f|_{V/W}, \quad f|_W \cong f|_{V/U}. \quad (4.1)$$

In order to prove (4.1) let's consider again a basis  $\mathbf{e}_1, \dots, \mathbf{e}_s$  in  $U$  and complement it with a basis  $\mathbf{h}_1, \dots, \mathbf{h}_{n-s}$  in  $U$ . Let's denote  $\mathbf{e}_{s+1} = \mathbf{h}_1, \dots, \mathbf{e}_n = \mathbf{h}_{n-s}$ . As a result of such a reunion of bases we get a basis in  $V$ . Let's define a mapping  $A: U \rightarrow V/W$  by setting

$$A\mathbf{x} = \text{Cl}_W(\mathbf{x}) \text{ for all } \mathbf{x} \in U. \quad (4.2)$$

From (4.2) one easily derives the values of the mapping  $A$  on basis vectors

$$A(\mathbf{h}_1) = \mathbf{E}_1, \quad \dots, \quad A(\mathbf{h}_{n-s}) = \mathbf{E}_{n-s}. \quad (4.3)$$

The mapping  $A$  establishes bijective correspondence of bases of  $U$  and  $V/W$ . For this reason it is bijective. Let's verify that it implements an isomorphism of representations, i.e. let's verify the relationship (1.1) for  $A$ :

$$A(f(g)\mathbf{h}_i) = \text{Cl}_W(f(g)\mathbf{h}_i) = \psi(g) \text{Cl}_W(\mathbf{h}_i) = \psi(g)A(\mathbf{h}_i).$$

This relationship proves the first isomorphism in (4.1). It is implemented by the mapping  $A$  defined in (4.2).

From the invariance of the subspace  $U$  we derive  $f(g)\mathbf{h}_i \in U$ . This fact and the relationship (4.3) let us write the matrix of the operator  $f(g)$ :

$$F(g) = \left\| \begin{array}{c|c} \varphi_j^i & 0 \\ \hline 0 & \psi_j^i \end{array} \right\|. \quad (4.4)$$

The matrix (4.4) is block-diagonal, therefore no loss of information occur when passing from  $f$  to the representations  $\varphi$  and  $\psi$  in the case of completely reducible representation  $f$ . Due to (4.1) the representation  $\psi$  can be treated as the restriction of  $f$  to the invariant complement  $U$ . The representations  $(\varphi, G, W)$  and  $(\psi, G, U)$  are not linked to each other, they are defined in their own spaces which intersect trivially and their sum is the complete space  $V$ . This situation is described by the following definition.

DEFINITION 4.2. A representation  $(f, G, V)$  of a group  $G$  is called an *inner direct sum* of the representations  $(\varphi, G, W)$  and  $(\psi, G, U)$  if  $V = W \oplus U$ , while the subspaces  $W$  and  $U$  are invariant subspaces for  $f$  and the restrictions of  $f$  to these subspaces coincide with  $\varphi$  and  $\psi$ .

Note that the splitting of  $f$  into a direct sum  $f = \varphi \oplus \psi$  can occur even in that case where  $f$  is not a completely reducible representation. However in this case such a splitting is rather an exception than a rule.

Having a pair of representations  $(\varphi, G, W)$  and  $(\psi, G, U)$  of the same group  $G$  in two distinct spaces, we can construct their *external direct sum*. Let's consider the external direct sum  $W \oplus U$ . Let's recall that it is the set of ordered pairs  $(\mathbf{w}, \mathbf{u})$ , where  $\mathbf{w} \in W$  and  $\mathbf{u} \in U$ , with the algebraic operations

$$\begin{aligned} (\mathbf{w}_1, \mathbf{u}_1) + (\mathbf{w}_2, \mathbf{u}_2) &= (\mathbf{w}_1 + \mathbf{w}_2, \mathbf{u}_1 + \mathbf{u}_2), \\ \alpha(\mathbf{w}, \mathbf{u}) &= (\alpha\mathbf{w}, \alpha\mathbf{u}), \quad \text{where } \alpha \in \mathbb{C}. \end{aligned}$$

The subspaces  $W$  and  $U$  in the external direct sum  $W \oplus U$  are assumed to be disjoint even in that case where they have non-zero intersection or do coincide. Let's define operators  $f(g)$  in

$W \oplus U$  as follows:

$$f(g)(\mathbf{w}, \mathbf{u}) = (\varphi(g)\mathbf{w}, \psi(g)\mathbf{u}). \quad (4.5)$$

The representation  $(f, G, W \oplus U)$  constructed according to the representation (4.5) is called the *external direct sum* of the representations  $(\varphi, G, W)$  and  $(\psi, G, W)$ . It is denoted  $f = \varphi \oplus \psi$ . The difference of internal and external direct sums is rather formal, their properties do coincide in most.

Assume that the space  $V$  of a representation  $(f, G, V)$  is expanded into a direct sum of subspaces  $V = W \oplus U$  (not necessarily invariant ones). Each such an expansion are uniquely associated with two projection operators  $P$  and  $Q$ . The projector  $P$  is the projection operator that projects onto the subspace  $W$  parallel to the subspace  $U$ , while  $Q$  projects onto  $U$  parallel to  $W$ . They satisfy the following relationships:

$$P^2 = P, \quad Q^2 = Q, \quad P + Q = 1. \quad (4.6)$$

Moreover,  $W = \text{Im } P$  and  $U = \text{Im } Q$ . These properties of the projection operators are well known (see [1]).

LEMMA 4.1. *The subspace  $W$  in an expansion  $V = W \oplus U$  is invariant with respect to the operators of a representation  $f, G, V$  if and only if the corresponding projector  $P$  obeys the relationship*

$$(P \circ f(g) - f(g) \circ P) \circ P = 0 \quad \text{for all } g \in G. \quad (4.7)$$

*The invariance of both subspaces  $W$  and  $U$  in an expansion  $V = W \oplus U$  is equivalent to the relationship*

$$(P \circ f(g) = f(g) \circ P) \quad \text{for all } g \in G, \quad (4.8)$$

*which means that the projector  $P$  commutes with all operators of the representation  $f$ .*

Let  $\mathbf{x}$  be an arbitrary vector. Then  $P\mathbf{x} \in W$ . In the case of an invariant subspace  $W$  the vector  $\mathbf{y} = f(g)P\mathbf{x}$  also belongs to  $W$ . For the vector  $\mathbf{y} \in W$  we have  $P\mathbf{y} = \mathbf{y}$ . Therefore

$$Pf(g)P\mathbf{x} = f(g)P\mathbf{x} = f(g)P^2\mathbf{x}. \quad (4.9)$$

Comparing the left and right hand sides of the equality (4.9) and taking into account the arbitrariness of the vector  $\mathbf{x}$ , we easily derive the relationship (4.7) in the statement of the lemma.

And conversely, from the relationship (4.7), using the property  $P^2 = P$  from (4.6), we easily derive (4.9). From (4.9) we derive that the vector  $f(g)P\mathbf{x}$  belongs to  $W$  for an arbitrary vector  $\mathbf{x}$ . Let's choose  $\mathbf{x} \in W$  and from  $P\mathbf{x} = \mathbf{x}$  for such a vector  $\mathbf{x}$  we find that  $f(g)\mathbf{x}$  belongs to  $W$ . The invariance of  $W$  is established. In order to prove the second statement of the lemma let's write the relationship (4.7) for the projector  $Q$ :

$$(Q \circ f(g) - f(g) \circ Q) \circ Q = 0. \quad (4.10)$$

from  $Q = 1 - P$  we get  $Q \circ f(g) - f(g) \circ Q = f(g) \circ P - P \circ f(g)$ . Therefore the relationship (4.10) is rewritten as

$$f(g) \circ P - P \circ f(g) + (P \circ f(g) - f(g) \circ P) \circ P = 0.$$

Then, taking into account (4.7), we reduce it to the relationship (4.8), which means that the operators  $P$  and  $f(g)$  do commute. The lemma is proved.

The second proposition of the lemma 4.1 can be generalized for the case where  $V$  is expanded into a direct sum of several subspaces. Assume that  $V = W_1 \oplus \dots \oplus W_s$ . Let's recall (see [1]) that this expansion uniquely fixes a concordant family of projection operators  $P_1, \dots, P_s$ . They obey the following concordance relationships:

$$\begin{aligned} (P_i)^2 &= P_i, & P_1 + \dots + P_s &= 1, \\ P_i \circ P_j &= 0 \text{ for } i \neq j, & P_i \circ P_j &= P_j \circ P_i. \end{aligned}$$

Moreover,  $W_i = \text{Im } P_i$ . The condition of invariance of all subspaces  $W_i$  in the expansion  $V = W_1 \oplus \dots \oplus W_s$  with respect to representation operators of a representation  $(f, G, V)$  in terms of the corresponding projection operators is formulated in the following lemma. We leave its proof to the reader.

LEMMA 4.2. *The expansion  $V = W_1 \oplus \dots \oplus W_s$  is an expansion of  $V$  into a direct sum of invariant subspaces of the representation  $(f, G, V)$  if and only if all projection operators  $P_i$  associated with the expansion  $V = W_1 \oplus \dots \oplus W_s$  commute with the representation operators  $f(g)$ .*

Let's consider a completely reducible finite-dimensional representation  $(f, G, V)$  split into a direct sum of its restrictions to invariant subspaces  $f = \varphi \oplus \psi$ . Assume that the restriction  $\varphi$  of  $f$  to the subspace  $W$  is reducible and assume that  $W_1$  is its non-trivial invariant subspace:  $\{0\} \subsetneq W_1 \subsetneq W$ . Then the subspace  $W_1$  is invariant with respect to  $f$ . It has an invariant complement  $U_1$ . Let's consider the expansions

$$V = W \oplus U, \quad V = W_1 \oplus U_1. \quad (4.11)$$

Note that  $W_1 \subset W$ , hence,  $W + U_1 = V$ . Therefore the dimensions of the subspaces in (4.11) are related as follows:

$$\dim(W) + \dim(U_1) - \dim(V) = \dim(W) - \dim(W_1) > 0.$$

From this relationship, we see that the intersection  $W_2 = W \cap U_1$  is non-zero and its dimension is given by the formula

$$\dim(W \cap U_1) = \dim(W) - \dim(W_1). \quad (4.12)$$

Now note that  $W_1 \cap W_2 = \{0\}$  since  $W_2 \subset U_1$ . Therefore, due to (4.12) the subspace  $W$  is expanded into the direct sum

$$W = W_1 \oplus W_2,$$



each summand in which is invariant with respect to  $f$  and, hence, with respect to  $\varphi$ . Thus, we have proved the following important theorem.

**THEOREM 4.1.** *The restriction of a completely reducible finite-dimensional representation to an invariant subspace is completely reducible.*

The next theorem on the expansion into a direct sum is an immediate consequence of the theorem 4.1.

**THEOREM 4.2.** *Each finite-dimensional completely reducible representation  $f$  is expanded into a direct sum of irreducible representations*

$$f = f_1 \oplus \dots \oplus f_k, \quad V = W_1 \oplus \dots \oplus W_k, \quad (4.13)$$

where each  $f_i$  is a restriction of  $f$  to the corresponding invariant subspace  $W_i$ .

Note that in general the expansion (4.13) is not unique. Let's consider two expansions of  $f$  into irreducible components:

$$\begin{aligned} f &= f_1 \oplus \dots \oplus f_k, & V &= W_1 \oplus \dots \oplus W_k, \\ f &= \tilde{f}_1 \oplus \dots \oplus \tilde{f}_k, & V &= \tilde{W}_1 \oplus \dots \oplus \tilde{W}_k. \end{aligned} \quad (4.14)$$

The extent of differences in two expansions (4.14) is determined by the following Jordan-Hölder theorem.

**THEOREM 4.3.** *The numbers of irreducible components in the expansions (4.14) are the same:  $q = k$ , and there is a transposition  $\sigma \in S_k$  such that  $(f_i, G, W_i) \cong (f_{\sigma i}, G, \tilde{W}_{\sigma i})$ .*

The expansions (4.14) are isomorphic up to a transposition of components. However, we should emphasize that the isomorphism does not mean the coincidence of these expansions.

We shall prove the Jordan-Hölder theorem by induction on the number of components  $k$  in the first expansion (4.14).

**The base of the induction:**  $k = 1$ ,  $V = W_1$ . In this case the representation  $f = f_1$  is irreducible. Therefore  $q = 1 = k$  and  $\tilde{W}_1 = V$ ,  $\tilde{f}_1 = f = f_1$ . The base of the induction is proved.

**The inductive step.** Assume that the theorem is valid for representations possessing at least one expansion of the form (4.13) with the length  $k - 1$ . For the representation  $f$  we introduce the following notations:

$$\begin{aligned} \tilde{V}_i &= \tilde{W}_1 \oplus \dots \oplus \tilde{W}_i \quad \text{where } i = 1, \dots, q, \\ U &= W_1 \oplus \dots \oplus W_{k-1} \quad \text{and } \tilde{U}_i = \tilde{V}_i \cap U. \end{aligned} \tag{4.15}$$

All of the subspaces in (4.15) are invariant with respect to  $f$ . Moreover,  $V = U \oplus W_k$  and there are two chains of inclusions:

$$\begin{aligned} \{0\} &\subsetneq \tilde{V}_1 \subsetneq \dots \subsetneq \tilde{V}_q = V, \\ \{0\} &\subsetneq \tilde{U}_1 \subsetneq \dots \subsetneq \tilde{U}_q = U. \end{aligned} \tag{4.16}$$

Let  $h: V \rightarrow V/U$  be a canonical projection onto the factorspace. Let's denote by  $\varphi$  the factorrepresentation in the factorspace  $V/U$  and consider the chain of invariant subspaces for it

$$\{0\} \subseteq h(\tilde{V}_1) \subseteq \dots \subseteq h(\tilde{V}_q) = h(V) = V/U \cong W_k.$$

Due to the isomorphism  $\varphi \cong f_k$  we conclude that  $\varphi$  is irreducible. Hence the above chain does actually look like

$$\{0\} = h(\tilde{V}_1) = \dots = h(\tilde{V}_s) \subsetneq h(\tilde{V}_{s+1}) = \dots = h(\tilde{V}_q) = h(V).$$

Therefore  $\tilde{V}_i \subseteq U$  and  $\tilde{V}_i = \tilde{V}_i$  for  $i \leq s$ . For  $i \geq s + 1$  we use the isomorphisms  $\tilde{V}_i/\tilde{U}_i \cong h(\tilde{V}_i) = h(\tilde{V}_{i+1}) \cong \tilde{V}_{i+1}/\tilde{U}_{i+1}$ . But  $\tilde{V}_i \subsetneq \tilde{V}_i$ , hence,  $\tilde{U}_i \subsetneq \tilde{U}_{i+1}$ . Then from (4.16) we get

$$\{0\} \subsetneq \tilde{U}_1 \subsetneq \dots \subsetneq \tilde{U}_s = \tilde{U}_{s+1} \subsetneq \dots \subsetneq \tilde{U}_q = U. \tag{4.17}$$

The equality  $\tilde{U}_s = \tilde{U}_{s+1}$  follows from the irreducibility of the factorrepresentation  $\varphi_{s+1} \cong \varphi \cong f_k$  in the factorspace  $\tilde{V}_{s+1}/\tilde{V}_s$

and from the inclusions  $\tilde{V}_s = \tilde{U}_s = \tilde{U}_{s+1} \subsetneq \tilde{V}_{s+1}$ . Relying upon complete reducibility of  $f$ , we choose invariant complements  $\tilde{W}_{i+1}$  for  $\tilde{U}_i$  in  $\tilde{U}_{i+1}$ , i. e.  $\tilde{U}_{i+1} = \tilde{U}_i \oplus \tilde{W}_{i+1}$ . Then  $\tilde{V}_i + \tilde{U}_{i+1} = \tilde{V}_i \oplus \tilde{W}_{i+1}$  is an invariant subspace in  $\tilde{V}_{i+1}$  containing  $\tilde{V}_i$  and not coinciding with it. But the factorrepresentation in  $\tilde{V}_{i+1}/\tilde{V}_i$  is isomorphic to  $\tilde{f}_{i+1}$  and, hence, is irreducible. Therefore  $\tilde{V}_i \oplus \tilde{W}_{i+1} = \tilde{V}_{i+1}$  and the restriction of  $f$  to  $\tilde{W}_{i+1}$  is isomorphic to  $\tilde{f}_{i+1}$ . From (4.15) and (4.17) we get

$$\begin{aligned} U &= W_1 \oplus \dots \oplus W_s \oplus W_{s+1} \oplus \dots \oplus W_{k-1}, \\ U &= \tilde{W}_1 \oplus \dots \oplus \tilde{W}_s \oplus \tilde{W}_{s+2} \oplus \dots \oplus \tilde{W}_q. \end{aligned} \tag{4.18}$$

Now it is sufficient to apply the inductive hypothesis to (4.18). As a result we get  $k = q$  and find  $\sigma i$  for  $i = 1, \dots, k - 1$ . The isomorphism of  $f_k$  and the factorrepresentation  $\varphi_{s+1}$  in  $\tilde{V}_{s+1}/\tilde{V}_s$  yields  $\sigma k = s + 1$  since  $\varphi_{s+1} \cong \tilde{f}_{s+1}$  by construction of the subspaces (4.15). The Jordan-Hölder theorem is proved.

Due to the theorems 4.1 and 4.2 it is natural to introduce the following terminology.

**DEFINITION 4.3.** An invariant subspace  $W \subseteq V$  is called an *irreducible subspace* for a representation  $(f, G, V)$  if the restriction of  $f$  to  $W$  is irreducible.

The following theorem yields a tool for verifying the complete reducibility of representations.

**THEOREM 4.4.** A finite-dimensional representation  $(f, G, V)$  is completely reducible if and only if the set of all its irreducible subspaces span the whole space  $V$ .

**PROOF.** Let  $\{W_\alpha\}_{\alpha \in A}$  be the set of all irreducible subspaces in  $V$ . The number of such subspaces could be infinite even in the case of a finite-dimensional representation. Let's denote by  $W$

the sum of all irreducible subspaces  $W_\alpha$ :

$$W = \sum_{\alpha \in A} W_\alpha = \left\langle \bigcup_{\alpha \in A} W_\alpha \right\rangle.$$

The theorem 4.4 says that the condition  $W = V$  is necessary and sufficient for the representation  $f$  to be completely reducible. The necessity of this condition follows from the theorem 4.2. Let's prove its sufficiency. Let  $U = U_0$  be an invariant subspace for  $f$  and  $\{0\} \neq U_0 \neq V$ . Then for each irreducible subspace  $W_\alpha$  we have two mutually exclusive options:

$$W_\alpha \subseteq U_0 \quad \text{or} \quad W_\alpha \cap U_0 = \{0\}.$$

And there is at least one subspace  $W_{\alpha_1}$  for which the second option is valid:  $W_{\alpha_1} \cap U_0 = \{0\}$ . Indeed, otherwise we would have  $W \subseteq U_0$ , which contradicts  $W = V$ . Let's consider the sum

$$U_1 = U_0 + W_{\alpha_1} = U_0 \oplus W_{\alpha_1}.$$

The subspace  $U_1$  is invariant with respect to  $f$ . If  $U_1 \neq V$ , we repeat our considerations with  $U_1$  instead of  $U_0$ . As a result we get a new invariant subspace

$$U_2 = U_1 \oplus W_{\alpha_2} = U_0 \oplus W_{\alpha_1} \oplus W_{\alpha_2}.$$

The process of adding new direct summand to  $U_0$  will terminate at some step since  $\dim V < \infty$ . As a result we get

$$U_k = U_0 \oplus (W_{\alpha_1} \oplus \dots \oplus W_{\alpha_k}) = V.$$

The subspace  $W = W_{\alpha_1} \oplus \dots \oplus W_{\alpha_k}$  is a required invariant direct complement for  $U = U_0$ . The theorem is proved.  $\square$

### § 5. Schur's lemma and some corollaries of it.

The theorem 4.2 proved in previous section says that each completely reducible representation is expanded into a direct sum of its irreducible components. In this section we study these irreducible components by themselves. Schur's lemma plays the central role in this. We formulate two versions of this lemma, the second version being a strengthening of the first one, but for a more special case.

**LEMMA 5.1 (Schur's lemma).** *Let  $(f, G, V)$  and  $(h, G, W)$  be two irreducible representations of a group  $G$ . Each homomorphism  $A$  relating these two representations either is identically zero or is a homomorphism.*

Before proving the Schur's lemma we consider the following theorem having a separate value.

**THEOREM 5.1.** *If a linear mapping  $A: V \rightarrow W$  is a homomorphism of representations from  $(f, G, V)$  to  $(h, G, W)$ , then its kernel  $\text{Ker } A$  is an invariant subspace for  $f$ , while its image  $\text{Im } A$  is an invariant subspace for  $h$ .*

**PROOF.** Let  $\mathbf{y} = A\mathbf{x}$  be a vector from the image of the mapping  $A$ . We apply the operator  $f(g)$  to it and use the relationship (1.1). Then we get

$$h(g)\mathbf{y} = h(g)A\mathbf{x} = Af(g)\mathbf{x}.$$

Now it is clear that  $h(g)\mathbf{y} \in \text{Im } A$ , i. e.  $\text{Im } A$  is an invariant subspace for  $h$ . The invariance of  $\text{Ker } A$  is proved similarly. Assume that  $\mathbf{x} \in \text{Ker } A$ . Let's apply the operator  $f(g)$  to  $\mathbf{x}$  and then apply the mapping  $A$ . Taking into account (1.1), we get

$$Af(g)\mathbf{x} = h(g)A\mathbf{x} = 0$$

since  $A\mathbf{x} = 0$ . Hence,  $f(g)\mathbf{x} \in \text{Ker } A$ . The invariance of the kernel  $\text{Ker } A$  with respect to  $f$  is proved.  $\square$

Now let's proceed to proving Schur's lemma 5.1. The case, where  $A = 0$ , is trivial. In order to exclude this case assume that  $A \neq 0$ . Then  $\text{Im } A \neq \{0\}$ . According to the theorem 5.1 proved just above,  $\text{Im } A$  is invariant with respect to the representation  $h$ . Since  $h$  is irreducible, we get  $\text{Im } A = W$ . This means that the homomorphism  $A$  is a surjective linear mapping  $A: V \rightarrow W$ .

The kernel  $\text{Ker } A$  of the mapping  $A: V \rightarrow W$  is invariant with respect to  $f$ . Since  $f$  is an irreducible representation, we have two options  $\text{Ker } A = \{0\}$  or  $\text{Ker } A = V$ . The second option leads to the trivial case  $A = 0$ , which is excluded. Therefore,  $\text{Ker } A = \{0\}$ . This means that  $A: V \rightarrow W$  is an injective linear mapping. Being surjective and injective simultaneously, this mapping is bijective. Hence, it is an isomorphism of  $(f, G, V)$  and  $(h, G, W)$ . Schur's lemma is proved.

Now let's consider an operator  $A: V \rightarrow V$  that interlaces an irreducible representation  $(f, G, V)$  with itself. The relationship (1.1) written for this case means that  $A$  commutes with all representation operators  $f(g)$ . The second version of Schur's lemma describes this special case  $f = g$ .

**LEMMA 5.2 (Schur's lemma).** *Each operator  $A: V \rightarrow V$  that commutes with all operators of an irreducible finite-dimensional representation  $(f, G, V)$  in a linear vector space  $V$  over the field of complex numbers  $\mathbb{C}$  is a scalar operator, i. e.  $A = \lambda \cdot 1$ , where  $\lambda \in \mathbb{C}$ .*

**PROOF.** Let  $\lambda$  be an eigenvalue of the operator  $A$  and let  $V_\lambda \neq \{0\}$  be the corresponding eigenspace. If  $\mathbf{x} \in V_\lambda$  then  $A\mathbf{x} = \lambda\mathbf{x}$ . Moreover, from the commutation condition of the operators  $A$  and  $f(g)$  we obtain

$$Af(g)\mathbf{x} = f(g)A\mathbf{x} = \lambda f(g)\mathbf{x}.$$

Hence,  $f(g)\mathbf{x}$  is also a vector of the eigenspace  $V_\lambda$ . In other words,  $V_\lambda$  is an invariant subspace. Since  $V_\lambda \neq \{0\}$  and since  $f$  is irreducible, we get  $V_\lambda = V$ . Hence,  $A\mathbf{x} = \lambda\mathbf{x}$  for an arbitrary vector  $\mathbf{x} \in V$ . This means that  $A = \lambda \cdot 1$ .  $\square$

Note that the condition of finite-dimensionality of the representation  $f$  and the condition of complexity of the vector space  $V$  in Schur's lemma 5.2 are essential. Without these conditions one cannot grant the existence of a non-trivial eigenspace.

Now we apply Schur's lemma for to investigate tensor products of representations of some special sort. For this reason we need to define the concept of tensor product for representations.

Let  $(f, g, V)$  and  $(h, G, W)$  be two representations of a group  $G$ . We define the operators  $\varphi(g)$  acting within the tensor product  $V \otimes W$  by setting

$$\varphi(g)(\mathbf{v} \otimes \mathbf{w}) = (f(g)\mathbf{v} \otimes h(g)\mathbf{w}) \quad \text{for all } g \in G. \quad (5.1)$$

The operators  $\varphi(g)$  constitute a new representation of the group  $G$ . The proof of this proposition and verifying the correctness of the formula (5.1) are left to the reader.

DEFINITION 5.1. The representation  $(\varphi, G, V \otimes W)$  given by the operators (5.1) is called the *tensor product* of the representations  $f$  and  $h$ . It is denoted  $\varphi = f \otimes h$ .

Note that the construction (5.1) is easily generalized for the case of several representations.

Assume that the representation  $f$  in the construction (5.1) is irreducible. As a second tensorial multiplicand in (5.1) we choose the trivial representation  $i$  given by the identity operators  $i(g) = 1$  for all  $g \in G$ . In the case where  $\dim W > 1$  the tensor product  $f \otimes i$  is a reducible representation. Let's prove this fact. Assume that  $\mathbf{e}_1, \dots, \mathbf{e}_m$  is a basis in the space  $W$ . Let's denote by  $W_k$  one-dimensional subspaces spanned by basis vectors  $\mathbf{e}_k$  and then consider the subspaces

$$V \otimes W_k, \quad k = 1, \dots, m. \quad (5.2)$$

The subspaces (5.2) are invariant within  $V \otimes W$  with respect to the operators of the representation  $f \otimes i$ . The restrictions of  $f \otimes i$  to  $V \otimes W_k$  all are isomorphic to  $f$ . Hence, the subspaces

(5.2) are irreducible. These results are simple. They are proved immediately.

No note that  $V \otimes W = V \otimes W_1 \oplus \dots \oplus V \otimes W_m$ . The space  $V \otimes W$  is a sum of its irreducible subspaces  $V \otimes W_k$ . Hence it is spanned by these subspaces. Therefore, it is sufficient to apply the theorem 4.4. As a result we get the following proposition.

**THEOREM 5.2.** *The tensor product  $f \otimes i$  of a finite-dimensional irreducible representation  $f$  and a trivial finite-dimensional representation  $i$  is completely reducible.*

By means of this theorem one can describe the structure of all invariant subspaces of the tensor product  $f \otimes i$ .

**THEOREM 5.3.** *If under the assumptions of the theorem 5.2  $f$  and  $i$  are representations in complex linear vector spaces  $V$  and  $W$ , then each invariant subspace of the representation  $f \otimes i$  is a tensor product  $U = V \otimes \tilde{W}$ , where  $\tilde{W}$  is some subspace of  $W$ .*

**PROOF.** Let  $U$  be some invariant subspace of  $f \otimes i$  in  $V \otimes W$ . Due to the theorem 5.2 the representation  $f \otimes i$  is completely reducible. Therefore  $U$  has an invariant complement  $\tilde{U}$ . The expansion  $V \times W = U \oplus \tilde{U}$  means that we can define a projection operator  $P$  that project onto  $U$  parallel to  $\tilde{U}$ . The invariance of both spaces  $U$  and  $\tilde{U}$  means that  $P$  commutes with all operators of the representation  $\varphi = f \times i$ .

Let  $A : V \times W \rightarrow V \times W$  be some operator in the tensor product  $V \times W$ . In our proof  $A = P$ , however, we return to this special case a little bit later. Let's apply the operator  $A$  to  $\mathbf{x} \otimes \mathbf{e}_j$  and write the result as

$$A(\mathbf{x} \otimes \mathbf{e}_j) = \sum_{k=1}^m A_j^k(\mathbf{x}) \otimes \mathbf{e}_k. \quad (5.3)$$

Here  $\mathbf{e}_1, \dots, \mathbf{e}_m$  is some basis in  $W$ . Since basis vectors are linearly independent, the coefficients  $A_j^k(\mathbf{x}) \in V$  in the expansion (5.3) are unique. They are changed only if we change a basis,



the transformation rule for them being analogous to that for components of a tensor. This fact is not important for us, since in our considerations we will not change the basis  $\mathbf{e}_1, \dots, \mathbf{e}_m$ .

Let's study the dependence of  $A_j^k(\mathbf{x})$  on  $\mathbf{x}$ . It is clear that it is linear. Therefore each coefficient  $A_j^k(\mathbf{x})$  in (5.3) defines a linear operator  $A_j^k: V \rightarrow V$ . Let's write the commutation relationship for  $A$  and  $\varphi(g)$ . It is sufficient write this relationship as applied to the vectors of the form  $\mathbf{x} \otimes \mathbf{e}_i$ . From (5.3) we derive

$$\begin{aligned} A \circ \varphi(g)(\mathbf{x} \otimes \mathbf{e}_i) &= \sum_{k=1}^m A_j^k \circ f(g)(\mathbf{x}) \otimes \mathbf{e}_k, \\ \varphi(g) \circ A(\mathbf{x} \otimes \mathbf{e}_i) &= \sum_{k=1}^m f(g) \circ A_j^k(\mathbf{x}) \otimes \mathbf{e}_k. \end{aligned} \quad (5.4)$$

For to write (5.4) it is sufficient to recall that  $\varphi = f \otimes i$ , where  $i$  is a trivial representation. From (5.4), it is easy to see that the commutation relationship  $A \circ \varphi(g) = \varphi(g) \circ A$  is equivalent to

$$f(g) \circ A_j^k = A_j^k \circ f(g). \quad (5.5)$$

Thus, all of the linear operators  $A_j^k$  commute with the operators  $f(g)$  of the representation  $f$ .

The next step in proof is to return to the projection operator  $A = P$  (see above) and to apply Schur's lemma to the operators  $A_j^k = P_j^k$ . The projector  $P$  commutes with  $\varphi(g)$ , therefore the relationships (5.5) hold for  $A_j^k = P_j^k$ . Applying Schur's lemma 5.2, we get

$$P_j^k = \lambda_j^k \times 1, \quad (5.6)$$

where  $\lambda_j^k$  are some complex numbers. Substituting (5.6) into the expansion (5.3), for the operator  $P$  we derive:

$$P(\mathbf{x} \otimes \mathbf{e}_j) = \sum_{k=1}^m A_j^k(\mathbf{x}) \otimes \mathbf{e}_k = \mathbf{x} \otimes \left( \sum_{k=1}^m \lambda_j^k \mathbf{e}_k \right). \quad (5.7)$$

The relationship (5.7) shows that it is natural to define a linear operator  $Q: W \rightarrow W$  given by its values on basis vectors:

$$Q(\mathbf{e}_j) = \sum_{k=1}^m \lambda_j^k \mathbf{e}_k. \quad (5.8)$$

Due to (5.8) we can rewrite (5.7) as

$$P(\mathbf{x} \otimes \mathbf{e}_j) = \mathbf{x} \otimes Q(\mathbf{e}_j). \quad (5.9)$$

Now let's remember that  $P$  is a projection operator. Hence,  $P^2 = P$  (see [1]). Combining this relationship with (5.9), we derive  $Q^2 = Q$ . Therefore,  $Q$  is also a projection operator. Let's denote  $\tilde{W} = \text{Im } Q \subseteq W$ . Then for the invariant subspace  $U \subseteq V \otimes W$  we get

$$U = \text{Im } P = V \otimes \text{Im } Q = V \otimes \tilde{W}. \quad (5.10)$$

The formula (5.10) describes the structure of all invariant subspaces if the representation  $f \otimes i$ . Thus, the theorem 5.3 is proved.  $\square$

The theorem 5.3 can be applied for proving the following extremely useful fact.

**THEOREM 5.4.** *Let  $f$  be a finite dimensional irreducible representation of some group  $G$  in a complex linear vector space  $V$ . Then the set of representation operators  $f(g)$  spans the whole space of linear operators  $\text{End}(V)$ .*

**PROOF.** Each representation  $f$  in  $V$  generates an associated representation in the space of linear operators  $\text{End}(V)$ . Indeed, if  $A \in \text{End}(V)$ , then we can define the action of  $\psi(g)$  to  $A$  as the composition of operators  $f(g)$  and  $A$ :

$$\psi(g)(A) = f(g) \circ A \quad \text{for all } A \in \text{End}(V). \quad (5.11)$$

Let  $F$  be the span of the set of all operators  $f(g)$ , i. e.

$$F = \langle \{f(g): g \in G\} \rangle.$$

It is easy to verify that the subspace  $F$  is invariant with respect to the representation  $\psi$  defined by the formula (5.11).

In order to apply the theorem 5.3 let's remember that there is a canonical isomorphism  $V \otimes V^* \cong \text{End}(V)$ , where  $V^*$  is the dual space for  $V$  (the space of linear functionals in  $V$ ). This isomorphism is established by the mapping  $\sigma: V \times V^* \rightarrow \text{End}(V)$  which is defined as follows:

$$\sigma(\mathbf{x} \otimes \lambda)\mathbf{y} = \lambda(\mathbf{y})\mathbf{x} \quad \text{for all } \mathbf{x}, \mathbf{y} \in V \quad \text{and } \lambda \in V^*. \quad (5.12)$$

The proof of the correctness of the definition (5.12) and verifying that  $\sigma$  is an isomorphism are left to the reader.

It is easy to verify that the canonical isomorphism  $\sigma$  is an isomorphism interlacing the representation  $\psi$  from (5.11) and the representation  $f \times i$ , where  $i$  is the trivial representation of the group  $G$  in  $V^*$ . The subspace  $F$  is mapped by  $\sigma$  onto some invariant subspace  $U_F \subseteq V \otimes V^*$ . Since  $f$  is irreducible, now we can apply the theorem 5.3. It yields  $U_F = V \otimes \tilde{W}$ , where  $\tilde{W} \subseteq V^*$  is a subspace in  $V^*$ .

If we assume that  $\tilde{W} \neq V^*$ , then there is some vector  $\mathbf{x} \neq 0$  in  $V$  such that  $\lambda(\mathbf{x}) = 0$  for all  $\lambda \in V^*$ . Applying this fact to  $F$  and taking into account (5.11) and (5.12), we find that this vector  $\mathbf{x} \neq 0$  belongs to the kernel of any operator  $A \in F$ . But the operators  $f(g) \in F$  are non-degenerate, their kernels are zero. This contradiction shows that  $\tilde{W} = V^*$  and  $U_F = V \otimes V^*$ . Due to the isomorphism  $\sigma$  then we derive  $F = \text{End}(V)$ .  $\square$

### § 6. Irreducible representations of the direct product of groups.

The direct product is the simplest construction for building new groups from those already available. Let's recall that the

group  $G_1 \times G_2$  is the set of ordered pairs  $(g_1, g_2)$  with the multiplication rule

$$(g_1, g_2) \cdot (\tilde{g}_1, \tilde{g}_2) = (g_1 \cdot \tilde{g}_1, g_2 \cdot \tilde{g}_2),$$

where  $g_1, \tilde{g}_1 \in G_1$  and  $g_2, \tilde{g}_2 \in G_2$ .

The construction of direct product of groups is in a good agreement with the construction of tensor product of their representations. Let  $(f_1, G_1, V_1)$  and  $(f_2, G_2, V_2)$  are representations of the groups  $G_1$  and  $G_2$  respectively. Let's define a representation of the group  $G = G_1 \times G_2$  in the space  $V_1 \otimes V_2$  by the formula

$$\begin{aligned} f(g)(\mathbf{x} \otimes \mathbf{y}) &= f(g_1, g_2)(\mathbf{x} \otimes \mathbf{y}) = \\ &= (f_1(g_1)\mathbf{x}) \otimes (f_2(g_2)\mathbf{y}). \end{aligned} \tag{6.1}$$

It is easy to verify that the definition (6.1) is correct.

DEFINITION 6.1. The representation  $(f, G_1 \times G_2, V_1 \otimes V_2)$  given by the formula (6.1) is called the *tensor product* of the representations  $(f_1, G_1, V_1)$  and  $(f_2, G_2, V_2)$ . It is denoted  $f = f_1 \otimes f_2$ .

Note that the earlier construction of the tensor product given by the definition 5.1 is embedded into the construction 6.1. Indeed, let's consider the diagonal in the direct product  $G \times G$ :

$$D = \{(g_1, g_2) \in G \times G: g_1 = g_2\}.$$

It is easy to see that  $G \cong G$ . The restriction of the representation (6.1) to the diagonal  $D$  coincides with the representation (5.1), where  $f = f_1$  and  $h = f_2$ .

THEOREM 6.1. *The tensor product  $(f, G_1 \times G_2, V_1 \otimes V_2)$  of two finite-dimensional representations  $(f_1, G_1, V_1)$  and  $(f_2, G_2, V_2)$  in complex vector spaces  $V_1$  and  $V_2$  is irreducible if and only if both multiplicands  $f_1$  and  $f_2$  are irreducible.*

PROOF. Let's begin with proving the necessity in the formulated proposition. Assume that  $(f, G_1 \times G_2, V_1 \otimes V_2)$  is irreducible.

And assume that the irreducibility condition for  $(f_1, G_1, V_1)$  and  $(f_2, G_2, V_2)$  is broken. For the sake of certainty assume that the second representation  $(f_2, G_2, V_2)$  is reducible. Then  $f_2$  has a non-trivial invariant subspace  $\{0\} \subsetneq W_2 \subsetneq V_2$ . But in this case  $V_1 \otimes W_2$  is a non-trivial invariant subspace for  $f = f_1 \otimes f_2$ . This contradicts the irreducibility of the representation  $f$ . The necessity is proved.

Let's prove the sufficiency. Assume that  $(f_1, G_1, V_1)$  and  $(f_2, G_2, V_2)$  are irreducible. In order to prove the irreducibility of  $(f, G_1 \times G_2, V_1 \otimes V_2)$  we use the irreducibility criterion in form of the theorem 3.1. Let's choose an arbitrary vector  $\mathbf{u} \neq 0$  in  $V_1 \times V_2$  and consider its orbit. The vector  $\mathbf{u}$  can be written as

$$\mathbf{u} = \mathbf{x}_1 \otimes \mathbf{y}_1 + \dots + \mathbf{x}_k \otimes \mathbf{y}_k. \tag{6.2}$$

Without loss of generality we can assume that the vectors  $\mathbf{y}_1, \dots, \mathbf{y}_k$  in (6.2) are linearly independent. The expansion (6.2) is not unique. However, if the linearly independent vectors  $\mathbf{y}_1, \dots, \mathbf{y}_k$  are fixed, then the corresponding vectors  $\mathbf{x}_1, \dots, \mathbf{x}_k$  are determined uniquely. Without loss of generality we can assume them to be nonzero.

Now let's apply the theorem 5.4. Let  $A: V_2 \rightarrow V_2$  be a linear operator satisfying the following condition:

$$A\mathbf{y}_1 = y_1, A\mathbf{y}_2 = 0, \dots, A\mathbf{y}_k = 0. \tag{6.3}$$

Since the vectors  $\mathbf{y}_1, \dots, \mathbf{y}_k$  in (6.3) are linearly independent, such an operator  $A$  does exist. Applying the theorem 5.4 to the representation  $f_2$ , we conclude that the operator  $A$  belong to the span of the representation operators, i. e.

$$A = \sum_{i=1}^q \alpha_i f_2(g_i), \text{ where } g_i \in G_2.$$

Let's apply the operator  $1 \otimes A$  to the vector (6.2). This yields

$$(1 \otimes A)\mathbf{u} = \sum_{i=1}^k \mathbf{x}_i \otimes A\mathbf{y}_i = \mathbf{x}_1 \otimes \mathbf{y}_1. \quad (6.4)$$

On the other hand, for the same quantity we get

$$(1 \otimes A)\mathbf{u} = \sum_{i=1}^q \alpha_i (1 \otimes f(g_i))\mathbf{u} = \sum_{i=1}^q \alpha_i f(e_1, g_i)\mathbf{u}. \quad (6.5)$$

Here  $e_1$  is the unit element of the group  $G_1$ . Comparing (6.4) and (6.5), we see that the vector  $\mathbf{x}_1 \otimes \mathbf{y}_1$  belongs to the orbit of the vector  $\mathbf{u}$  from (6.3). Due to the irreducibility of the representation  $f_1$  the orbit of the vector  $\mathbf{x}_1$  spans  $V_1$ . For the similar reasons the orbit of the vector  $\mathbf{y}_1$  spans  $V_2$ . These facts mean that any two vectors  $\mathbf{x} \in V_1$  and  $\mathbf{y} \in V_2$  can be obtained as

$$\begin{aligned} \mathbf{x} &= \sum_{i=1}^r \beta_i f(g_i)\mathbf{x}_1, \quad \text{where } g_i \in G_1; \\ \mathbf{y} &= \sum_{j=1}^s \gamma_j f(g_j)\mathbf{y}_1, \quad \text{where } g_j \in G_2. \end{aligned} \quad (6.6)$$

From (6.6) we immediately derive

$$\mathbf{x} \otimes \mathbf{y} = \sum_{i=1}^r \sum_{j=1}^s \beta_i \gamma_j f(g_i, g_j)(\mathbf{x}_1 \otimes \mathbf{y}_1),$$

where  $(g_i, g_j) \in G_1 \times G_2$ . Hence, an arbitrary vector of the form  $\mathbf{x} \otimes \mathbf{y}$  belongs to the orbit of the vector  $\mathbf{x}_1 \otimes \mathbf{y}_1$  from (6.4), and this vector in turn belongs to the orbit of the vector  $\mathbf{u}$  from (6.2). However, we know that the vectors of the form  $\mathbf{x} \otimes \mathbf{y}$  spans the whole space  $V_1 \otimes V_2$ . As a result we have proved that the orbit of an arbitrary vector  $\mathbf{u} \in V_1 \otimes V_2$  spans the whole space of the representation  $f = f_1 \otimes f_2$ . According to the theorem 3.1, this representation is irreducible. Thus, the theorem 6.1 is proved.  $\square$

**THEOREM 6.2.** *Any finite-dimensional irreducible representation  $\varphi$  of the direct product of two groups  $G_1$  and  $G_2$  in a complex space  $U$  is isomorphic to the tensor product of two irreducible representations  $(f_1, G_1, V_1)$  and  $(f_2, G_2, V_2)$  of the groups  $G_1$  and  $G_2$ .*

Let  $\varphi(g_1, g_2)$  be the representation operators for the representation  $\varphi$  of the group  $G_1 \times G_2$  in the space  $U$ . Then the operators of the form  $\varphi(g_1, e_2)$ , where  $e_2$  is the unit element of the group  $G_2$ , define a representation of the group  $G_1$ . In general case it is reducible. Let  $V_1 \subseteq U$  be some irreducible subspace in  $U$ . Denote

$$\varphi_1(g_1) = \varphi(g_1, e_2), \text{ where } g_1 \in G_1. \quad (6.7)$$

The restrictions of the operators (6.7) to  $V_1$  define some irreducible representation of the group  $G_1$ . We denote them as

$$f_1(g_1) = \varphi_1(g_1) \Big|_{V_1} = \varphi_1(g_1, e_2) \Big|_{V_1}, \text{ where } g_1 \in G_1.$$

By analogy to (6.7) we introduce the following operators defining some representation of the group  $G_2$  in  $U$ :

$$\varphi_2(g_2) = \varphi(e_1, g_2), \text{ where } g_2 \in G_2. \quad (6.8)$$

Then we denote by  $F_2$  the span of the set of all operators (6.8). It is a subspace in the space of the operators  $\text{End}(U)$ :

$$F_2 = \langle \{\varphi_2(g_2) : g_2 \in G_2\} \rangle.$$

The operators from  $F_2$  commute with all operators (6.7) since the operators  $\varphi_2(g_2)$  spanning  $F_2$  commute with  $\varphi_1(g_1)$ . For each operator  $A \in F_2$  we denote by  $\tilde{A}$  the restriction of  $A$  to the subspace  $V_1$ . The operators  $\tilde{A}$  should be treated as the elements of the linear space  $\tilde{F}_2 \subseteq \text{Hom}(V_1, U)$ :

$$\tilde{A}: V_1 \rightarrow U.$$

The operators  $A \in F_2$  and  $\tilde{A} \in \tilde{F}_2$  deserve a special consideration. Let's define a subspace  $V_A = AV_1 = \tilde{A}V_1 = \text{Im } \tilde{A} \subseteq U$ . Since  $A$  commute with operators (6.7), the subspace  $V_A$  is invariant with respect to the operators  $\varphi_1(g_1)$ . Therefore we have a representation of the group  $G_1$  in  $V_A$ :

$$f_A(g_1) = \varphi_1(g_1) \Big|_{V_A} = \varphi(g_1, e_2) \Big|_{V_A}, \quad \text{where } g_1 \in G_1.$$

The mapping  $\tilde{A}: V_1 \rightarrow V_A$  interlaces the representations  $f_1$  and  $f_A$  in  $V_1$  and  $V_A$ . Indeed, we have

$$\tilde{A} \circ f_1(g_1) = A \circ \varphi(g_1) \Big|_{V_1} = \varphi(g_1) \circ A \Big|_{V_1} = f_A(g_1) \circ \tilde{A}.$$

The mapping  $\tilde{A}: V_1 \rightarrow V_A$  is surjective by its definition. The kernel  $\text{Ker } \tilde{A} \subseteq V_1$  of this mapping is invariant with respect to the representation  $f_1$ . Since  $f_1$  is irreducible, we have two mutually excluding options:

$$\begin{aligned} \text{Ker } \tilde{A} = V_1 &\Rightarrow \tilde{A} = 0 \quad \text{and} \quad V_A = \{0\}; \\ \text{Ker } \tilde{A} = \{0\} &\Rightarrow \tilde{A} \text{ is an isomorphism and } f_1 \cong f_A. \end{aligned} \tag{6.9}$$

Let's study the second option in (6.9). Denote  $W = V_1 \cap V_A$ . The operators  $f_1(g_1)$  and  $f_A(g_1)$  upon restricting to  $W$  do coincide. Therefore  $W \subseteq V_1$  is invariant with respect to  $f_1$ . Applying the irreducibility of  $f_1$  again, we get the following two options:

$$\begin{aligned} W = V_1 &\Rightarrow V_A = V_1 \quad \text{and} \quad f_A = f_1; \\ W = \{0\} &\Rightarrow V_A \cap V_1 = \{0\} \quad \text{and} \quad f_A \cong f_1. \end{aligned} \tag{6.10}$$

Combining (6.9) and (6.10), we find

$$\begin{aligned} V_A = \{0\}, \quad f_A = 0, \quad \tilde{A} = 0; \\ V_A = V_1, \quad f_A = f_1, \quad \tilde{A} = \lambda \cdot 1; \\ V_A \cap V_1 = \{0\}, \quad f_A \cong f_1, \quad \tilde{A} \text{ is an isomorphism.} \end{aligned} \tag{6.11}$$



The condition  $\tilde{A} = \lambda \cdot 1$  in the second option of (6.11) follows from Schur's lemma 5.2.

Let  $\mathbf{u}$  be some nonzero vector in  $V_1$ . We fix this vector and consider the subspace  $V_2 \subseteq U$  obtained by applying the operators  $A \in F_2$  upon the vector  $\mathbf{u}$ :

$$V_2 = F_2\mathbf{u} = \{\mathbf{v} \in U : \mathbf{v} = A\mathbf{u} \text{ for some } A \in F_2\}. \quad (6.12)$$

The subspace  $V_2$  is invariant with respect to the operators (6.8). Therefore we have a representation  $(f_2, G_2, V_2)$  of the group  $G_2$ . It is given by the operators

$$f_2(g_2) = \varphi_2(g_2) \Big|_{V_2} = \varphi_1(e_1, g_2) \Big|_{V_2}. \quad (6.13)$$

Due to the definition (6.12) for any vector  $\mathbf{y} \in V_2$  there is a mapping  $\tilde{A} \in \tilde{F}_2$  such that  $\mathbf{y} = \tilde{A}\mathbf{u}$ . Let's prove that such a mapping is uniquely fixed by the vector  $\mathbf{y} \in V_2$ . According to (6.11), we study three possible options.

If  $\mathbf{y} = 0$ , then  $\text{Ker } \tilde{A} \neq 0$ . Due to (6.9) the only operator  $\tilde{A} \in F_2$  satisfying the condition  $\mathbf{y} = \tilde{A}\mathbf{u}$  is the identically zero mapping  $\tilde{A} = 0$ . This case corresponds to the first option in (6.11).

If  $\mathbf{y} \neq 0$  and  $\mathbf{y} \in V_1$ , then from  $\mathbf{y} = \tilde{A}\mathbf{u}$  we derive that  $\mathbf{y} \in V_1 \cap V_A$ . Hence the intersection  $V_1 \cap V_A$  is nonzero and we have the first option in (6.10), which is equivalent to the second option of (6.11). Hence,  $\tilde{A} = \lambda \cdot 1$  and  $\mathbf{y} = \lambda\mathbf{u}$ . The number  $\lambda$  relating two collinear vectors is uniquely fixed by these two vectors. Therefore, the mapping  $\tilde{A} = \lambda \cdot 1$  is also unique.

And finally, the third case, where  $\mathbf{y} \notin V_1$ . Due to (6.11) in this case we have  $V_1 \cap V_A = \{0\}$  and the mapping  $\tilde{A}: V_1 \rightarrow V_A$  is bijective. Assume for a while that the condition  $\mathbf{y} = \tilde{A}\mathbf{u}$  does not fix the mapping  $\tilde{A} \in F_2$  uniquely. Let  $\tilde{A}_1$  and  $\tilde{A}_2$  be two such mappings. Their associated subspaces  $V_{A_1}$  and  $V_{A_2}$  do coincide. Indeed,  $\mathbf{y} \in V_{A_1} \cap V_{A_2} \neq \{0\}$ . Hence,  $V_{A_1} \cap V_{A_2}$  is a non-trivial

invariant subspace for the irreducible representations  $f_{A_1} \cong f_1$  and  $f_{A_2} \cong f_1$ . So,  $V_{A_1} \cap V_{A_2} = V_{A_1} = V_{A_2}$ . Using  $V_{A_1} = V_{A_2}$  and the bijectivity of the mappings

$$\tilde{A}_1: V_1 \rightarrow V_{A_1}, \quad \tilde{A}_2: V_1 \rightarrow V_{A_2},$$

we invert one of them and consider the operator  $\tilde{A}_3 = \tilde{A}_2^{-1} \circ \tilde{A}_1$ . This is a non-degenerate operator in  $V_1$ . It implements the automorphism of the representation  $f_1$ , i.e. it interlaces the operators  $f_1(g_1)$  with themselves:

$$\tilde{A}_3 \circ f_1(g_1) = f_1(g_1) \circ \tilde{A}_3.$$

Using the irreducibility of  $f_1$  and applying Schur's lemma 5.2, we get  $\tilde{A}_3 = \lambda \cdot 1$ . This yields  $\tilde{A}_1 = \lambda \tilde{A}_2$ . Now from the conditions  $\mathbf{y} = \tilde{A}_1 \mathbf{u}$  and  $\mathbf{y} = \tilde{A}_2 \mathbf{u}$  we derive  $\lambda = 1$ . Hence,  $\tilde{A}_2 = \tilde{A}_1$ . Thus, the uniqueness of  $\tilde{A}$  is established.

For the mapping  $\tilde{A} \in \tilde{F}_2$ , which we uniquely determine from the condition  $\mathbf{y} = \tilde{A} \mathbf{u}$ , we use the notation  $\tilde{A} = \tilde{A}(\mathbf{y})$ . The dependence of  $\tilde{A}$  on the vector  $\mathbf{y}$  can be treated as a mapping  $\tilde{A}: V_2 \rightarrow \text{Hom}(V_1, U)$ . It is easy to verify that this mapping is linear. It satisfies the equality

$$\tilde{A}(f_2(g_2)\mathbf{y}) = \varphi_2(g_2) \circ \tilde{A}(\mathbf{y}), \quad (6.14)$$

where the operator  $f_2(g_2)$  is given by (6.13). Let's prove the equality (6.14). Remember that  $\tilde{A}(\mathbf{y})$  is the restriction to  $V_1$  of some operator  $A_1 \in F_2$  such that

$$A_1 \mathbf{u} = \tilde{A}(\mathbf{y}) \mathbf{u} = \mathbf{y}.$$

But the operator  $A_2 = \varphi_2 \circ A_1$  also belongs to  $F_2$  (see the definition of the space  $F_2$  above). For  $A_2$  we derive

$$A_2 \mathbf{u} = \varphi_2(g_2) A_1 \mathbf{u} = \varphi_2(g_2) \mathbf{y} = f_2(g_2) \mathbf{y}.$$

Therefore the restriction of  $A_2$  to  $V_1$  coincides with  $\tilde{A}(f_2(g_2)\mathbf{y})$ . The equality (6.14) is proved.

The next step in proving the theorem 6.2 is to apply the mapping  $\tilde{A}(\mathbf{y})$  for building the isomorphism of the representation  $f = f_1 \otimes f_2$  and the representation  $\varphi$ . But before doing it note that we have no information on whether the representation  $f_2$  in (6.13) is irreducible or not. Fortunately we can assume  $f_2$  to be irreducible due to the following reasons. Let  $\tilde{V}_2 \subseteq V_2$  be some irreducible invariant subspace for the representation (6.13). If  $\mathbf{u} \in \tilde{V}_2$ , then  $\tilde{V}_2 = V_2$ . This fact follows from the theorem 3.1. In the case, where  $\mathbf{u} \notin \tilde{V}_2$ , we choose a nonzero vector  $\tilde{\mathbf{u}}$  and take a mapping  $\tilde{A} \in \tilde{F}_2$  such that  $\tilde{A}\mathbf{u} = \tilde{\mathbf{u}}$ . We have already proved the existence and uniqueness of such a mapping  $\tilde{A} = \tilde{A}(\tilde{\mathbf{u}})$ . In our case  $\tilde{A} \neq 0$ . Therefore, due to (6.11) we see that the mapping  $\tilde{A}$  is bijective, it establishes the isomorphism of representations  $f_1 \cong f_A$ . Because of the isomorphism  $f_1 \cong f_A$  we can replace  $f_1$  by  $f_A$ , which is also irreducible. The latter representation is preferable since its space  $V_A$  comprises the vector  $\tilde{\mathbf{u}}$ . The orbit of the vector  $\tilde{\mathbf{u}}$  spans the irreducible subspace  $\tilde{V}_2$  within the space of the representation  $\varphi_2$ . For this reason we should come back to the beginning of our constructions and assume that  $V_1$  is exactly that irreducible subspace of  $\varphi_1$  which comprises some vector  $\mathbf{u}$  generating an irreducible subspace of the representation  $\varphi_2$ . Just above we have demonstrated that such a choice of the subspace  $V_1$  is possible.

Thus, under a proper choice of the subspace  $V_1$  both representations  $(f_1, G_1, V_1)$  and  $(f_2, G_2, V_2)$  are irreducible. We consider their tensor product  $f = f_1 \otimes f_2$  and then construct the mapping  $\sigma: V_1 \otimes V_2 \rightarrow U$  by means of the following formula:

$$\sigma(\mathbf{x} \otimes \mathbf{y}) = \tilde{A}(\mathbf{y})\mathbf{x}, \quad \text{where } \mathbf{x} \in V_1, \mathbf{y} \in V_2. \quad (6.15)$$

Let's show that the mapping (6.15) is an interlacing mapping for the representations  $(f_1 \otimes f_2, G_1 \times G_2, V_1 \otimes V_2)$  and  $(\varphi, G_1 \times G_2, U)$ .

Indeed, we easily derive

$$\begin{aligned}\varphi(g_1, g_2) \sigma(\mathbf{x} \otimes \mathbf{y}) &= \varphi_1(g_1) \varphi_2(g_2) \tilde{A}(\mathbf{y}) \mathbf{x} = \\ &= \varphi_2(g_2) \tilde{A}(\mathbf{y}) \varphi_1(g_1) \mathbf{x},\end{aligned}\tag{6.16}$$

$$\begin{aligned}\sigma f(g_1, g_2)(\mathbf{x} \otimes \mathbf{y}) &= \sigma((f_1(g_1) \mathbf{x}) \otimes (f_2(g_2) \mathbf{y})) = \\ &= \tilde{A}(f_2(g_2) \mathbf{y}) f_1(g_1) \mathbf{x}.\end{aligned}$$

The values of the right hand sides in two above formulas (6.16) do coincide due to (6.14). Therefore, from (6.16) we extract

$$\varphi(g_1, g_2) \circ \sigma = \sigma \circ f(g_1, g_2).$$

This is exactly the equality (1.1) written for the representations  $f$  and  $\varphi$ . The mapping  $\sigma$  implements an isomorphism of these two representations. Note that

$$\sigma(\mathbf{u} \times \mathbf{u}) = \tilde{A}(\mathbf{u}) \mathbf{u} = \mathbf{u} \neq 0.$$

Therefore  $\sigma \neq 0$ . Now it is sufficient to use the irreducibility of representations  $f = f_1 \otimes f_2$  and  $\varphi$ . Applying Schur's lemma 5.1, we conclude that  $\sigma$  is an isomorphism. The irreducibility of  $f$  is derived from the irreducibility of  $f_1$  and  $f_2$  due to the previous theorem. Thus, the proof of the theorem 6.2 is completed.

## § 7. Unitary representations.

DEFINITION 7.1. A finite-dimensional complex linear vector space  $V$  equipped with a symmetric positive sesquilinear form is called a *Hermitian space*.

Let's recall that a *sesquilinear form* in  $V$  is a complex-valued numeric function  $\varphi(\mathbf{x}, \mathbf{y})$  with two vectorial arguments  $\mathbf{x}, \mathbf{y} \in V$  such that it satisfies the following four conditions:

$$(1) \quad \varphi(\mathbf{x}_1 + \mathbf{x}_2, \mathbf{y}) = \varphi(\mathbf{x}_1, \mathbf{y}) + \varphi(\mathbf{x}_2, \mathbf{y});$$

- (2)  $\varphi(\alpha \mathbf{x}, \mathbf{y}) = \bar{\alpha} \varphi(\mathbf{x}, \mathbf{y});$   
 (3)  $\varphi(\mathbf{x}, \mathbf{y}_1 + \mathbf{y}_2) = \varphi(\mathbf{x}, \mathbf{y}_1) + \varphi(\mathbf{x}, \mathbf{y}_2);$   
 (4)  $\varphi(\mathbf{x}, \alpha \mathbf{y}) = \alpha \varphi(\mathbf{x}, \mathbf{y}).$

The bar sign over  $\alpha$  in the second condition is the complex conjugation sign. The conditions (1)–(4) are usually complemented with the conditions of symmetry and positivity:

- (5)  $\varphi(\mathbf{x}, \mathbf{y}) = \overline{\varphi(\mathbf{y}, \mathbf{x})};$   
 (6)  $\varphi(\mathbf{x}, \mathbf{x}) > 0$  for all  $\mathbf{x} \neq 0$ .

The condition (5) implies that  $\varphi(\mathbf{x}, \mathbf{x})$  is a real number. The condition (6) strengthens condition (5) requiring  $\varphi(\mathbf{x}, \mathbf{x})$  to be a positive number. A form  $\varphi(\mathbf{x}, \mathbf{y})$  is called *non-degenerate* if  $\varphi(\mathbf{x}, \mathbf{y}) = 0$  for all  $\mathbf{y} \in V$  implies  $\mathbf{x} = 0$ . Note that the positivity of a form implies its non-degeneracy.

The symmetric positive form declared in the definition of a Hermitian space is called the *Hermitian scalar product*. For this form we fix the following notation:

$$\varphi(\mathbf{x}, \mathbf{y}) = \langle \mathbf{x} | \mathbf{y} \rangle.$$

Let  $\mathbf{e}_1, \dots, \mathbf{e}_n$  be a basis in a space  $V$ . The quantities  $g_{ij} = \langle \mathbf{e}_i | \mathbf{e}_j \rangle$  compose the Gram matrix of the basis  $\mathbf{e}_1, \dots, \mathbf{e}_n$ . They satisfy the relationship  $g_{ij} = \overline{g_{ji}}$ . It follows from the symmetry of the scalar product.

A basis, the Gram matrix of which is the unit matrix, is called an *orthonormal basis*. Orthonormal bases do exist because each symmetric sesquilinear in a finite-dimensional space is diagonalizable.

**DEFINITION 7.2.** A linear operator  $A: V \rightarrow V$  in a Hermitian space  $V$  is called a *Hermitian operator* if  $\langle \mathbf{x} | A\mathbf{y} \rangle = \langle A\mathbf{x} | \mathbf{y} \rangle$  for any two vectors  $\mathbf{x}$  and  $\mathbf{y}$  in  $V$ .

There is the standard theory of Hermitian operators in finite-dimensional Hermitian spaces. We give basic facts of this theory without proofs for the reader to recall them.

**THEOREM 7.1.** *Hermitian operators of a finite-dimensional Hermitian space are in a one-to-one correspondence with symmetric sesquilinear forms:*

$$\varphi_A(\mathbf{x}, \mathbf{y}) = \langle \mathbf{x} | A\mathbf{y} \rangle. \quad (7.1)$$

*Non-degenerate operators correspond to non-degenerate forms.*

**DEFINITION 7.3.** A Hermitian operator  $A$  is called a *positive operator* if the corresponding form  $\varphi_A$  is positive.

**THEOREM 7.2.** *Each Hermitian operator  $A$  is diagonalizable, its eigenvalues are real numbers, and eigenvectors corresponding to distinct eigenvalues  $\lambda_i \neq \lambda_j$  are perpendicular to each other.*

**THEOREM 7.3.** *An operator  $A$  is a Hermitian operator if and only if its eigenvalues are real numbers and if it diagonalizes in some orthonormal basis.*

The proofs of the theorems 7.1, 7.2, and 7.3 can be found in many standard textbooks on linear algebra. Apart from them, we need one more theorem, which also can be found in some textbooks, but it is less standard.

**THEOREM 7.4.** *Let  $A$  be a diagonalizable operator such that its eigenvalues  $\lambda_1, \dots, \lambda_n$  are real non-negative numbers. Then there is a unique operator  $B$  with eigenvalues  $\mu_i \geq 0$  such that  $B^2 = A$  and  $B$  commutes with any operator  $C$  that commutes with  $A$ . If  $A$  is a Hermitian operator, then the corresponding operator  $B$  is a Hermitian operator too.*

The operator  $B$  declared in the theorem 7.4 is naturally called the *square root* of the operator  $A$ . Let's prove its existence. Let  $\mathbf{e}_1, \dots, \mathbf{e}_n$  be a basis composed by eigenvectors of the operator  $A$  corresponding to its eigenvalues  $\lambda_1, \dots, \lambda_n$ . The operator  $B$  is defined through its action upon basis vectors:

$$B\mathbf{e}_i = \sqrt{\lambda_i} \mathbf{e}_i, \quad i = 1, \dots, n. \quad (7.2)$$

Due to this definition the operator  $B$  is diagonalized in the same basis as the operator  $A$ , its eigenvalues  $\mu_i = \sqrt{\lambda_i}$  are real and non-negative numbers.

Let's study the problem of commuting for the operators  $B$  and  $C$ . If the operator  $A$  commutes with  $C$ , this means that

$$(\lambda_i - \lambda_j) C_j^i = 0, \quad (7.3)$$

where  $C_j^i$  are the matrix elements for the operator  $C$  in the basis  $\mathbf{e}_1, \dots, \mathbf{e}_n$ . The condition (7.3) is equivalent to  $C_j^i = 0$  for all  $\lambda_i \neq \lambda_j$ . But  $\lambda_i \neq \lambda_j$  implies  $\mu_i \neq \mu_j$ . Therefore the operator  $B$  defined by the formula (7.2) commutes with any operators  $C$  that commutes with  $A$ . If the operator  $A$  is a Hermitian operator, then the basis  $\mathbf{e}_1, \dots, \mathbf{e}_n$  can be chosen to be an orthonormal basis. In this case, applying the theorem 7.3, we find that  $B$  is a Hermitian operator too.

Now we need to prove the uniqueness of the operator  $B$  declared in the theorem 7.4. The condition  $C_j^i = 0$  for all  $\lambda_i \neq \lambda_j$ , which follows from  $A \circ C = C \circ A$ , can be formulated in an invariant (basis-free) way.

**PROPOSITION 7.1.** *An operator  $C$  commutes with a diagonalizable operator  $A$  if and only if all eigenspaces of the operator  $A$  are invariant with respect to the operator  $C$ .*

Under the assumptions of the theorem 7.4 let's take  $C = A$  and apply the proposition 7.1 to the operator  $B$ . From  $B \circ A = A \circ B$  in this case we derive that all eigenspaces of the operator  $A$  are invariant under the action of the operator  $B$ . The requirement that  $B$  is diagonalizable now means that both  $A$  and  $B$  can be diagonalized simultaneously in some basis. The conditions  $B^2 = A$  and  $\mu_i \geq 0$  then fix the unique choice of the operator  $B$  defined by the relationships (7.2).

**DEFINITION 7.4.** A linear mapping  $T : V \rightarrow W$  from some Hermitian space  $V$  to another Hermitian space  $W$  is called an

*isometry* if  $\langle T\mathbf{x}|T\mathbf{y} \rangle = \langle \mathbf{x}|\mathbf{y} \rangle$  for all  $\mathbf{x}, \mathbf{y} \in V$ , i. e. if it preserves the scalar product.

Due to the non-degeneracy of the sesquilinear forms determining scalar products in  $V$  and  $W$  each isometry  $T: V \rightarrow W$  is an injective mapping.

DEFINITION 7.5. A linear operator  $T \in \text{End}(V)$  is called a *unitary operator* if it implements an isometry  $T: V \rightarrow V$ .

Unitary operators are non-degenerate. Their determinants and their eigenvalues satisfy the following relationships:

$$|\det T| = 1, \quad |\lambda| = 1.$$

Unitary operators in a Hermitian space  $V$  form the group  $U(V)$ , which is a subgroup in  $\text{Aut}(U)$ . Unitary operators with unit determinant, in turn, form the group  $SU(V) \subsetneq U(V)$ .

DEFINITION 7.6. A representation  $(f, G, V)$  of a group  $G$  in a Hermitian space  $V$  is called a *unitary representation* if all operators of this representation  $f(g)$  are unitary operators.

Unitary representations constitute an important subclass in the class of general representations of groups. First of all this is because unitary representations arise in applications of the theory of representations to the quantum mechanics. The role of the following useful fact is also substantial.

THEOREM 7.5. *Each unitary representation  $(f, G, V)$  is completely reducible.*

Indeed, let  $U \subseteq V$  be an invariant subspace for the operators of the representation  $f$ . In the case of a unitary operator  $f(g)$  the orthogonal complement to an invariant subspace

$$U_{\perp} = \{\mathbf{x} \in V: \langle \mathbf{x}|\mathbf{y} \rangle = 0 \text{ for all } \mathbf{y} \in U\}$$

is also an invariant subspace. The subspaces  $U$  and  $U_{\perp}$  intersect trivially (i. e. at zero vector only), their direct sum coincides with  $V$ . Therefore,  $U_{\perp}$  is a required invariant direct complement for  $U$ . The complete reducibility of  $f$  is shown.



COROLLARY 7.1. *Each representation  $f$  which is equivalent to some unitary representation  $h$  is completely reducible.*

Let  $A: V \rightarrow W$  be an interlacing mapping which implements the isomorphism of  $f$  and  $h$ . Then each invariant subspace  $U$  of  $f$  has the invariant direct complement  $A^{-1}((AU)_{\perp})$ .

Along with the concept of equivalence, in the class of unitary representations we have the concept of *unitary equivalence*.

DEFINITION 7.7. Two unitary representations  $(f, G, V)$  and  $(h, G, W)$  are called unitary equivalent, if there is an isometry  $A: V \rightarrow W$  implementing an isomorphism of them.

The following theorem shows that despite the difference in definitions the concepts of equivalence and unitary equivalence do coincide.

THEOREM 7.6. *If two unitary representations  $f$  and  $h$  are equivalent, then they are unitary equivalent.*

In order to prove this theorem we need an auxiliary fact which is formulated as a lemma.

LEMMA 7.1. *Let  $A: V \rightarrow W$  be a bijective linear mapping from a Hermitian space  $V$  to another Hermitian space  $W$ . Then it can be expanded as a composition  $A = T \circ B$ , where  $T: V \rightarrow W$  is an isometry and  $B$  is a positive Hermitian operator in  $V$ .*

PROOF OF THE LEMMA. Let's consider the following sesquilinear form in the space  $V$ :

$$\varphi(\mathbf{x}, \mathbf{y}) = \langle A\mathbf{x} | A\mathbf{y} \rangle. \quad (7.4)$$

It is easy to see that the form (7.4) is symmetric and positive. We apply the theorem 7.1 in order to define a Hermitian positive operator  $D$  in  $V$ . The associated sesquilinear form (7.1) of this operator coincides with (7.4). This condition yields

$$\langle \mathbf{x} | D\mathbf{y} \rangle = \langle A\mathbf{x} | A\mathbf{y} \rangle. \quad (7.5)$$

Using the operator  $D$  and applying the theorem 7.4 to it, we construct a positive Hermitian operator  $B$  being the square root of  $D$ , i.e.  $B^2 = D$ . Now let's consider a mapping  $T: V \rightarrow W$  defined as the composition  $T = A \circ B^{-1}$ . Note that  $B$  is non-degenerate since it is positive. Therefore it is invertible. The rest is to show that  $T$  is an isometry. Indeed, we have

$$\langle T\mathbf{x}|T\mathbf{y} \rangle = \langle A \circ B^{-1}\mathbf{x}|A \circ B^{-1}\mathbf{y} \rangle = \langle B^{-1}\mathbf{x}|D \circ B^{-1}\mathbf{y} \rangle. \quad (7.6)$$

The last equality in the chain (7.6) is provided by (7.5). The further calculations are obvious:

$$\langle B^{-1}\mathbf{x}|D \circ B^{-1}\mathbf{y} \rangle = \langle B^{-1}\mathbf{x}|B\mathbf{y} \rangle = \langle B \circ B^{-1}\mathbf{x}|\mathbf{y} \rangle = \langle \mathbf{x}|\mathbf{y} \rangle.$$

Combining this equality with (7.6), we get  $\langle T\mathbf{x}|T\mathbf{y} \rangle = \langle \mathbf{x}|\mathbf{y} \rangle$  for any two vectors  $\mathbf{x}, \mathbf{y} \in V$ . Hence,  $T$  is an isometry. The lemma is proved.  $\square$

**PROOF OF THE THEOREM 7.6.** The mapping  $A = T \circ B$  in this case implements an isomorphism of two unitary representations  $f$  and  $h$ . Therefore, we have

$$T \circ B \circ f(g) = h(g) \circ T \circ B \quad \text{for all } g \in G. \quad (7.7)$$

Let's show that the operator  $B$  commutes with  $f(g)$ . For this purpose we show that  $D = B^2$  commutes with  $f(g)$ :

$$\langle \mathbf{x}|f(g)B^2\mathbf{y} \rangle = \langle f(g^{-1})\mathbf{x}|BB\mathbf{y} \rangle = \langle Bf(g^{-1})\mathbf{x}|B\mathbf{y} \rangle.$$

Here we used the facts that  $f(g)$  is a unitary operator and  $B$  is a Hermitian operator. Let's continue our calculations using the isometry of the mapping  $T$ :

$$\langle Bf(g^{-1})\mathbf{x}|B\mathbf{y} \rangle = \langle Bf(g^{-1})\mathbf{x}|T\mathbf{B}\mathbf{y} \rangle = \langle TBf(g^{-1})\mathbf{x}|B\mathbf{y} \rangle.$$

But  $T \circ B \circ f(g^{-1}) = h(g^{-1}) \circ T \circ B$ . This fact follows from (7.7). Taking into account this equality and taking into account that  $h(g)$  is a unitary operator, we get

$$\langle TBf(g^{-1})\mathbf{x}|B\mathbf{y} \rangle = \langle h(g^{-1})TB\mathbf{x}|B\mathbf{y} \rangle = \langle TB\mathbf{x}|h(g)TB\mathbf{y} \rangle.$$

Now let's use again the relationship (7.7) written as  $h(g) \circ T \circ B = T \circ B \circ h(g)$ . Then we take into account the isometry of  $T$ :

$$\langle TB\mathbf{x}|h(g)TB\mathbf{y}\rangle = \langle TB\mathbf{x}|TBf(g)\mathbf{y}\rangle = \langle T^{-1}TB\mathbf{x}|Bf(g)\mathbf{y}\rangle.$$

In order to complete this series of calculations, we remember that  $B$  is a Hermitian operator:

$$\begin{aligned} \langle T^{-1}TB\mathbf{x}|Bf(g)\mathbf{y}\rangle &= \langle B\mathbf{x}|Bf(g)\mathbf{y}\rangle = \\ &= \langle \mathbf{x}|BBf(g)\mathbf{y}\rangle = \langle \mathbf{x}|B^2f(g)\mathbf{y}\rangle. \end{aligned}$$

As a result we have got  $\langle \mathbf{x}|f(g)B^2\mathbf{y}\rangle = \langle \mathbf{x}|B^2f(g)\mathbf{y}\rangle$ . Since  $\mathbf{x}$  and  $\mathbf{y}$  are arbitrary two vectors, we conclude that the operators  $f(g)$  commute with the operator  $D = B^2$ . But the positive Hermitian operator  $B$  is a square root of the positive Hermitian operator  $D$ . Therefore  $B$  commutes with all operators that commute with the operator  $D$  (see theorem 7.4). As a result we get  $f(g) \circ B = B \circ f(g)$ . Substituting this equality into (7.7), we derive  $T \circ f(g) \circ B = h(g) \circ T \circ B$ . Canceling the non-degenerate operator  $B$ , we find

$$T \circ f(g) = h(g) \circ T \quad \text{for all } g \in G. \quad (7.8)$$

The equality (7.8) means that the isometric mapping  $T$  implements an isomorphism of the unitary representations  $f$  and  $h$ . Hence, the representations  $f$  and  $h$  are unitary equivalent. Thus, the theorem 7.6 is proved.  $\square$

## CHAPTER II

# REPRESENTATIONS OF FINITE GROUPS

### § 1. Regular representations of finite groups.

Let  $G$  be a finite group and  $N = |G|$  be the number of elements in this group. Let's consider the set of complex-valued numeric functions on  $G$ . We denote it by  $L_2(G)$ . It is clear that  $L_2(G)$  is a complex linear vector space of the dimension  $\dim(L_2(G)) = N$ . Let's equip  $L_2(G)$  with the structure of a Hermitian space. For this purpose we consider the scalar product of two functions  $u(g)$  and  $v(g)$  given by the formula

$$\langle u|v \rangle = \frac{1}{N} \sum_{g \in G} \overline{u(g)} v(g). \quad (1.1)$$

Now we define an action of the group  $G$  in  $L_2(G)$  by defining linear operators  $R(g): L_2(G) \rightarrow L_2(G)$ . Let's set

$$R(g)v(a) = v(ag) \quad \text{for all } a, g \in G \text{ and } v \in L_2(G). \quad (1.2)$$

The operators  $R(g)$  act upon functions of  $L_2(g)$  by means of right shifts of their arguments. It is easy to verify that these operators satisfy the following relationship:

$$R(g_1) \circ R(g_2) = R(g_1 g_2).$$

Hence, the operators  $R(g)$ , which act according to (1.2), form a representation of the group  $G$  in the space  $L_2(G)$ . This

representation is called the *right regular representation* of the group  $G$ .

Along with the right regular representation there is the *left regular representation*  $(L, G, L_2(G))$  of the group  $G$ . Its operators are defined as follows:

$$L(g)v(a) = v(g^{-1}a) \text{ for all } a, g \in G \text{ and } v \in L_2(G). \quad (1.3)$$

**THEOREM 1.1.** *The right regular representation defined by the formula (1.2) and the left regular representation defined by the formula (1.3) are unitary representations with respect to the Hermitian structure given by the scalar product (1.1).*

**PROOF.** Let's verify by means of direct calculations that  $R(g)$  and  $L(g)$  are unitary operators. Assume that  $u$  and  $v$  are two arbitrary functions from  $L_2(G)$ . Then we have

$$\begin{aligned} \langle R(g)u | R(g)v \rangle &= \frac{1}{N} \sum_{a \in G} \overline{u(ag)} v(ag) = \\ &= \frac{1}{N} \sum_{b \in G} \overline{u(b)} v(b) = \langle u | v \rangle; \\ \langle L(g)u | L(g)v \rangle &= \frac{1}{N} \sum_{a \in G} \overline{u(g^{-1}a)} v(g^{-1}a) = \\ &= \frac{1}{N} \sum_{b \in G} \overline{u(b)} v(b) = \langle u | v \rangle; \end{aligned}$$

Here we used the fact that the right shift  $a \mapsto b = ag$  and the left shift  $a \mapsto g^{-1}a$  are two bijective mappings of the group  $G$  onto itself.  $\square$

**THEOREM 1.2.** *The right regular representation and the left regular representation are unitary equivalent to each other.*

**PROOF.** In order to prove the theorem it is necessary to construct the unitary operator  $A: L_2(G) \rightarrow L_2(G)$  interlacing

the representations  $(R, G, L_2(G))$  and  $(L, G, L_2(G))$ . We define this operator as follows:

$$Av(g) = v(g^{-1}) \text{ for all } g \in G \text{ and } v \in L_2(G). \quad (1.4)$$

The fact that  $A$  is a unitary operator with respect to the Hermitian structure (1.1) is shown by the following calculations:

$$\langle Au|Av \rangle = \frac{1}{N} \sum_{a \in G} \overline{u(a^{-1})} v(a^{-1}) = \frac{1}{N} \sum_{b \in G} \overline{u(b)} v(b) = \langle u|v \rangle.$$

In carrying out these calculations we used the fact that the inversion operation  $a \mapsto b = a^{-1}$  is a bijective mapping of the group  $G$  onto itself.

Now let's show that the operator  $A$  introduced by means of the formula (1.4) interlaces right and left regular representations. Let  $v$  be some arbitrary function from  $L_2(G)$ . Assume that  $u = L(g)v$  and  $w = Av$ . Then we have

$$\begin{aligned} AL(g)v(a) &= Au(a) = u(a^{-1}) = v(g^{-1}a^{-1}) = \\ &= v((ag)^{-1}) = w(ag) = R(g)w(a) = R(g)Av(a). \end{aligned}$$

Since  $v$  is an arbitrary function from  $L_2(G)$ , this sequence of calculations shows that  $A \circ L(g) = R(g) \circ A$ . The proof is over.  $\square$

## § 2. Invariant averaging over a finite group.

In previous section we have noted that the idea of considering numeric functions on a finite group is fruitful enough. The finiteness of a group  $G$  provides the opportunity to define the operation of invariant averaging for such functions. For an arbitrary function  $v \in L_2(G)$  we denote by  $M[v]$  the number determined by the following relationship:

$$M[v] = \frac{1}{N} \sum_{g \in G} v(g), \text{ where } N = |G|. \quad (2.1)$$

We used the symbol  $M$  for denoting the operation of invariant averaging (2.1) since it is analogous to the mathematical expectation or the mean value in the theory of probability.

Note that the operation of invariant averaging (2.1) can be applied not only to numeric functions, but to vector-valued, operator-valued, and matrix-valued functions on a group  $G$ . In order to apply this operation to a function it should take its values in some linear vector space. Then the result of averaging it  $M[v]$  is an element of the same linear vector space. The operation of invariant averaging satisfies the following obvious conditions of linearity:

- (1)  $M[u + v] = M[u] + M[v]$ ;
- (2)  $M[\alpha u] = \alpha M[u]$ , where  $\alpha$  is a number.

The invariance of the averaging (2.1) reveals in the form of the following relationships:

- (3)  $M[R(g)u] = M[u]$ , the invariance with respect to right shifts;
- (4)  $M[L(g)u] = M[u]$ , the invariance with respect to left shifts;
- (5)  $M[Au] = M[u]$ , the invariance with respect to the inversion.

The proof of the properties (3)–(5) is reduced to verifying the following relationships:

$$M[R(g)u] = \frac{1}{N} \sum_{a \in G} u(ag) = \frac{1}{N} \sum_{b \in G} u(b) = M[u],$$

$$M[L(g)u] = \frac{1}{N} \sum_{a \in G} u(g^{-1}a) = \frac{1}{N} \sum_{b \in G} u(b) = M[u],$$

$$M[Au] = \frac{1}{N} \sum_{a \in G} u(a^{-1}) = \frac{1}{N} \sum_{b \in G} u(b) = M[u].$$

Remember that the inversion operator  $A$  in the property (5) above is defined by the relationship (1.4).

If  $u$  is a vector-valued function with the values in a linear vector space  $V$ , then the properties (1)–(5) can be complemented with one more property:

- (6)  $M[Bu] = BM[u]$ , where  $B$  is an arbitrary linear mapping with the domain  $V$ .

The relationship like (6) is fulfilled for operator-valued functions with the values in  $\text{End}(V)$ :

- (6)  $M[B \circ u] = B \circ M[u]$ , where  $B$  is an arbitrary linear mapping with the domain  $V$ .

Moreover, for such functions with values in  $\text{End}(V)$  we can write the following two additional properties:

- (7)  $\text{tr } M[u] = M[\text{tr } u]$ ;  
 (8)  $M[B \circ u] = B \circ M[u]$ , where  $B$  is an arbitrary linear mapping with the domain  $V$ .

The operation of invariant averaging (2.1) plays an important role in the theory of representations of finite groups. As the first example of its usage we prove the following fact.

**THEOREM 2.1.** *Each finite-dimensional representation of a finite group is equivalent to some unitary representation of it.*

Let  $(f, G, V)$  be some finite-dimensional representation of a finite group  $G$ . Generally speaking, in order to prove the theorem we should construct a unitary representation  $(h, G, W)$  of the same group in some Hermitian space  $W$  and find a linear mapping  $A: V \rightarrow W$  being an isomorphism of representations  $f$  and  $h$ . Assume for a while that we managed to do it. Then we have the following relationships:

$$A \circ f(g) = h(g) \circ A, \quad \langle h(g)\mathbf{u} | h(g)\mathbf{v} \rangle = \langle \mathbf{u} | \mathbf{v} \rangle.$$

The space  $V$  is not equipped with its own scalar product. However, we equip it with a scalar product as follows:

$$\langle \mathbf{u} | \mathbf{v} \rangle = \langle A\mathbf{u} | A\mathbf{v} \rangle. \quad (2.2)$$



All properties of a scalar product for the sesquilinear form (2.2) are verified immediately. The positivity is present because  $A$  is a bijective mapping and  $\text{Ker } A = \{0\}$ . The representation  $f$  appears to be a unitary representation with respect to the Hermitian scalar product (2.2):

$$\begin{aligned} \langle f(g)\mathbf{u}|f(g)\mathbf{v} \rangle &= \langle Af(g)\mathbf{u}|Af(g)\mathbf{v} \rangle = \\ &= \langle h(g)A\mathbf{u}|h(g)A\mathbf{v} \rangle = \langle A\mathbf{u}|A\mathbf{v} \rangle = \langle \mathbf{u}|\mathbf{v} \rangle, \end{aligned}$$

while  $A$  establishes the unitary equivalence for  $f$  and  $h$ . These considerations show that in order to prove the theorem 2.1 there is no need to construct a separate unitary representation  $h, G, W$  and find an isomorphism  $A$ . It is sufficient to define a proper scalar product in  $V$  such that  $f$  is a unitary representation with respect to it.

Let  $\langle\langle f(g)\mathbf{u}|f(g)\mathbf{v} \rangle\rangle$  be some arbitrary scalar product in  $V$ . For instance, it can be defined using the coordinates of vectors  $\mathbf{u}$  and  $\mathbf{v}$  in some fixed basis:

$$\langle\langle f(g)\mathbf{u}|f(g)\mathbf{v} \rangle\rangle = \sum_{i=1}^n \overline{u^i} v^i.$$

The operators  $f(g)$  should not be unitary operators with respect to such a scalar product. So, we need to improve it. Let's define another scalar product in  $V$  by means of the operation of invariant averaging:

$$\langle \mathbf{u}|\mathbf{v} \rangle = M[\langle\langle f(g)\mathbf{u}|f(g)\mathbf{v} \rangle\rangle] = \frac{1}{N} \sum_{g \in G} \langle\langle f(g)\mathbf{u}|f(g)\mathbf{v} \rangle\rangle. \quad (2.3)$$

It is easy to see that the form (2.3) is sesquilinear and symmetric. It is also a positive form:

$$\langle \mathbf{u}|\mathbf{u} \rangle = \sum_{g \in G} \frac{\langle\langle f(g)\mathbf{u}|f(g)\mathbf{u} \rangle\rangle}{N} = \sum_{g \in G} \frac{\|f(g)\mathbf{u}\|^2}{N} > 0 \text{ for all } \mathbf{u} \neq 0.$$

The operators  $f(g)$  are unitary operators with respect to the scalar product (2.3). This fact follows from the property (3) of the invariant averaging. Indeed, we have

$$\begin{aligned} \langle f(g)\mathbf{u}|f(g)\mathbf{v} \rangle &= \sum_{a \in G} \frac{\langle\langle f(a)f(g)\mathbf{u}|f(a)f(g)\mathbf{v} \rangle\rangle}{N} = \\ &= \sum_{a \in G} \frac{\langle\langle f(ag)\mathbf{u}|f(ag)\mathbf{v} \rangle\rangle}{N} = \sum_{b \in G} \frac{\langle\langle f(b)\mathbf{u}|f(b)\mathbf{v} \rangle\rangle}{N} = \langle \mathbf{u} | \mathbf{v} \rangle. \end{aligned}$$

The above considerations prove that each finite-dimensional representation of a finite group can be transformed to a unitary representation by means of the proper choice (2.3) of a scalar product. Thus, the theorem 2.1 is proved.

As an immediate corollary of the theorem 2.1 we get the following important proposition concerning finite dimensional representations of finite groups.

**THEOREM 2.2.** *Each finite-dimensional representation of a finite group is completely reducible.*

The proof of this theorem is based on the theorem 7.5 from Chapter I. This theorem says that each unitary representation is completely reducible. As for the finite-dimensional representations of finite groups, we have proved their equivalence to some unitary representations.

### § 3. Characters of group representations.

Let  $(f, G, V)$  be some finite-dimensional representation of a group  $G$ . Each such representation is associated with the numeric function  $\chi_f$  on the group  $G$  defined through the traces of representation operators:

$$\chi_f(g) = \text{tr } f(g). \quad (3.1)$$

The numeric function  $\chi_f(g)$  on  $G$  introduced by the formula (3.1) is called the *character* of the representation  $f$ .

**THEOREM 3.1.** *The characters of finite-dimensional representations possess the following properties:*

- (1) *the characters of two equivalent representations do coincide;*
- (2) *a character is constant within each conjugacy class;*
- (3) *if  $f$  is a unitary representation, then  $\chi_f(g^{-1}) = \overline{\chi_f(g)}$ ;*
- (4) *the character of the direct sum of representations is equal to the sum of characters of separate direct summands;*
- (5) *the character of the tensor product of representations is equal to the product of characters of its multiplicands.*

Let's begin with proving the first item of the theorem. Assume that we have two equivalent representation  $(f, G, V)$  and  $(h, G, W)$  and let  $A: V \rightarrow W$  be an isomorphism of these representations. Let's choose a basis  $\mathbf{e}_1, \dots, \mathbf{e}_n$  in  $V$ . Then the vectors  $\tilde{\mathbf{e}}_1 = A\mathbf{e}_1, \dots, \tilde{\mathbf{e}}_n = A\mathbf{e}_n$  constitute a basis in the space  $W$ . Let's calculate the matrices of the operators  $f(g)$  and  $h(g)$  in these bases. They are defined by the following relationships:

$$f(g)\mathbf{e}_i = \sum_{j=1}^n F_i^j(g)\mathbf{e}_j, \quad h(g)\tilde{\mathbf{e}}_i = \sum_{j=1}^n H_i^j(g)\tilde{\mathbf{e}}_j. \quad (3.2)$$

Let's substitute  $\tilde{\mathbf{e}}_i = A\mathbf{e}_i$  and  $\tilde{\mathbf{e}}_j = A\mathbf{e}_j$  into the second formula (3.2) and take into account the relationship  $A \circ f(g) = h(g) \circ A$ . As a result we obtain

$$Af(g)\mathbf{e}_i = \sum_{j=1}^n H_i^j(g)A\mathbf{e}_j. \quad (3.3)$$

The mapping  $A$  is a bijective mapping. Therefore, we can cancel it in (3.3). Upon canceling  $A$ , we compare (3.3) with the first formula (3.2). This comparison yields  $H_i^j(g) = F_i^j(g)$ , i. e. the matrices of the operators  $f(g)$  and  $h(g)$  do coincide. Hence,  $\text{tr } f(g) = \text{tr } h(g)$  and  $\chi_f(g) = \chi_h(g)$ . The first item in the theorem 3.1 is proved.

Assume that  $g$  and  $\tilde{g}$  are two elements of the same conjugacy class in  $G$ . Then  $\tilde{g} = a g a^{-1}$  for some  $a \in G$ . Therefore, we get

$$f(\tilde{g}) = f(a g a^{-1}) = f(a) \circ f(g) \circ f(a)^{-1}.$$

Now it is sufficient to apply the formula  $\text{tr}(B \circ A \circ B^{-1}) = \text{tr}(A)$  setting  $A = f(g)$  and  $B = f(a)$  in it. The second item in the theorem 3.1 is proved.

In order to prove the third item we consider a unitary representation  $(f, G, V)$  and choose some orthonormal basis in  $V$ . The condition that  $f$  is unitary is written as  $\langle f(g)\mathbf{u}|\mathbf{v} \rangle = \langle \mathbf{u}|f(g)^{-1}\mathbf{v} \rangle$ . Upon substituting  $\mathbf{u} = \mathbf{e}_i$  and  $\mathbf{v} = \mathbf{e}_j$  we take into account the first formula (3.2). As a result we get

$$\overline{F_i^j(g)} = F_j^i(g^{-1}). \quad (3.4)$$

The relationship (3.4) means that the matrices  $\overline{F(g)}$  and  $F(g^{-1})$  are obtained from each other by transposing. The traces of any two such matrices do coincide. The third item is proved.

The fourth item is trivial. Let  $f = f_1 \oplus f_2$  and  $V = V_1 \oplus V_2$ . We choose a basis in  $V$  composed by two bases in  $V_1$  and  $V_2$  respectively. The matrix of the operator  $f(g)$  in such a basis is a blockwise diagonal matrix, the diagonal blocks of it coincides with the matrices of the operators  $f_1(g)$  and  $f_2(g)$ . Therefore,  $\text{tr } f(g) = \text{tr } f_1(g) + \text{tr } f_2(g)$ .

Now let's proceed to the fifth item of the theorem. Let's denote  $\varphi = f \otimes h$  and  $V = U \otimes W$ . We choose some basis  $\mathbf{e}_1, \dots, \mathbf{e}_n$  in  $U$  and some basis  $\tilde{\mathbf{e}}_1, \dots, \tilde{\mathbf{e}}_m$  in  $W$ . The matrices of the operators  $f(g)$  and  $h(g)$  are determined by the relationships

$$f(g)\mathbf{e}_i = \sum_{j=1}^n F_i^j(g)\mathbf{e}_j, \quad h(g)\tilde{\mathbf{e}}_q = \sum_{p=1}^m H_q^p(g)\tilde{\mathbf{e}}_p, \quad (3.5)$$

which are analogous to (3.2). The vectors  $\mathbf{E}_{iq} = \mathbf{e}_i \otimes \tilde{\mathbf{e}}_q$  constitute a basis in the tensor product  $V = U \otimes W$ . The vectors of this

basis are enumerated by two indices, therefore the matrix of the operator  $\varphi(g)$  in this basis is represented by a four-index array. It is determined by the following relationships:

$$\varphi(g)\mathbf{E}_{iq} = \sum_{j=1}^n \sum_{p=1}^m \Phi_{iq}^{jp}(g) \mathbf{E}_{jp}. \quad (3.6)$$

The action of the operator  $\varphi(g)$  upon the basis vectors  $\mathbf{E}_{iq} = \mathbf{e}_i \otimes \tilde{\mathbf{e}}_q$  is determined by the formula (5.1) from Chapter I:

$$\varphi(g)(\mathbf{e}_i \otimes \tilde{\mathbf{e}}_q) = (f(g)\mathbf{e}_i) \otimes (h(g)\tilde{\mathbf{e}}_q). \quad (3.7)$$

Combining (3.7) with the formulas (3.5), we find

$$\varphi(g)\mathbf{E}_{iq} = \sum_{j=1}^n \sum_{p=1}^m F_i^j(g) H_q^p(g) \mathbf{E}_{jp}. \quad (3.8)$$

Now, comparing the relationships (3.6) and (3.8), we determine the matrix components for the operator  $\varphi(g)$ :

$$\Phi_{iq}^{jp}(g) = F_i^j(g) H_q^p(g). \quad (3.9)$$

The rest is to calculate the trace of the operator  $\varphi(g)$  as the trace of a matrix array (3.9):

$$\begin{aligned} \operatorname{tr} \varphi(g) &= \sum_{i=1}^n \sum_{q=1}^m \Phi_{iq}^{iq}(g) = \sum_{i=1}^n \sum_{q=1}^m F_i^i(g) H_q^q(g) = \\ &= \left( \sum_{i=1}^n F_i^i(g) \right) \left( \sum_{q=1}^m H_q^q(g) \right) = \operatorname{tr} f(g) \operatorname{tr} h(g). \end{aligned}$$

This relationship completes the proof of the fifth item in the theorem 3.1 and the proof of the theorem in whole.

In the end of this section one should remark that the properties of representation character considered above are valid for finite-dimensional representations of arbitrary groups, not for finite groups only.

### § 4. Orthogonality relationships.

Let  $(f, G, V)$  and  $(h, G, W)$  are two complex finite-dimensional representations of a finite group  $G$ . We choose some linear mapping  $B: V \rightarrow W$  and, using it, we define a function  $\varphi_B(g)$  on  $G$  with the values in  $\text{Hom}(V, W)$ . Let's set

$$\varphi_B(g) = h(g) \circ B \circ f(g^{-1}).$$

The result of invariant averaging of  $\varphi_B(g)$  over the group  $G$  is some element  $C \in \text{Hom}(V, W)$ :

$$C = M[\varphi_B(g)] = \frac{1}{N} \sum_{a \in G} h(a) \circ B \circ f(a^{-1}). \quad (4.1)$$

It is easy to verify that the mapping  $C: V \rightarrow W$  is a homomorphism of the representations  $f$  and  $h$ . Indeed, we have

$$\begin{aligned} C \circ f(g) &= M[\varphi_B(g)] \circ f(g) = \frac{1}{N} \sum_{a \in G} h(a) \circ B \circ f(a^{-1}) \circ f(g) = \\ &= \frac{1}{N} \sum_{a \in G} h(a) \circ B \circ f(a^{-1}g) = \frac{1}{N} \sum_{b \in G} h(gb) \circ B \circ f(b^{-1}) = \\ &= \frac{1}{N} \sum_{b \in G} h(g) \circ h(b) \circ B \circ f(b^{-1}) = h(g) \circ M[\varphi_B(g)] = h(g) \circ C. \end{aligned}$$

Assume that the representations  $(f, G, V)$  and  $(h, G, W)$  are irreducible. If they are not equivalent, applying Schur's lemma 5.1, we get  $C = 0$ .

Let's study the case  $f \cong h$ . For each pair of equivalent irreducible representations we fix some bijective mapping  $A_{fh}: V \rightarrow W$  implementing an isomorphism of these representations. Then the following lemma determines the structure of the mapping  $C$  in (4.1).

LEMMA 4.1. *A homomorphism  $C: V \rightarrow W$  of two equivalent irreducible finite-dimensional complex representations  $(f, G, V)$  and  $(h, G, W)$  is fixed up to a numeric factor, i. e.  $C = \lambda A_{fh}$ .*

PROOF. Let's consider the operator  $A = A_{fh}^{-1} \circ C$  in the space  $V$ . Being the composition of two homomorphisms, this operator implements an isomorphism of  $f$  with itself. Therefore we have  $A \circ f(g) = f(g) \circ A$  for all  $f(g)$ . Applying Schur's lemma 5.2, we get  $A = \lambda \cdot 1$ . Hence,  $C = \lambda A_{fh}$ . The lemma is proved.  $\square$

In order to calculate the numeric factor  $\lambda$  we use the trace of the operator  $A$ , which is its numeric invariant:

$$\lambda = \frac{\text{tr } A}{\text{tr } 1} = \frac{\text{tr } A}{\dim V} = \frac{1}{\dim V} \text{tr}(A_{fh}^{-1} \circ C).$$

Let's substitute the expression (4.1) for  $C$  into this formula:

$$\begin{aligned} \lambda &= \frac{1}{N \dim V} \sum_{a \in G} \text{tr}(A_{fh}^{-1} \circ h(a) \circ B \circ f(a^{-1})) = \\ &= \frac{1}{N \dim V} \sum_{a \in G} \text{tr}(f(a) \circ A_{fh}^{-1} \circ B \circ f(a^{-1})) = \frac{\text{tr}(A_{fh}^{-1} \circ B)}{\dim V}. \end{aligned}$$

Here again we used the formula  $\text{tr}(F \circ D \circ F^{-1}) = \text{tr}(D)$  with  $F = f(a)$  and  $D = A_{fh}^{-1} \circ B$ . The result of calculating the parameter  $\lambda$  enables us to formula the following proposition.

THEOREM 4.1. *For arbitrary two irreducible finite-dimensional complex representations  $(f, G, V)$  and  $(h, G, W)$  of a finite group  $G$  the relationship*

$$\sum_{a \in G} \frac{h(a) \circ B \circ f(a^{-1})}{N} = \begin{cases} 0 & \text{for } f \not\cong h; \\ \frac{\text{tr}(A_{fh}^{-1} \circ B)}{\dim V} & \text{for } f \cong h \end{cases} \quad (4.2)$$

is fulfilled. It is valid for an arbitrary choice of a linear mapping  $B: V \rightarrow W$  in  $\text{Hom}(V, W)$ .

The relationship (4.2) is the basic orthogonality relationship in the theory of representations of finite groups. Let's consider the matrix form of this relationship. Assume that the bases  $\mathbf{e}_1, \dots, \mathbf{e}_n$  and  $\tilde{\mathbf{e}}_1, \dots, \tilde{\mathbf{e}}_m$  in the spaces  $V$  and  $W$  are chosen. They determine the matrices  $F_i^p(a)$  and  $H_q^j(a)$  for the operators  $f(a)$  and  $h(a)$  respectively. They also determine the matrix  $B_p^q$  for the mapping  $B \in \text{Hom}(V, W)$ . In the case  $f \not\cong h$  the bases in  $V$  and  $W$  are not related to each other. In the case  $f \cong h$  it is convenient to choose the first basis arbitrarily and then define the other by means of the relationship

$$\tilde{\mathbf{e}}_i = A_{fh} \mathbf{e}_i, \quad i = 1, \dots, n. \quad (4.3)$$

Under such a choice of bases the mapping  $A_{fh}$  is represented by the unit matrix, while the matrices of the operators  $f(a)$  and  $h(a)$  do coincide:  $F_i^p(a) = H_i^p(a)$ . As for the mapping  $B$ , we choose it so that the only nonzero element in its matrix  $B_p^q = 1$  is placed in the crossing of the  $q$ -th row and the  $p$ -th column. If all these provisions are made, then the orthogonality relationship (4.2) is rewritten as follows:

$$\frac{1}{N} \sum_{a \in G} H_q^j(a) F_i^p(a^{-1}) = \begin{cases} 0 & \text{for } f \not\cong h, \\ \frac{\delta_i^j \delta_q^p}{n} & \text{for } f \cong h. \end{cases} \quad (4.4)$$

Assume that the representations  $f$  and  $h$  are unitary ones. Above we have already proved that any finite-dimensional complex representation of a finite group can be replaced by some unitary representation equivalent to it. And if  $f$  and  $h$  are equivalent, then they are unitary equivalent as well. For this reason we can assume that the mapping  $A_{fh}$  is an isometry, while the bases  $\mathbf{e}_1, \dots, \mathbf{e}_n$  and  $\tilde{\mathbf{e}}_1, \dots, \tilde{\mathbf{e}}_m$  are orthonormal bases. Then



the relationship (4.4) is rewritten as follows:

$$\frac{1}{N} \sum_{a \in G} H_q^j(a) \overline{F_p^i(a)} = \begin{cases} 0 & \text{for } f \not\cong h, \\ \frac{\delta_i^j \delta_q^p}{n} & \text{for } f \cong h. \end{cases} \quad (4.5)$$

Note that the equality (4.3) is compatible with the orthonormality of the bases  $\mathbf{e}_1, \dots, \mathbf{e}_n$  and  $\tilde{\mathbf{e}}_1, \dots, \tilde{\mathbf{e}}_m$  since  $A_{fh}$  is an isometry. In writing (4.5) we used the relationship

$$F_i^p(a^{-1}) = \overline{F_p^i(a)}$$

because the matrices of unitary operators  $f(a)$  in an orthonormal basis are unitary matrices.

Let's set  $q = j$  and  $p = i$  in the formula (4.5) and then sum over the indices  $i$  and  $j$ . As a result we derive from (4.5) the following relationship for the characters of irreducible representations  $f$  and  $h$  of a finite group:

$$\frac{1}{N} \sum_{a \in G} \text{tr}(h(a)) \overline{\text{tr}(f(a))} = \begin{cases} 0 & \text{for } f \not\cong h, \\ 1 & \text{for } f \cong h. \end{cases} \quad (4.6)$$

Note that the representations  $f$  and  $h$  in (4.6) are not unitary ones. The matter is that the characters of equivalent representations do coincide, while  $f$  and  $h$ , according to the theorem 2.1, are equivalent to some unitary representations.

**THEOREM 4.2.** *The characters of two non-equivalent irreducible finite-dimensional complex representations of a finite group  $G$  are orthogonal as the elements of the space  $L_2(G)$ .*

The relationship (4.6) is a proof of the theorem 4.2. In order to see it one should compare this relationship with (1.1). From the finiteness  $\dim L_2(G) = N \leq \infty$  we conclude that the number of non-equivalent irreducible finite-dimensional complex representations of a finite group  $G$  is finite. Therefore, considering a *complete set* of such representations is a true idea.

DEFINITION 4.1. The representations  $f_1, \dots, f_m$  form a complete set of non-equivalent irreducible finite-dimensional complex representations of a finite group  $G$  if

- (1) any two of them are not equivalent to each other;
- (2) each irreducible finite-dimensional complex representation of  $G$  is equivalent to one of the representations  $f_1, \dots, f_m$ .

The number  $m$  of the representations in a complete set is a numeric invariant of a finite group  $G$ . It is not greater than the order of the group  $N = |G|$ .

Let  $(f_1, G, V_1), \dots, (f_m, G, V_m)$  be a complete set of non-equivalent irreducible representations. Without loss of generality we can assume these representations to be unitary ones. Let  $n_1, \dots, n_m$  be the dimensions of these representations. We choose some orthonormal basis in each of the spaces  $V_1, \dots, V_m$ . Then we have a set of matrices with the components

$$F_i^j(g, r), \quad r = 1, \dots, m; \quad 1 \leq i, j \leq n_r.$$

Each component in these matrices depend on  $g \in G$ , therefore, it can be treated as a function from  $L_2(G)$ . From (4.5) we derive the following orthogonality relationships for these functions:

$$\frac{1}{N} \sum_{a \in G} F_q^j(a, r) \overline{F_p^i(a, s)} = \frac{1}{n_r} \delta_{rs} \delta_{ij} \delta_{pq}. \quad (4.7)$$

The relationships (4.7) mean that the functions  $F_i^j(g, r)$  treated as the elements of the space  $L_2(G)$  are pairwise orthogonal to each other. Apparently, they are not only orthogonal, but form a complete orthogonal set of functions in this space.

THEOREM 4.3. For an arbitrary complete set  $f_1, \dots, f_m$  of irreducible unitary representations of a finite group  $G$  the matrix elements of the operators  $f_r(g)$  calculated in some orthonormal bases form a complete orthogonal system of functions in  $L_2(G)$ .

PROOF. The orthogonality of the functions  $F_j^i(g, r)$  follows from the relationship (4.7). We need to prove their completeness. For this purpose we consider the right regular representation  $(R, G, L_2(G))$ . It is unitary with respect to the Hermitian structure given by the scalar product (1.1) (see theorem 1.1). For this reason the right regular representation  $(R, G, L_2(G))$  is completely reducible, it is expanded into the direct sum of unitary irreducible representations:

$$R = R_1 \oplus \dots \oplus R_k. \quad (4.8)$$

The expansion (4.8) of the right regular representation  $R$  is associated with the expansion of the space  $L_2(G)$  into the direct sum of irreducible  $R$ -invariant subspaces

$$L_2(G) = W_1 \oplus \dots \oplus W_k.$$

Each of the irreducible representations  $R_q$  in (4.8) is equivalent to one of the irreducible unitary representations  $(f_{r(q)}, G, V_{r(q)})$  from our complete set  $f_1, \dots, f_m$ . Applying the theorem 7.6 from Chapter I, we conclude that the representations  $R_q$  and  $f_{r(q)}$  are unitary equivalent. Therefore, in each subspace  $W_q \subseteq L_2(G)$  we can choose some orthonormal basis of functions

$$\varphi_i(g, q), \quad 1 \leq i \leq n_{r(q)}, \quad (4.9)$$

such that the matrices of the operators  $R_q(g)$  coincide with the matrix components  $F_i^j(g, r(q))$  of the operators for the corresponding representation  $f_{r(q)}$  in the complete set. Let's write this fact as a formula:

$$R_q(g)\varphi_i(q, q) = \sum_{j=1}^{n_{r(q)}} F_i^j(g, r(q)) \varphi_j(a, q). \quad (4.10)$$

But  $R_q(g)$  is the restriction of the operator  $R(g)$  from (4.8) to its invariant subspace  $W_q$ , while  $\varphi_i(a, q)$  is an element of this

subspace. Therefore, we have

$$R_q(g)\varphi_i(a, q) = R(g)\varphi_i(a, q) = \varphi_i(ag, q). \quad (4.11)$$

Let's substitute (4.11) into (4.10). Then in the relationship obtained we set  $a = e$ . The quantities  $\varphi_i(e, q)$  are some constant numbers, we denote them  $c_{iq} = \varphi_i(e, q)$ . As a result we get

$$\varphi_i(g, q) = \sum_{j=1}^{n_{r(q)}} c_{jq} F_i^j(q, r(q)). \quad (4.12)$$

The formula (4.12) is an expansion of the function  $\varphi_i(g, q)$  in the set of functions  $F_i^j(q, r(q))$ . But the set of functions  $\varphi_i(g, q)$  in (4.9) constitute a basis in  $L_2(G)$ . It is a complete set, each element  $\varphi(g) \in L_2(G)$  has an expansion in this set of functions. Due to the formula (4.12) such an expansion can be transformed into the expansion of  $\varphi(g)$  in the set of functions  $F_i^j(q, r(q))$ . Hence, the set of functions  $F_i^j(q, r(q))$  also is a complete set for  $L_2(G)$ . The theorem 4.3 is proved.  $\square$

Let  $(\varphi, G, V)$  be some finite-dimensional complex representation of a finite group  $G$ . It is completely reducible (see theorem 2.2). It is expanded into a sum of irreducible representations:

$$\varphi = \varphi_1 \oplus \dots \oplus \varphi_\nu. \quad (4.13)$$

Each of the irreducible representation  $\varphi_q$  in (4.13) is equivalent to one of the representations  $f_{r(q)}$  in our complete set  $f_1, \dots, f_m$ . Let's denote by  $k_r$  the number of irreducible representations in the expansion (4.13) which are equivalent to the representation  $f_r$ . Then the expansion (4.13) is rewritten as

$$\varphi \cong k_1 f_1 \oplus \dots \oplus k_m f_m. \quad (4.14)$$

The number  $k_r$  in (4.14) is called the *multiplicity* of the entry of the irreducible representation  $f_r$  in  $\varphi$ . The orthogonality relationship (4.6) for characters enables us to calculate the multiplicities

without performing the expansion (4.13) itself:

$$k_r = \frac{1}{N} \sum_{g \in G} \operatorname{tr} \varphi(g) \overline{\operatorname{tr} f_r(g)}. \quad (4.15)$$

The relationship (4.15) is derived from the following expansion for the function  $\operatorname{tr} \varphi(g)$ :

$$\operatorname{tr} \varphi(g) = k_1 \operatorname{tr} f_1 + \dots + k_m \operatorname{tr} f_m. \quad (4.16)$$

The relationship (4.16) in turn is derived from (4.14).

Let's find the expansion (4.14) in the case of the right regular representation  $R(g)$ . The corresponding expansion for the left regular representation is the same because these two representations are equivalent. In order to calculate the trace of the operator  $R(g)$  it is necessary to choose a basis in  $L_2(G)$  and calculate the matrix elements of the operator  $R(g)$  in this basis. The theorem 4.3, which was proved above, says that the set of functions  $F_i^j(g, r)$  corresponding to some complete set of irreducible representations can be used as a basis in  $L_2(G)$ . The basis functions  $F_i^j(g, r)$  are enumerated with three indices  $i$ ,  $j$ , and  $r$ . Therefore, the matrix of the operator  $R(g)$  in this basis is represented by a six-index array  $R_{qi}^{jp}(r, s)$ . This array is determined as follows:

$$R(g)F_i^j(a, r) = \sum_{s=1}^m \sum_{q=1}^{n_s} \sum_{p=1}^{n_s} R_{qi}^{jp}(r, s) F_p^q(a, s). \quad (4.17)$$

The trace of the six-index array  $R_{qi}^{jp}(r, s)$  representing a matrix is calculated according to the following formula:

$$\operatorname{tr} R(g) = \sum_{r=1}^m \sum_{i=1}^{n_s} \sum_{j=1}^{n_s} R_{ji}^{ji}(r, r). \quad (4.18)$$

Let's calculate the left hand side of the equality (4.17) directly from the relationship (1.2) that define the operator  $R(g)$ :

$$R(g)F_i^j(a, r) = F_i^j(a g, r) = \sum_{p=1}^{n_r} F_p^j(a, r) F_i^p(g, r). \quad (4.19)$$

Here, deriving the formula (4.19), we used the relationship  $f_r(a g) = f_r(a) \circ f_r(g)$  written in the matrix form. Comparing the formulas (4.17) and (4.19), we find

$$R_{qi}^{jp}(r, s) = \delta_{rs} \delta_q^j F_i^p(g, r).$$

The rest is to substitute this expression into (4.18) and perform the summations prescribed by that formula:

$$\text{tr } R(g) = \sum_{r=1}^m n_r \text{tr } f_r(g). \quad (4.20)$$

Let's compare (4.20) and (4.16). The result of this comparison is formulated in the following theorem.

**THEOREM 4.4.** *Each irreducible representation  $f_r$  from some complete set  $f_1, \dots, f_m$  of irreducible finite-dimensional complex representations of a finite group  $G$  enters the right regular representation  $(R, G, L_2(G))$  with the multiplicity  $k_r$  equal to its dimension, i. e.  $k_r = n_r = \dim V_r$ .*

Exactly the same proposition is valid for the left regular representation  $(L, G, L_2(G))$  of the group  $G$  either. The theorem 4.4 has the following immediate corollary that follows from the fact that  $\dim L_2(G) = |G|$ .

**COROLLARY 4.1.** *The order of a finite group  $N = |G|$  is equal to the sum of squares of the dimensions of all its non-equivalent irreducible finite-dimensional complex representations  $f_1, \dots, f_m$ , i. e.  $N = (n_1)^2 + \dots + (n_m)^2$ .*

The same result can be obtained if we calculate the total number of functions entering the orthogonality relationships (4.7) and forming a complete orthogonal set of functions in  $L_2(G)$ .

Let's consider the set of characters  $\chi_1, \dots, \chi_m$  for irreducible representations from some complete set  $f_1, \dots, f_m$ . Due to the theorem 4.2 and the relationships (4.6) they are orthogonal. But in general case they do not form a complete set of such functions in  $L_2(G)$ . From the theorem 3.1 we know that these functions are constants within conjugacy classes of  $G$ . Let's denote by  $M_2(G)$  the set of complex numeric functions on  $G$  constant within each conjugacy class of  $G$ . It is clear that  $M_2(G)$  is a linear subspace in  $L_2(G)$ . It inherits the Hermitian scalar product (1.1) from the space  $L_2(G)$ .

**THEOREM 4.5.** *The characters  $\chi_1, \dots, \chi_m$  of the representations  $f_1, \dots, f_m$  forming a complete set of irreducible finite-dimensional representations of a finite group  $G$  form a complete set of orthogonal functions (a basis) in the space  $M_2(G) \subseteq L_2(G)$ .*

Due to the theorem 3.1 all of the characters  $\chi_1, \dots, \chi_m$  belong to  $M_2(G)$ . They are orthogonal to each other and normalized to the unity. It follows from the theorem 4.2. Let  $\varphi(g)$  be some arbitrary element of the space  $M_2(G)$ . Then we can expand it in the set of functions  $F_i^j(g, r)$  (see theorem 4.3):

$$\varphi(g) = \sum_{r=1}^m \sum_{i=1}^{n_r} \sum_{j=1}^{n_r} c_j^i(r) F_i^j(g, r). \quad (4.21)$$

Let's perform the conjugation  $g \mapsto a g a^{-1}$  in the argument of the function  $\varphi(g)$ . This operation does not change its value since  $\varphi(g) \in M_2(G)$ . This value is not changed upon averaging over conjugations by means of all elements of the group  $G$  either:

$$\varphi(g) = \frac{1}{N} \sum_{a \in G} \varphi(a g a^{-1}). \quad (4.22)$$

Let's substitute the expansion (4.21) into (4.22). This yields

$$\varphi(g) = \sum_{r=1}^m \sum_{i=1}^{n_r} \sum_{j=1}^{n_r} c_j^i(r) \left( \frac{1}{N} \sum_{a \in G} F_i^j(a g a^{-1}, r) \right). \quad (4.23)$$

We denote by  $\psi_i^j(g, r)$  the expression enclosed into round brackets in right hand side of (4.23). For this quantity we get

$$\psi_i^j(g, r) = \frac{1}{N} \sum_{a \in G} \sum_{p=1}^{n_r} \sum_{q=1}^{n_r} F_p^j(a, r) F_q^p(g, r) F_i^q(a^{-1}, r).$$

Here we used the relationship  $f_r(a g a^{-1}) = f_r(a) \circ f_r(g) \circ f_r(a^{-1})$  written in the matrix form. Now let's recall that  $F_i^j(a, r)$  are unitary matrices and apply the orthogonality relationship (4.7):

$$\begin{aligned} \psi_i^j(g, r) &= \frac{1}{N} \sum_{a \in G} \sum_{p=1}^{n_r} \sum_{q=1}^{n_r} F_p^j(a, r) F_q^p(g, r) \overline{F_i^q(a, r)} = \\ &= \frac{1}{n_r} \sum_{p=1}^{n_r} \sum_{q=1}^{n_r} F_q^p(g, r) \delta_i^j \delta_p^q. \end{aligned}$$

Upon performing the summation in the right hand side of the above formula for the quantity  $\psi_i^j(g, r)$  we get

$$\psi_i^j(g, r) = \frac{1}{n_r} \operatorname{tr} f_r(g) \delta_i^j = \frac{1}{n_r} \chi_r(g) \delta_i^j.$$

The rest is to substitute this expression back into the formula (4.23). As a result we find

$$\varphi(g) = \sum_{r=1}^m \left( \sum_{i=1}^{n_r} \frac{c_i^i(r)}{n_r} \right) \chi_r(g). \quad (4.24)$$

From (4.24) it is clear that any function  $\varphi(g) \in M_2(G)$  is expanded in the set of functions  $\chi_1, \dots, \chi_m$ .



The following theorem on the number of irreducible representations in a complete set of such representations for a finite group is an obvious corollary of the theorem 4.5, which is proved just above.

**THEOREM 4.6.** *The number of representations in a complete set  $f_1, \dots, f_m$  of irreducible finite-dimensional complex representations of a finite group  $G$  coincides with the number of conjugacy classes in the group  $G$ .*

### § 5. Expansion into irreducible components.

Let  $G$  be a finite group. The theorem 4.6 determines the number of irreducible representations in a complete set  $f_1, \dots, f_m$ , while the theorem 4.4 yields a way for finding such representations. Indeed, each of the representations  $f_1, \dots, f_m$  enters the right regular representation  $(R, G, L_2(G))$  at least once. Therefore, in order to find it one should construct the expansion (4.13) for  $\varphi = R$ . For each particular finite group  $G$  this could be done with the tools of linear algebra.

Suppose that this part of work is already done and some complete set of unitary representations  $f_1, \dots, f_m$  is constructed. Then upon choosing orthonormal bases in the spaces  $V_1, \dots, V_m$  of these representations we can assume that the matrix elements  $F_i^j(g, r)$  of the operators  $f_r(g)$  are known. Under these assumptions we consider the problem of expanding of a given representation  $\varphi, G, V$  into its irreducible components. Let's begin with defining the following operators:

$$P_j^i(r) = \frac{n_r}{N} \sum_{a \in G} \overline{F_i^j(a, r)} \varphi(a). \tag{5.1}$$

The number of such operators coincides with the number of functions  $F_i^j(a, r)$ . However, a part of these operators can be equal to zero. The operators  $P_j^i(r)$  are interpreted as the coefficients for the Fourier expansion of the operator-valued function in orthogonal system of functions in  $L_2(G)$ . The following expansion

approves such interpretation:

$$\varphi(g) = \sum_{r=1}^m \sum_{i=1}^{n_r} \sum_{j=1}^{n_r} F_i^j(a, r) P_j^i(r). \quad (5.2)$$

It is easily derived from the orthogonality relationship (4.7). On the base of the same relationship (4.7) one can derive a number of other relationship for the operators (5.1). First of all we consider the following ones:

$$\varphi(g) \circ P_j^i(r) = \sum_{q=1}^{n_r} F_j^q(g, r) P_q^i(r), \quad (5.3)$$

$$P_j^i(r) \circ \varphi(g) = \sum_{q=1}^{n_r} F_q^i(g, r) P_j^q(r). \quad (5.4)$$

We prove the relationship (5.3) by means of direct calculations. In order to transform the left hand side of (5.3) we use the formula (5.1) for the operator  $P_j^i(r)$ :

$$\begin{aligned} \varphi(g) \circ P_j^i(r) &= \frac{n_r}{N} \sum_{a \in G} \overline{F_i^j(a, r)} \varphi(g) \circ \varphi(a) = \\ &= \frac{n_r}{N} \sum_{a \in G} \overline{F_i^j(a, r)} \varphi(ga). \end{aligned}$$

Replacing  $a$  with  $b = ga$  in summation over the group, we get

$$\begin{aligned} \varphi(g) \circ P_j^i(r) &= \frac{n_r}{N} \sum_{b \in G} \overline{F_i^j(g^{-1}b, r)} \varphi(b) = \\ &= \frac{n_r}{N} \sum_{b \in G} \sum_{q=1}^{n_r} \overline{F_q^j(g^{-1}, r) F_i^q(b, r)} \varphi(b). \end{aligned}$$

Here we used the relationship  $f_r(g^{-1}b) = f_r(g^{-1}) \circ f_r(b)$  written in the matrix form. Now the rest is to use the relationship

$f_r(g^{-1}) = f_r(g)^{-1}$  and the unitarity of the matrix  $F(g, r)$ :

$$\begin{aligned} \varphi(g) \circ P_j^i(r) &= \sum_{q=1}^{n_r} F_j^q(g, r) \left( \frac{n_r}{N} \sum_{b \in G} \overline{F_i^q(b, r)} \varphi(b) \right) = \\ &= \sum_{q=1}^{n_r} F_j^q(g, r) P_q^i(r). \end{aligned}$$

Comparing the left and right hand sides of the above equalities, we see that the formula (5.3) is proved. The formula (5.4) is proved in a similar way, therefore, we do not give its proof here.

Let's set  $j = i$  in the formulas (5.3) and (5.4) and then sum up these equalities over the index  $i$ . The double sums in right hand sides of the resulting equalities do coincide. Therefore, the result can be written as

$$\begin{aligned} \sum_{i=1}^{n_r} \varphi(g) \circ P_i^i(r) &= \sum_{i=1}^{n_r} P_i^i(r) \circ \varphi(g) = \\ &= \sum_{i=1}^{n_r} \sum_{j=1}^{n_r} F_i^j(g, r) P_j^i(r). \end{aligned} \tag{5.5}$$

The right hand side of (5.5) differs from that of (5.2) by the absence of the sum over  $r$ . Combining (5.5) and (5.2), we obtain

$$\varphi(g) = \sum_{r=1}^m \sum_{i=1}^{n_r} \varphi(g) \circ P_i^i(r) \sum_{r=1}^m \sum_{i=1}^{n_r} P_i^i(r) \circ \varphi(g). \tag{5.6}$$

Due to (5.6) it is natural to introduce new operators by means of the following formula:

$$P(r) = \sum_{i=1}^{n_r} P_i^i(r) = \frac{n_r}{N} \sum_{a \in G} \overline{\text{tr } f_r(a)} \varphi(a). \tag{5.7}$$

In terms of (5.7) the relationship (5.6) itself is rewritten as

$$\varphi(g) = \sum_{r=1}^m \varphi(g) \circ P(r) \sum_{r=1}^m P(r) \circ \varphi(g). \quad (5.8)$$

Setting  $g = e$  in (5.8), we get an expansion of the unity (of the identical operator) in operators (5.7):

$$1 = \sum_{r=1}^m P(r). \quad (5.9)$$

**THEOREM 5.1.** *The operators  $P(r): V \rightarrow V$ ,  $r = 1, \dots, m$ , given by the formula (5.7) possess the following properties:*

- (1) *they satisfy the relationships  $P(r)^2 = P(r)$ , because of which those of them being nonzero  $P(r) \neq 0$  are projectors onto the subspaces  $V(r) = \text{Im } P(r)$ ;*
- (2) *they commute with the representation operators  $\varphi(g)$ , because of which the subspaces  $V(r)$  are invariant with respect to  $\varphi(g)$ ;*
- (3) *they satisfy the relationship (5.9) and the relationships  $P(r) \circ P(s) = 0$  for  $r \neq s$ , because of which the expansion  $V = V(1) \oplus \dots \oplus V(m)$  is an expansion into the direct sum of invariant subspaces.*

The relationships  $P(r)^2 = P(r)$  from the first item of the theorem and the relationships  $P(r) \circ P(s) = 0$  for  $r \neq s$  from the third item of the theorem can be combined into one relationship:

$$P(r) \circ P(s) = P(r) \delta_{rs} = \begin{cases} 0 & \text{for } r \neq s, \\ P(r) & \text{for } r = s. \end{cases} \quad (5.10)$$

The relationship (5.10) is easily derived from the following more general relationship for the operators  $P_j^i(r)$  defined in (5.1):

$$P_j^i(r) \circ P_q^k(s) = \delta_{rs} \delta_q^i P_j^k(r). \quad (5.11)$$

It is convenient to prove (5.11) by direct calculations. From the formula (5.1) for the product of operators in the left hand side of (5.11) we derive

$$P_j^i(r) \circ P_q^k(s) = \frac{n_r n_s}{N^2} \sum_{a \in G} \sum_{b \in G} \overline{F_i^j(a, r) F_k^q(b, s)} \varphi(ab).$$

Let's denote  $c = ab$  and choose  $c$  as a new parameter in summation over the group  $G$  in place of  $b$ . This yields

$$\begin{aligned} P_j^i(r) \circ P_q^k(s) &= \frac{n_r n_s}{N^2} \sum_{a \in G} \sum_{c \in G} \overline{F_i^j(a, r) F_k^q(a^{-1}c, s)} \varphi(c) = \\ &= \frac{n_r n_s}{N^2} \sum_{a \in G} \sum_{c \in G} \overline{F_i^j(a, r)} \sum_{p=1}^{n_s} \overline{F_p^q(a^{-1}, s) F_k^p(c, s)} \varphi(c). \end{aligned}$$

In the next step we use the unitarity of the matrix  $F(a, s)$ :

$$P_j^i(r) \circ P_q^k(s) = \sum_{p=1}^{n_s} \sum_{a \in G} \frac{\overline{F_i^j(a, r) F_q^p(a, s)}}{N} \sum_{c \in G} \frac{n_s \overline{F_k^p(c, s)}}{N} \varphi(c).$$

And finally, we use (5.1) and the orthogonality relationship (4.7):

$$P_j^i(r) \circ P_q^k(s) = \sum_{p=1}^{n_s} \delta_{rs} \delta_j^p \delta_q^i P_p^k(s) = \delta_{rs} \delta_q^i P_j^k(r).$$

Comparing the left and right hand sides of this formula with (5.11), we see that the formula (5.11) is proved. Hence, the relationship (5.10) is also proved.

The relationship  $P(r)^2 = P(r)$ , which follows from (5.10), in the case of a nonzero operator  $P(r) \neq 0$  means that  $P(r)$  is a projection operator. It projects the space  $V$  onto the subspace  $V(r) = \text{Im } P(r)$  parallel to the subspace  $\text{Ker } P(r)$  (see more details in [1]). The relationship

$$P(r) \circ P(s) = 0 = P(s) \circ P(r) \quad \text{for } r \neq s \tag{5.12}$$

means that the projectors  $P(1), \dots, P(m)$  commute and that  $\text{Im } P(r) \subseteq \text{Ker } P(s)$  for  $r \neq s$ . Due to (5.12) and (5.9) the set of projectors  $P(1), \dots, P(m)$  is a concordant and complete set of projectors. This set of projectors determines an expansion of  $V$  into a direct sum of subspaces:

$$V = V(1) \oplus \dots \oplus V(m) = \bigoplus_{r=1}^m V(r). \quad (5.13)$$

The second item of the theorem 5.1 claiming the commutativity of  $P(r)$  and  $\varphi(g)$  follows immediately from (5.5). Due to this fact all subspaces in the expansion (5.13) are invariant subspaces of the representation  $(\varphi, G, V)$ .

The matrix elements  $F_i^j(a, r)$  of the operators  $f_r(a)$  depend on a choice of bases in spaces where they act. The operators  $P_j^i(r)$  calculated according to the formula (5.1) also depend on a choice of these bases. However, the operators  $P(r)$  do not depend on bases since the traces of the operators  $f_r(a)$  in the formula (5.7) are their (basis-free) scalar invariants. Therefore, the expansion (5.13) is also invariant, it is determined by the group  $G$  itself and by its representation  $\varphi$ . Let's study how the expansions (5.13) and (4.14) are related to each other.

**THEOREM 5.2.** *The restriction of the representation  $\varphi$  to the invariant subspace  $V(r) = \text{Im } P(r)$  is isomorphic to the irreducible representation  $f_r$  taken with the multiplicity  $k_r$ , where  $k_r$  is the coefficient of  $f_r$  in the expansion (4.14).*

In order to prove the theorem 5.2 we use the following relationship whose left hand side coincides with the operator  $P(r)$  in the case of  $\varphi = f_s$  (see formula (5.7)):

$$\frac{n_r}{N} \sum_{a \in G} \sum_{i=1}^{n_r} \overline{F_i^i(a, r)} f_s(a) = \begin{cases} 0 & \text{for } s \neq r, \\ 1 & \text{for } s = r. \end{cases} \quad (5.14)$$

In order to verify that (5.14) is valid it is sufficient to pass from

the operators  $f_s(a)$  to their matrices  $F(a, s)$  and then to apply the orthogonality relationship (4.7).

Now let's consider the expansion (4.13). According to (4.13), the space  $V$  is a direct sum of irreducible subspaces  $V = V_1 \oplus \dots \oplus V_\nu$ , the restriction of  $\varphi$  to each such subspace is isomorphic to some irreducible representation from the complete set  $f_1, \dots, f_r$ :

$$\varphi \Big|_{V_q} \cong f_{r(q)}. \tag{5.15}$$

Substituting (5.15) into (5.7) and taking into account (5.14), we find that  $V(r)$  is the sum of those subspaces  $V_q$  in the expansion  $V = V_1 \oplus \dots \oplus V_\nu$  for which  $r(q) = r$ . The number of such subspaces is equal to  $k_r$ , while the restriction of  $\varphi$  to each of them is isomorphic to  $f_r$ . The theorem 5.2 is proved. According to the theorem 5.2, the operators  $P(r)$  yield a constructive way to build the expansion (4.14), while the expansion itself is unique up to a permutation of the summands.

Let's consider a separate subspace  $V(r)$  corresponding to the component  $k_r f_r$  in the expansion (4.14). If  $k_r = 0$ , the subspace  $V(r) = \{0\}$  is trivial. If  $k_r = 1$  the subspace  $V(r)$  is irreducible, it does not require a further expansion. The rest is the case  $k_r > 1$ , it should be especially treated. In this case the subspace  $V(r)$  is expanded into the sum of several irreducible subspaces

$$V(r) = \bigoplus_{q=1}^{k_r} W_q, \text{ where } \dim W_q = n_r. \tag{5.16}$$

In contrast to the expansion (5.13), the expansion (5.16) is not unique. One of the ways for constructing such an expansion is due to the operators  $P_i^i(r)$ . Their sum is equal to  $P(r)$  according to the formula (5.7). From (5.11) for these operators we derive

$$P_i^i(r)^2 = P_i^i(r), \quad P_i^i(r) \circ P_j^j(r) = 0 \text{ for } i \neq j. \tag{5.17}$$

There is no summation over  $i$  and  $j$  in these formulas. Due to (5.17) the operators  $P_i^i(r)$ ,  $i = 1, \dots, n_r$  form a concordant and complete set of projection operators. They define an expansion of  $V(r)$  into a direct sum of smaller subspaces:

$$V(r) = \bigoplus_{i=1}^{n_r} V_i(r). \quad (5.18)$$

The projection operators  $P_i^i(r)$  do not commute with  $\varphi(g)$ . Therefore, the subspaces  $V_i(i) = \text{Im } P_i^i(r)$  in the expansion (5.18) are not invariant for the representation  $\varphi$ . However, we can overcome this difficulty. For this purpose we use the following equalities easily derived from (5.11):

$$P_k^k(r) \circ P_k^i(r) = P_k^i(r), \quad P_i^i(r) \circ P_i^i(r) = P_i^i(r), \quad (5.19)$$

$$P_i^k(r) \circ P_k^i(r) = P_i^i(r), \quad P_i^i(r) \circ P_i^k(r) = P_i^k(r). \quad (5.20)$$

**THEOREM 5.3.** *For  $i \neq k$  the operator  $P_k^i(r)$  performs a bijective mapping from  $V_i(r)$  onto  $V_k(r)$ .*

The proof of the theorem 5.3 is based on the formulas (5.19) and (5.20). Indeed, let  $\mathbf{v} \in V_i(r)$  and let  $\mathbf{u} = P_k^i(r)\mathbf{v}$ . Then, using the first relationship (5.19), we derive  $P_k^k\mathbf{u} = \mathbf{u}$ . Hence,  $\mathbf{u} \in V_k(r)$ , i.e.  $P_k^i(r)$  maps  $V_i(r)$  into  $V_k(r)$ . The mapping  $P_k^i(r): V_i(r) \rightarrow V_k(r)$  is bijective since it is invertible. According to (5.20), the mapping  $P_i^k(r): V_k(r) \rightarrow V_i(r)$  is inverse to it.

The equality  $\dim V_i(r) = \dim V_k(r)$  is an immediate corollary of the theorem 5.3, which is proved just above. Using this equality, from the expansions (5.16) and (5.18) we derive

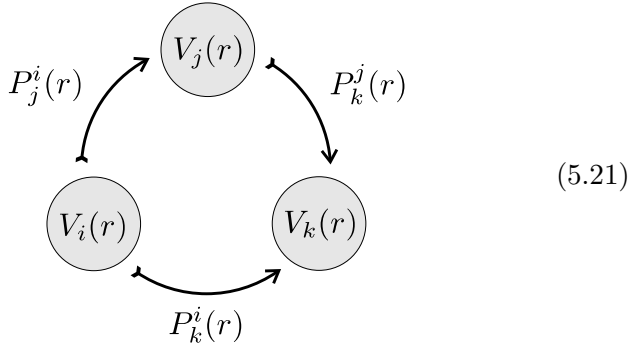
$$\dim V(r) = k_r n_r, \quad \dim V(r) = n_r \dim V_i(r).$$

Comparing these two formulas, we find that

$$\dim V_i(r) = k_r, \quad i = 1, \dots, n_r.$$



The subspaces  $V_1, \dots, V_{n_r}$  are connected by virtue of the mappings  $P_k^i(r)$ . It is easy to verify that the diagram



is commutative. Indeed,  $P_k^j(r) \circ P_j^i(r) = P_k^i(r)$ . This equality is derived from (5.11). Let's choose a basis  $\mathbf{e}_1^1(r), \dots, \mathbf{e}_{k_r}^1(r)$  in the subspace  $V_1(r)$  and then replicate it to the other subspaces  $V(r)$  by means of the mappings  $P_i^1(r)$ :

$$\mathbf{e}_1^i(r) = P_i^1(r)\mathbf{e}_1^1(r), \dots, \mathbf{e}_{k_r}^i(r) = P_i^1(r)\mathbf{e}_{k_r}^1(r).$$

Due to the commutativity of the diagram (5.21) we get

$$P_k^i \mathbf{e}_s^i(r) = \mathbf{e}_s^k(r), \quad s = 1, \dots, k_r. \tag{5.22}$$

The whole set of vectors  $\mathbf{e}_s^i(r)$ ,  $i = 1, \dots, n_r$ ,  $s = 1, \dots, k_r$  is a linear independent set since the sum of subspaces in (5.18) is a direct sum. Let's define new subspaces as the spans of the following sets of vectors:

$$U_s(r) = \langle \mathbf{e}_s^1(r), \dots, \mathbf{e}_s^{n_r}(r) \rangle, \quad s = 1, \dots, k_r. \tag{5.23}$$

Due to the formula (5.22) the subspaces (5.23) are invariant with respect to the operators  $P_i^k(r)$ . However, a stronger proposition is also valid.

**THEOREM 5.4.** *The subspace  $U_s(r)$  in (5.23) is an invariant subspace of the representation  $(\varphi, G, V)$ . It is irreducible and the restriction of  $\varphi$  to  $U_s(r)$  is isomorphic to  $f_r$ .*

In order to prove the invariance of  $U_s(r)$  with respect to  $\varphi$  we use the relationship (5.3). We have proved it in the very beginning of this section. Applying it, we get

$$\begin{aligned} \varphi(g)\mathbf{e}_s^i(r) &= \varphi(g)P_i^1(r)\mathbf{e}_s^1(r) = \\ &= \sum_{q=1}^{n_r} F_i^q(g, r) P_q^1(r)\mathbf{e}_s^1(r) = \sum_{q=1}^{n_r} F_i^q(g, r)\mathbf{e}_s^q(r). \end{aligned} \quad (5.24)$$

Not only does the relationship (5.24) prove the invariance of  $U_s(r)$  with respect to the operator  $\varphi(g)$ , but it shows that the matrix of the operator  $\varphi(g)$  in the basis  $\mathbf{e}_s^1(r), \dots, \mathbf{e}_s^{n_r}(r)$  coincides with the matrix of the operator  $f_r(g)$ . Thus, the second proposition of the theorem 5.4 is also proved.

As a result we have found the constructive way for expanding the space  $V$  into a direct sum of subspaces

$$V = \bigoplus_{r=1}^m \bigoplus_{s=1}^{k_r} U_s(r)$$

that corresponds to the expansion (4.14) of the representation  $\varphi$  into its irreducible components.

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## APPENDIX

### List of publications by the author for the period 1986–2006.

#### Part 1. Soliton theory.

1. Sharipov R. A., *Finite-gap analogs of  $N$ -multiplet solutions of the KdV equation*, Uspehi Mat. Nauk **41** (1986), no. 5, 203–204.
2. Sharipov R. A., *Soliton multiplets of the Korteweg-de Vries equation*, Dokladi AN SSSR **292** (1987), no. 6, 1356–1359.
3. Sharipov R. A., *Multiplet solutions of the Kadomtsev-Petviashvili equation on a finite-gap background*, Uspehi Mat. Nauk **42** (1987), no. 5, 221–222.
4. Bikbaev R. F. & Sharipov R. A., *Magnetization waves in Landau-Lifshits model*, Physics Letters A **134** (1988), no. 2, 105–108; see [solv-int/9905008](#).
5. Bikbaev R. F. & Sharipov R. A., *Asymptotics as  $t \rightarrow \infty$  for a solution of the Cauchy problem for the Korteweg-de Vries equation in the class of potentials with finite-gap behaviour as  $x \rightarrow \pm\infty$* , Theor. and Math. Phys. **78** (1989), no. 3, 345–356.
6. Sharipov R. A., *On integration of the Bogoyavlensky chains*, Mat. zametki **47** (1990), no. 1, 157–160.
7. Cherdantsev I. Yu. & Sharipov R. A., *Finite-gap solutions of the Bullough-Dodd-Jiber-Shabat equation*, Theor. and Math. Phys. **82** (1990), no. 1, 155–160.
8. Cherdantsev I. Yu. & Sharipov R. A., *Solitons on a finite-gap background in Bullough-Dodd-Jiber-Shabat model*, International. Journ. of Modern Physics A **5** (1990), no. 5, 3021–3027; see [math-ph/0112045](#).
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10. Sharipov R. A., *Minimal tori in five-dimensional sphere in  $\mathbb{C}^3$* , Theor. and Math. Phys. **87** (1991), no. 1, 48–56; see [math.DG/0204253](#).
11. Safin S. S. & Sharipov R. A., *Backlund autotransformation for the equation  $u_{xt} = e^u - e^{-2u}$* , Theor. and Math. Phys. **95** (1993), no. 1, 146–159.
12. Boldin A. Yu. & Safin S. S. & Sharipov R. A., *On an old paper of Tzitzeika and the inverse scattering method*, Journal of Mathematical Physics **34** (1993), no. 12, 5801–5809.
13. Pavlov M. V. & Svinolupov S. I. & Sharipov R. A., *Invariant criterion of integrability for a system of equations of hydrodynamical type*, «Integrability in dynamical systems», Inst. of Math. UrO RAN, Ufa, 1994, pp. 27–48; Funk. Anal. i Pril. **30** (1996), no. 1, 18–29; see [solv-int/9407003](#).
14. Ferapontov E. V. & Sharipov R. A., *On conservation laws of the first order for a system of equations of hydrodynamical type*, Theor. and Math. Phys. **108** (1996), no. 1, 109–128.

## Part 2. Geometry of the normal shift.

1. Boldin A. Yu. & Sharipov R. A., *Dynamical systems accepting the normal shift*, Theor. and Math. Phys. **97** (1993), no. 3, 386–395; see [chaodyn/9403003](#).
2. Boldin A. Yu. & Sharipov R. A., *Dynamical systems accepting the normal shift*, Dokladi RAN **334** (1994), no. 2, 165–167.
3. Boldin A. Yu. & Sharipov R. A., *Multidimensional dynamical systems accepting the normal shift*, Theor. and Math. Phys. **100** (1994), no. 2, 264–269; see [patt-sol/9404001](#).
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5. Sharipov R. A., *Dynamical systems accepting the normal shift*, Uspehi Mat. Nauk **49** (1994), no. 4, 105; see [solv-int/9404002](#).
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7. Boldin A. Yu. & Bronnikov A. A. & Dmitrieva V. V. & Sharipov R. A., *Complete normality conditions for the dynamical systems on Riemannian manifolds*, «Dynamical systems accepting the normal shift», Bashkir State University, 1994, pp. 20–30; see also Theor. and Math. Phys. **103** (1995), no. 2, 267–275 and [astro-ph/9405049](#).
8. Sharipov R. A., *Higher dynamical systems accepting the normal shift*, «Dynamical systems accepting the normal shift», Bashkir State University, 1994, pp. 41–65.

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