A NOTE ON CONNECTIONS OF THE STANDARD MODEL IN A GRAVITATION FIELD.

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Abstract. The Standard Model of the theory of elementary particles is based on the U(1) × SU(2) × SU(3) symmetry. In the presence of a gravitation field, i. e. in a non-flat space-time manifold, this symmetry is implemented through three special vector bundles. Connections associated with these vector bundles are studied in this paper. In the Standard Model they are interpreted as gauge fields.

1. The U and SU-bundles and their basic fields.

In its canonical form the Standard Model describes elementary particles in the flat Minkowski space-time (see [1–6]). When passing to a non-flat space-time M we introduce three special vector bundles UM, SUM, and SUM, each equipped with its own basic tensorial fields (see [7]). They are listed in the following table:

Symbol	Name	Bundle	Tensorial type
$\mathbf{D}$	Hermitian metric tensor	UM	$(0,1\|0,1)$
$\mathbf{D}$	Hermitian metric tensor	SUM	$(0,1\|0,1)$
$\mathbf{d}$	Skew-symmetric metric tensor		$(0,2\|0,0)$
$\mathbf{D}$	Hermitian metric tensor	SUM	$(0,1\|0,1)$
$\mathbf{d}$	Completely skew-symmetric tensor		$(0,3\|0,0)$

(1.1)

The bundles UM, SUM, and SUM are complex bundles over a real base. For this reason they possess the involution of complex conjugation τ:

$$U^{\varepsilon}_{\eta}\bar{U}^{\sigma}_{\zeta}M \underset{\longleftarrow}{\xrightarrow{\ \tau\ }} U^{\sigma}_{\zeta}\bar{U}^{\varepsilon}_{\eta}M, \tag{1.2}$$

$$SU^{\pi}_{\rho}\overline{SU}^{\omega}_{\mu}M \underset{\longleftarrow}{\xrightarrow{\ \tau\ }} SU^{\omega}_{\mu}\overline{SU}^{\pi}_{\rho}M, \tag{1.3}$$

$$SU^{\varkappa}_{\lambda}\overline{SU}^{\varsigma}_{\chi}M \underset{\longleftarrow}{\xrightarrow{\ \tau\ }} SU^{\varsigma}_{\chi}\overline{SU}^{\varkappa}_{\lambda}M. \tag{1.4}$$

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The tensor bundles $\not{U}^{\varepsilon}_{\eta}\bar{\not{U}}^{\sigma}_{\zeta}M$, $S\not\not{U}^{\pi}_{\rho}\overline{S\not\not{U}}^{\omega}_{\mu}M$, and $S\not\not\not{U}^{\varkappa}_{\lambda}\overline{S\not\not\not{U}}^{\varsigma}_{\chi}M$ from (1.2), (1.3), and (1.4) are defined in [7] (see formulas (2.5), (2.11), and (3.4) there).

The Hermitian metric tensors $\not{\mathbf{D}}$, $\not\not{\mathbf{D}}$, and $\not\not\not{\mathbf{D}}$ in the table (1.1) are introduced through the corresponding Hermitian forms:

$$\not{D}(\mathbf{X},\mathbf{Y}) = \not{D}_{11}\,\overline{X^1}\,Y^1 = \overline{\not{D}(\mathbf{Y},\mathbf{X})}, \tag{1.5}$$

$$\not\not{D}(\mathbf{X},\mathbf{Y}) = \sum_{i=1}^{2}\sum_{\bar{j}=1}^{2}\not\not{D}_{i\bar{j}}\,\overline{X^{\bar{j}}}\,Y^i = \overline{\not\not{D}(\mathbf{Y},\mathbf{X})}, \tag{1.6}$$

$$\not\not\not{D}(\mathbf{X},\mathbf{Y}) = \sum_{i=1}^{3}\sum_{\bar{j}=1}^{3}\not\not\not{D}_{i\bar{j}}\,\overline{X^{\bar{j}}}\,Y^i = \overline{\not\not\not{D}(\mathbf{Y},\mathbf{X})}. \tag{1.7}$$

The Hermitian forms (1.5), (1.6), and (1.7) are positive, i. e. their signatures are $(+)$, $(+,+)$, and $(+,+,+)$ respectively. The components of tensor fields $\not{D}_{11}$, $\not\not{D}_{i\bar{j}}$, and $\not\not\not{D}_{i\bar{j}}$ in the above formulas are frame-relative.

Definition 1.1. *A frame* of a q-dimensional vector bundle over the base manifold M is an ordered set of q smooth sections of this bundle linearly independent at each point p of some open domain $U \subset M$.

Let's denote by $(U, \mathbf{\Psi}_1)$, $(U,\, \mathbf{\Psi}_1,\, \mathbf{\Psi}_2)$, $(U,\, \mathbf{\Psi}_1,\, \mathbf{\Psi}_2,\, \mathbf{\Psi}_3)$ some frames of the bundles $\not{U}M$, $S\not\not{U}M$, and $S\not\not\not{U}M$ respectively. Apart from these three frames, we choose some frame $(U,\, \mathbf{\Upsilon}_0,\, \mathbf{\Upsilon}_1,\, \mathbf{\Upsilon}_2,\, \mathbf{\Upsilon}_3)$ of the tangent bundle TM.

Definition 1.2. A frame $(U, \mathbf{\Psi}_1)$ of the electro-weak bundle $\not{U}M$ is called an *orthonormal frame* if $\not{D}_{11} = 1$ in this frame.

Definition 1.3. A frame $(U,\, \mathbf{\Psi}_1,\, \mathbf{\Psi}_2)$ of the bundle $S\not\not{U}M$ is called an *orthonormal frame* if the tensor fields $\not\not{\mathbf{D}}$ and $\not\not{\mathbf{d}}$ are given by the following matrices in this frame:

$$\not\not{D}_{i\bar{j}} = \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix}, \qquad\qquad \not\not{d}_{ij} = \begin{Vmatrix} 0 & 1 \\ -1 & 0 \end{Vmatrix}.$$

Definition 1.4. A frame $(U,\, \mathbf{\Psi}_1,\, \mathbf{\Psi}_2,\, \mathbf{\Psi}_3)$ of the bundle $S\not\not\not{U}M$ is called an *orthonormal frame* if $\not\not\not{d}_{123} = 1$ and $\not\not\not{\mathbf{D}}$ is given by the unit matrix in this frame:

$$\not\not\not{D}_{i\bar{j}} = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Vmatrix}.$$

Once the component $\not\not\not{d}_{123}$ is fixed by the condition $\not\not\not{d}_{123} = 1$, other components of the tensor $\not\not\not{\mathbf{d}}$ are determined by the skew-symmetry condition:

$$\not\not\not{d}_{ijk} = -\not\not\not{d}_{jik}, \qquad \not\not\not{d}_{ijk} = -\not\not\not{d}_{ikj}, \qquad \not\not\not{d}_{ijk} = -\not\not\not{d}_{kji}. \tag{1.8}$$

Orthonormal frames of the bundle $\not{U}M$ do always exist. As for the other two bundles $S\not\not{U}M$ and $S\not\not\not{U}M$, orthonormal frames for them do exist provided some concordance conditions are fulfilled (see theorems 2.1 and 3.1 in [7]). In the case of the electroweak bundle SUM the concordance condition of $\mathbf{D}$ and $\mathbf{d}$ is written as

$$\sum_{i=1}^{2}\sum_{j=1}^{2} d^{ij}\, D_{i\bar{i}}\, D_{j\bar{j}} = \overline{d_{\bar{i}\bar{j}}}. \tag{1.9}$$

By d^{ij} in (1.9) we denote the components of the matrix inverse to the matrix d_{ij}. They define the inverse skew-symmetric metric tensor in SUM. By tradition this tensor is denoted by the same symbol $\mathbf{d}$ as the initial one.

In the case of the color bundle SUM the concordance condition for the tensor fields $\mathbf{D}$ and $\mathbf{d}$ is given by the following equality:

$$\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} d^{ijk}\, D_{i\bar{i}}\, D_{j\bar{j}}\, D_{k\bar{k}} = \overline{d_{\bar{i}\bar{j}\bar{k}}}. \tag{1.10}$$

The equality (1.10) is a three-dimensional analog of the equality (1.9). By d^{ijk} in (1.10) we denote the components of the completely skew-symmetric tensor of the type $(3,0|0,0)$ inverse to the tensor $\mathbf{d}$. This tensor is denoted by the same symbol $\mathbf{d}$ as the initial one. Due to the skew-symmetry conditions

$$d^{ijk} = -d^{jik}, \qquad\qquad d^{ijk} = -d^{ikj}, \qquad\qquad d^{ijk} = -d^{kji}$$

similar to (1.8) the inverse tensor $\mathbf{d}$ is fixed by the equality $d^{123} = 1/d_{123}$. The concordance conditions (1.9) and (1.10) are postulated to be valid for the bundles SUM and SUM of the Standard Model.

2. A little bit of the general theory.

Let UM be an arbitrary q-dimensional complex vector bundle over the space-time manifold M. A connection in UM is a geometric structure which is used for to differentiate tensor fields obtaining other tensor fields from them:

$$\nabla\colon\ U^{\varepsilon}_{\eta}\bar{U}^{\sigma}_{\zeta}M \to U^{\varepsilon}_{\eta}\bar{U}^{\sigma}_{\zeta}M \otimes T^{*}M. \tag{2.1}$$

The operator of covariant differential (2.1) acts upon a tensorial field of the type $(\varepsilon,\eta|\sigma,\zeta)$ and produces a tensorial field of the type $(\varepsilon,\eta|\sigma,\zeta|0,1)$. In other words, the operator ∇ adds one lower (covariant) index to the components of a tensor. But this additional index is associated with the tangent bundle TM, not with the vector bundle UM itself. It is convenient to extend the domain of the operator (2.1) by adding more indices associated with the tangent bundle:

$$\nabla\colon\ U^{\varepsilon}_{\eta}\bar{U}^{\sigma}_{\zeta}M \otimes T^{m}_{n}M \to U^{\varepsilon}_{\eta}\bar{U}^{\sigma}_{\zeta}M \otimes T^{m}_{n+1}M. \tag{2.2}$$

The operator of covariant differential (2.2) acts upon a tensorial field of the type $(\varepsilon,\eta|\sigma,\zeta|m,n)$ and produces a tensorial field of the type $(\varepsilon,\eta|\sigma,\zeta|m,n+1)$. Assume that $(U,\,\mathbf{\Psi}_1,\,\ldots\,,\,\mathbf{\Psi}_q)$ is a frame of the bundle UM and let $(U,\,\mathbf{\Upsilon}_0,\,\mathbf{\Upsilon}_1,\,\mathbf{\Upsilon}_2,\,\mathbf{\Upsilon}_3)$ be some frame of the tangent bundle TM. Assume that the domain $U \subset M$ is equipped with some local coordinates x^0, x^1, x^2, x^3. Then we have the expansion

$$\boldsymbol{\Upsilon}_i=\sum_{j=0}^{3}\Upsilon^j_i\ \frac{\partial}{\partial x^j} \tag{2.3}$$

for the frame vector fields $\boldsymbol{\Upsilon}_0$, $\boldsymbol{\Upsilon}_1$, $\boldsymbol{\Upsilon}_2$, $\boldsymbol{\Upsilon}_3$. Apart from (2.3), we consider the following commutation relationships for these vector fields:

$$[\boldsymbol{\Upsilon}_i,\,\boldsymbol{\Upsilon}_j]=\sum_{k=0}^{3}c^k_{ij}\,\boldsymbol{\Upsilon}_k. \tag{2.4}$$

The quantities c^k_{ij} in (2.4) are similar to structural constants of Lie algebras. For this reason they are called the *structural constants* of the frame $(U,\,\boldsymbol{\Upsilon}_0,\,\boldsymbol{\Upsilon}_1,\,\boldsymbol{\Upsilon}_2,\,\boldsymbol{\Upsilon}_3)$, though actually they are not constants, but smooth real-valued functions within the domain U.

Let $\mathbf{X}\in U^\varepsilon_\eta\bar U^\sigma_\zeta M\!\otimes\! T^m_n M$ be a tensor field of the type $(\varepsilon,\eta|\sigma,\zeta|m,n)$. In the frame pair $(U,\,\boldsymbol{\Psi}_1,\,\ldots\,,\,\boldsymbol{\Psi}_q)$ and $(U,\,\boldsymbol{\Upsilon}_0,\,\boldsymbol{\Upsilon}_1,\,\boldsymbol{\Upsilon}_2,\,\boldsymbol{\Upsilon}_3)$ this tensor field is represented by its components $X^{i_1\ldots\,i_\varepsilon\,\bar i_1\ldots\,\bar i_\sigma\,h_1\ldots\,h_m}_{j_1\ldots\,j_\eta\,\bar j_1\ldots\,\bar j_\zeta\,k_1\ldots\,k_n}$. Then ∇ is represented by the formula

$$\begin{aligned}
&\nabla_{k_{n+1}}X^{i_1\ldots\,i_\varepsilon\,\bar i_1\ldots\,\bar i_\sigma\,h_1\ldots\,h_m}_{j_1\ldots\,j_\eta\,\bar j_1\ldots\,\bar j_\zeta\,k_1\ldots\,k_n}=\sum_{s=0}^{3}\Upsilon^s_{k_{n+1}}\,\frac{\partial X^{i_1\ldots\,i_\varepsilon\,\bar i_1\ldots\,\bar i_\sigma\,h_1\ldots\,h_m}_{j_1\ldots\,j_\eta\,\bar j_1\ldots\,\bar j_\zeta\,k_1\ldots\,k_n}}{\partial x^s}\,-\\
&+\sum_{\mu=1}^{\varepsilon}\sum_{v_\mu=1}^{q}\mathrm{A}^{i_\mu}_{k_{n+1}\,v_\mu}\ X^{i_1\ldots\,v_\mu\,\ldots\,i_\varepsilon\,\bar i_1\ldots\,\bar i_\sigma\,h_1\ldots\,h_m}_{j_1\ldots\,\ldots\,\ldots\,j_\eta\,\bar j_1\ldots\,\bar j_\zeta\,k_1\ldots\,k_n}\,-\\
&\qquad-\sum_{\mu=1}^{\eta}\sum_{w_\mu=1}^{q}\mathrm{A}^{w_\mu}_{k_{n+1}\,j_\mu}\ X^{i_1\ldots\,\ldots\,\ldots\,i_\varepsilon\,\bar i_1\ldots\,\bar i_\sigma h_1\ldots\,h_m}_{j_1\ldots\,w_\mu\,\ldots\,j_\eta\,\bar j_1\ldots\,\bar j_\zeta k_1\ldots\,k_n}\,+\\
&+\sum_{\mu=1}^{\sigma}\sum_{v_\mu=1}^{q}\bar{\mathrm{A}}^{\bar i_\mu}_{k_{n+1}\,v_\mu}\ X^{i_1\ldots\,i_\varepsilon\,\bar i_1\ldots\,v_\mu\,\ldots\,\bar i_\sigma\,h_1\ldots\,h_m}_{j_1\ldots\,j_\eta\,\bar j_1\ldots\,\ldots\,\ldots\,\bar j_\zeta\,k_1\ldots\,k_n}\,-\\
&\qquad-\sum_{\mu=1}^{\zeta}\sum_{w_\mu=1}^{q}\bar{\mathrm{A}}^{w_\mu}_{k_{n+1}\,\bar j_\mu}\ X^{i_1\ldots\,i_\varepsilon\,\bar i_1\ldots\,\ldots\,\ldots\,\bar i_\sigma\,h_1\ldots\,h_m}_{j_1\ldots\,j_\eta\,\bar j_1\ldots\,w_\mu\,\ldots\,\bar j_\zeta\,k_1\ldots\,k_n}\,+\\
&+\sum_{\mu=1}^{m}\sum_{v_\mu=0}^{3}\Gamma^{h_\mu}_{k_{n+1}\,v_\mu}\ X^{i_1\ldots\,i_\varepsilon\,\bar i_1\ldots\,\bar i_\sigma\,h_1\ldots\,v_\mu\,\ldots\,h_m}_{j_1\ldots\,j_\eta\,\bar j_1\ldots\,\bar j_\zeta\,k_1\ldots\,\ldots\,\ldots\,k_n}\,-\\
&\qquad-\sum_{\mu=1}^{n}\sum_{w_\mu=0}^{3}\Gamma^{w_\mu}_{k_{n+1}\,k_\mu}\ X^{i_1\ldots\,i_\varepsilon\,\bar i_1\ldots\,\bar i_\sigma\,h_1\ldots\,\ldots\,\ldots\,h_m}_{j_1\ldots\,j_\eta\,\bar j_1\ldots\,\bar j_\zeta\,k_1\ldots\,w_\mu\,\ldots\,k_n}.
\end{aligned}\tag{2.5}$$

The quantities $\Upsilon^s_{k_{n+1}}$ for (2.5) are taken from the expansion (2.3). In (2.5) these quantities form the so-called *Lie derivative* $L_{\boldsymbol{\Upsilon}_k}$:

$$L_{\boldsymbol{\Upsilon}_k}(f)=\sum_{s=0}^{3}\Upsilon^s_k\,\frac{\partial f}{\partial x^s}. \tag{2.6}$$

The quantities A^i_{kj}, $\bar{\mathrm{A}}^i_{kj}$, and Γ^i_{kj} determine the coordinate representation of a connection in the frame pair $(U,\,\boldsymbol{\Psi}_1,\,\ldots\,,\,\boldsymbol{\Psi}_q)$ and $(U,\,\boldsymbol{\Upsilon}_0,\,\boldsymbol{\Upsilon}_1,\,\boldsymbol{\Upsilon}_2,\,\boldsymbol{\Upsilon}_3)$.

Definition 2.1. A connection $(\Gamma,\mathrm{A},\bar{\mathrm{A}})$ of the bundle UM is called a *real connection* if the corresponding covariant differential (2.2) commute with the involution of complex conjugation τ, i. e. if $\nabla(\tau(\mathbf{X}))=\tau(\nabla\mathbf{X})$ for any tensor field $\mathbf{X}$.

In the case of a real connection $(\Gamma,\mathrm{A},\bar{\mathrm{A}})$ we have the following relationships:

$$\Gamma^i_{kj}=\overline{\Gamma^i_{kj}},\qquad\qquad \bar{\mathrm{A}}^i_{kj}=\overline{\mathrm{A}^i_{kj}}.\tag{2.7}$$

With the use of the Γ-components of a real connection we define the real quantities

$$T^k_{ij}=\Gamma^k_{ij}-\Gamma^k_{j\,i}-c^k_{ij}.\tag{2.8}$$

The quantities (2.8) are the components of a tensor $\mathbf{T}$. It is called the *torsion tensor*. If $\mathbf{T}=0$, then we say that $(\Gamma,\mathrm{A},\bar{\mathrm{A}})$ is a *torsion-free* connection or a *symmetric* connection.

Definition 2.2. A real connection $(\Gamma,\mathrm{A},\bar{\mathrm{A}})$ of the bundle UM is called *concordant with the metric tensor* $\mathbf{g}$ if $\nabla\mathbf{g}=0$. If it is symmetric, i. e. if $\mathbf{T}=0$, then such a connection is called a *metric connection*.

The Γ-components of a metric connection are uniquely determined by the metric tensor $\mathbf{g}$. In a coordinate representation we have the following formula:

$$\begin{aligned}\Gamma^k_{ij}=\sum^3_{r=0}\frac{g^{kr}}{2}\bigl(L_{\boldsymbol{\Upsilon}_i}(g_{jr})+L_{\boldsymbol{\Upsilon}_j}(g_{ri})-L_{\boldsymbol{\Upsilon}_r}(g_{ij})\bigr)-\\ -\frac{c^k_{ij}}{2}+\sum^3_{r=0}\sum^3_{s=0}g^{kr}\,\frac{c^s_{ir}}{2}\,g_{sj}+\sum^3_{r=0}\sum^3_{s=0}g^{kr}\,\frac{c^s_{jr}}{2}\,g_{si}.\end{aligned}\tag{2.9}$$

The proof of the formula (2.9) can be found in [8], while the formula (2.8) for the components of the torsion tensor $\mathbf{T}$ is taken from [9] (see formula (6.22) there).

Each connection $(\Gamma,\mathrm{A},\bar{\mathrm{A}})$ of the bundle UM produces three curvature tensors. The components of these curvature tensors are given by the following formulas:

$$R^p_{kij}=L_{\boldsymbol{\Upsilon}_i}(\Gamma^p_{j\,k})-L_{\boldsymbol{\Upsilon}_j}(\Gamma^p_{i\,k})+\sum^3_{h=0}\Bigl(\Gamma^p_{i\,h}\,\Gamma^h_{j\,k}-\Gamma^p_{j\,h}\,\Gamma^h_{i\,k}\Bigr)-\sum^3_{h=0}c^h_{ij}\,\Gamma^p_{hk},\tag{2.10}$$

$$\mathfrak{R}^p_{kij}=L_{\boldsymbol{\Upsilon}_i}(\mathrm{A}^p_{j\,k})-L_{\boldsymbol{\Upsilon}_j}(\mathrm{A}^p_{i\,k})+\sum^q_{h=1}\Bigl(\mathrm{A}^p_{i\,h}\;\mathrm{A}^h_{j\,k}-\mathrm{A}^p_{j\,h}\;\mathrm{A}^h_{i\,k}\Bigr)-\sum^3_{h=0}c^h_{ij}\;\mathrm{A}^p_{hk},\tag{2.11}$$

$$\bar{\mathfrak{R}}^p_{kij}=L_{\boldsymbol{\Upsilon}_i}(\bar{\mathrm{A}}^p_{j\,k})-L_{\boldsymbol{\Upsilon}_j}(\bar{\mathrm{A}}^p_{i\,k})+\sum^q_{h=1}\Bigl(\bar{\mathrm{A}}^p_{i\,h}\,\bar{\mathrm{A}}^h_{j\,k}-\bar{\mathrm{A}}^p_{j\,h}\,\bar{\mathrm{A}}^h_{i\,k}\Bigr)-\sum^3_{h=0}c^h_{ij}\,\bar{\mathrm{A}}^p_{hk}.\tag{2.12}$$

In the case of a real connection $(\Gamma,\mathrm{A},\bar{\mathrm{A}})$ the first curvature tensor (2.10) is a real tensor field, i. e. its components are real functions:

$$R^p_{kij}=\overline{R^p_{kij}}.\tag{2.13}$$

Similarly, in the case of a real connection $(\Gamma, \mathrm{A}, \bar{\mathrm{A}})$ the third curvature tensor (2.12) is expressed through the second curvature tensor (2.11):

$$\bar{\mathfrak{R}}^{p}_{kij} = \overline{\mathfrak{R}^{p}_{kij}}. \tag{2.14}$$

The formulas (2.13) and (2.14) are easily derived from (2.7) since the parameters c^k_{ij} introduced by the relationships (2.4) are real-valued functions.

3. The U(1)-connections.

Having resumed some facts from the general theory, now we return back to three special bundles considered in section 1. Let's begin with the bundle UM. A metric connection for this bundle is denoted by $(\Gamma, \mathrm{A}, \bar{\mathrm{A}})$. The Γ-components of this connection are given by the formula (2.9).

Definition 3.1. A real connection $(\Gamma, \mathrm{A}, \bar{\mathrm{A}})$ of the bundle UM is called *concordant* with the Hermitian metric tensor $\mathbf{D}$ if $\nabla\mathbf{D} = 0$.

In a coordinate form the concordance condition $\nabla\mathbf{D} = 0$ is written as

$$L_{\boldsymbol{\Upsilon}_k}(D_{11}) - D_{11}\,\mathrm{A}^1_{k1} - D_{11}\,\bar{\mathrm{A}}^1_{k1} = 0 \tag{3.1}$$

(see (2.5)). The equality (3.1) does not fix the A-components of the connection $(\Gamma, \mathrm{A}, \bar{\mathrm{A}})$. It is equivalent to the following formula for their real parts:

$$\mathrm{Re}(\mathrm{A}^1_{k1}) = \frac{L_{\boldsymbol{\Upsilon}_k}(D_{11})}{2\,D_{11}}. \tag{3.2}$$

Their imaginary parts are not fixed at all. Assume that $(U, \boldsymbol{\Psi}_1)$ is an orthonormal frame in the sense of the definition 1.2. Then (3.2) simplifies to $\mathrm{Re}(\mathrm{A}^1_{k1}) = 0$.

Let $(U, \boldsymbol{\Psi}_1)$ and $(\tilde{U}, \tilde{\boldsymbol{\Psi}}_1)$ be two orthonormal frames in the sense of the definition 1.2 and assume that $U \cap \tilde{U} \neq \varnothing$. Then at each point $p \in U \cap \tilde{U}$ we have

$$\tilde{\boldsymbol{\Psi}}_1 = e^{i\phi}\,\boldsymbol{\Psi}_1. \tag{3.3}$$

If $\boldsymbol{\psi}$ is a wave-function being a vector with respect to the bundle UM, then

$$\boldsymbol{\psi} = \psi^1\,\boldsymbol{\Psi}_1, \qquad\qquad \boldsymbol{\psi} = \tilde{\psi}^1\,\tilde{\boldsymbol{\Psi}}_1. \tag{3.4}$$

From (3.3) and (3.4) we derive the following formula for the coefficients ψ^1 and $\tilde{\psi}^1$:

$$\psi^1 = e^{i\phi}\,\tilde{\psi}^1. \tag{3.5}$$

Under the frame transformation (3.3) the connection components are transformed according to the formula similar to (19.8) in [10]:

$$\mathrm{A}^1_{k1} = \tilde{\mathrm{A}}^1_{k1} - i\,L_{\boldsymbol{\Upsilon}_k}(\phi) \tag{3.6}$$

The equality (3.6) is an analog of the gauge transformation for the vector-potential of the electromagnetic field (see [11]).

Now let's apply the general formula (2.11) to the connection $(\Gamma, \mathbb{A}, \bar{\mathbb{A}})$ given by the formula (3.3). As a result we get the curvature tensor with the components

$$\mathfrak{R}^1_{1ij} = L_{\mathbf{\Upsilon}_i}(\mathbb{A}^1_{j1}) - L_{\mathbf{\Upsilon}_j}(\mathbb{A}^1_{i1}) - \sum_{h=0}^{3} c^h_{ij}\,\mathbb{A}^1_{h1}. \tag{3.7}$$

It is the feature of one-dimensional bundles that the upper and lower indices do cancel each other (a contraction is performed without summation). Therefore, the quantities (3.7) can be understood as the components of a purely spacial tensor:

$$\mathfrak{R}_{ij} = L_{\mathbf{\Upsilon}_i}(\mathbb{A}^1_{j1}) - L_{\mathbf{\Upsilon}_j}(\mathbb{A}^1_{i1}) - \sum_{h=0}^{3} c^h_{ij}\,\mathbb{A}^1_{h1}. \tag{3.8}$$

The quantities (3.8) are the components of a physical field corresponding to the gauge field with the components $\mathbb{A}^1_{i1}$.

4. The SU(2)-connections.

Let $(\Gamma, \mathbb{A}, \bar{\mathbb{A}})$ be a metric connection for the bundle SUM. Its Γ-components are given by the formula (2.9) like in the previous case of metric U(1)-connections.

Definition 4.1. A real connection $(\Gamma, \mathbb{A}, \bar{\mathbb{A}})$ of the bundle SUM is called *concordant* with the Hermitian metric tensor $\mathbf{D}$ and with the skew-symmetric metric tensor $\mathbf{d}$ if $\nabla\mathbf{D} = 0$ and if $\nabla\mathbf{d} = 0$.

In a coordinate form the conditions $\nabla\mathbf{D} = 0$ and $\nabla\mathbf{d} = 0$ are written as

$$L_{\mathbf{\Upsilon}_k}(D_{i\bar{j}}) - \sum_{a=1}^{2} D_{a\bar{j}}\,\mathbb{A}^a_{ki} - \sum_{\bar{a}=1}^{2} D_{i\bar{a}}\,\bar{\mathbb{A}}^{\bar{a}}_{k\bar{j}} = 0, \tag{4.1}$$

$$L_{\mathbf{\Upsilon}_k}(d_{ij}) - \sum_{a=1}^{2} d_{aj}\,\mathbb{A}^a_{ki} - \sum_{a=1}^{2} d_{ia}\,\mathbb{A}^a_{kj} = 0 \tag{4.2}$$

(see formula (2.5)). Let's write the concordance conditions (4.1) and (4.2) in some orthonormal frame $(U,\ \mathbf{\Psi}_1,\ \mathbf{\Psi}_2)$ (see definition 1.3):

$$\overline{\mathbb{A}^i_{kj}} = -\mathbb{A}^j_{ki}, \qquad\qquad \sum_{i=1}^{2} \mathbb{A}^i_{ki} = 0. \tag{4.3}$$

The equalities (4.3) mean that for each fixed k the connection components $\mathbb{A}^i_{kj}$ are represented by skew-Hermitian traceless matrices. Such matrices compose the Lie algebra su(2) associated with the Lie group SU(2).

Let $(U,\ \mathbf{\Psi}_1,\ \mathbf{\Psi}_2)$ and $(\tilde{U},\ \tilde{\mathbf{\Psi}}_1,\ \tilde{\mathbf{\Psi}}_2)$ be two orthonormal frames in the sense of the definition 1.3. Assume that $U \cap \tilde{U} \neq \varnothing$. Then at each point $p \in U \cap \tilde{U}$ we have

$$\tilde{\mathbf{\Psi}}_i = \sum_{j=1}^{2} \mathfrak{S}^j_i\,\mathbf{\Psi}_j. \tag{4.4}$$

If $\boldsymbol{\psi}$ is a wave-function being a vector with respect to the bundle SUM, then it can be expanded in each of the above two frames:

$$\boldsymbol{\psi}=\sum_{i=1}^{2}\psi^i\,\boldsymbol{\Psi}_i,\qquad\qquad \boldsymbol{\psi}=\sum_{i=1}^{2}\tilde\psi^i\,\tilde{\boldsymbol{\Psi}}_i. \tag{4.5}$$

From (4.4) and (4.5) we derive the following formula for the coefficients ψ^i and $\tilde\psi^i$:

$$\psi^i=\sum_{j=1}^{2}\mathfrak{S}^i_j\;\tilde\psi^j. \tag{4.6}$$

Under the frame transformation (4.4) the connection components $\mathbb{A}^i_{kj}$ are transformed according to the formula similar to (19.8) in [10]:

$$\mathbb{A}^i_{kj}=\sum_{b=1}^{2}\sum_{a=1}^{2}\mathfrak{S}^i_a\,\mathfrak{T}^b_j\;\tilde{\mathbb{A}}^a_{k\,b}+\vartheta^i_{kj}. \tag{4.7}$$

The theta-parameters ϑ^i_{kj} in (4.7) are defined by the formula

$$\vartheta^i_{kj}=\sum_{a=1}^{2}\mathfrak{S}^i_a\,L_{\boldsymbol{\Upsilon}_k}(\mathfrak{T}^a_j)=-\sum_{a=1}^{2}L_{\boldsymbol{\Upsilon}_k}(\mathfrak{S}^i_a)\,\mathfrak{T}^a_j. \tag{4.8}$$

(see (9.31) in [10]). By $\mathfrak{T}^i_j$ in (4.7) and (4.8) we denote the components of the inverse matrix $\mathfrak{T}=\mathfrak{S}^{-1}$, while $L_{\boldsymbol{\Upsilon}_k}$ is the Lie derivative introduced in (2.6). Note that $\mathfrak{S}$ and $\mathfrak{T}$ both are special unitary matrices, i. e. $\mathfrak{S}\in\mathrm{SU}(2)$ and $\mathfrak{T}\in\mathrm{SU}(2)$.

The curvature tensor for a SU(2)-connection $(\Gamma,\mathbb{A},\bar{\mathbb{A}})$ is given by the general formula (2.11) upon substituting $\mathrm{A}^i_{kj}=\mathbb{A}^i_{kj}$ into this formula:

$$\mathfrak{R}^p_{kij}=L_{\boldsymbol{\Upsilon}_i}(\mathbb{A}^p_{j\,k})-L_{\boldsymbol{\Upsilon}_j}(\mathbb{A}^p_{i\,k})+\sum_{h=1}^{2}\Bigl(\mathbb{A}^p_{i\,h}\,\mathbb{A}^h_{j\,k}-\mathbb{A}^p_{j\,h}\,\mathbb{A}^h_{i\,k}\Bigr)-\sum_{h=0}^{3}c^{\,h}_{ij}\,\mathbb{A}^p_{hk}. \tag{4.9}$$

The quantities (4.9) are the components of a physical field corresponding to the gauge field with the components $\mathbb{A}^i_{kj}$.

5. The SU(3)-connections.

The SU(3)-connections correspond to the Quantum Chromodynamics which now is a part of the Standard Model. They are associated with the third special bundle SUM. Let $(\Gamma,\mathbb{A},\bar{\mathbb{A}})$ be a metric connection for the bundle SUM. Its Γ-components are given by the formula (2.9), while $\mathbb{A}$-components are interpreted as gluon fields.

Definition 5.1. A real connection $(\Gamma,\mathbb{A},\bar{\mathbb{A}})$ of the bundle SUM is called *concordant* with the Hermitian metric tensor $\mathbf{D}$ and with the skew-symmetric metric tensor $\mathbf{d}$ if $\nabla\mathbf{D}=0$ and if $\nabla\mathbf{d}=0$.

In a coordinate form the conditions $\nabla\mathbf{D}=0$ and $\nabla\mathbf{d}=0$ are written as

$$L_{\boldsymbol{\Upsilon}_k}(D_{i\bar j})-\sum_{a=1}^{3}D_{a\bar j}\,\mathbb{A}^a_{ki}-\sum_{\bar a=1}^{3}D_{i\bar a}\,\bar{\mathbb{A}}^{\bar a}_{k\bar j}=0, \tag{5.1}$$

$$L_{\boldsymbol{\Upsilon}_s}(d_{ijk}) - \sum_{a=1}^{3} d_{ajk}\,\mathbb{A}^a_{si} - \sum_{a=1}^{3} d_{iak}\,\mathbb{A}^a_{sj} - \sum_{a=1}^{3} d_{ija}\,\mathbb{A}^a_{sk} = 0. \tag{5.2}$$

(see formula (2.5)). Let's write the concordance conditions (5.1) and (5.2) in some orthonormal frame $(U,\ \boldsymbol{\Psi}_1,\ \boldsymbol{\Psi}_2,\ \boldsymbol{\Psi}_3)$ (see definition 1.4):

$$\overline{\mathbb{A}^i_{kj}} = -\mathbb{A}^j_{ki}, \qquad\qquad \sum_{i=1}^{3}\mathbb{A}^i_{ki} = 0. \tag{5.3}$$

In deriving the second formula (5.3) we used the following two well-known identities for a nonzero completely skew-symmetric tensor in a three-dimensional space:

$$\sum_{k=1}^{3} d_{ijk}\, d^{abk} = \delta^a_i\,\delta^b_j - \delta^a_j\,\delta^b_i, \qquad\qquad \sum_{j=1}^{3}\sum_{k=1}^{3} d_{ijk}\, d^{ajk} = 2\,\delta^a_i.$$

The equalities (5.3) mean that for each fixed k the connection components $\mathbb{A}^i_{kj}$ are represented by skew-Hermitian traceless matrices. Such matrices compose the Lie algebra su(3) associated with the Lie group SU(3).

Let $(U,\ \boldsymbol{\Psi}_1,\ \boldsymbol{\Psi}_2,\ \boldsymbol{\Psi}_3)$ and $(\tilde U,\ \tilde{\boldsymbol{\Psi}}_1,\ \tilde{\boldsymbol{\Psi}}_2,\ \tilde{\boldsymbol{\Psi}}_3)$ be two orthonormal frames in the sense of the definition 1.4. Assume that $U\cap\tilde U\neq\varnothing$. Then for each point p in the intersection of the domains U and $\tilde U$ we have

$$\tilde{\boldsymbol{\Psi}}_i = \sum_{j=1}^{3}\mathfrak{S}^j_i\,\boldsymbol{\Psi}_j. \tag{5.4}$$

If $\boldsymbol{\psi}$ is a wave-function being a vector with respect to the bundle $S\mathbb{U}M$, then it can be expanded in each of the above two frames:

$$\boldsymbol{\psi} = \sum_{i=1}^{3}\psi^i\,\boldsymbol{\Psi}_i, \qquad\qquad \boldsymbol{\psi} = \sum_{i=1}^{3}\tilde\psi^i\,\tilde{\boldsymbol{\Psi}}_i. \tag{5.5}$$

From (5.4) and (5.5) we derive the following formula for the coefficients ψ^i and $\tilde\psi^i$:

$$\psi^i = \sum_{j=1}^{3}\mathfrak{S}^i_j\ \tilde\psi^j. \tag{5.6}$$

Under the frame transformation (5.4) the connection components $\mathbb{A}^i_{kj}$ are transformed according to the formula similar to (4.7):

$$\mathbb{A}^i_{kj} = \sum_{b=1}^{3}\sum_{a=1}^{3}\mathfrak{S}^i_a\,\mathfrak{T}^b_j\ \tilde{\mathbb{A}}^a_{k\,b} + \vartheta^i_{kj}. \tag{5.7}$$

The theta-parameters ϑ^i_{kj} in (5.7) are defined by the formula similar to (4.8):

$$\vartheta^i_{kj} = \sum_{a=1}^{3}\mathfrak{S}^i_a\,L_{\boldsymbol{\Upsilon}_k}(\mathfrak{T}^a_j) = -\sum_{a=1}^{3}L_{\boldsymbol{\Upsilon}_k}(\mathfrak{S}^i_a)\,\mathfrak{T}^a_j. \tag{5.8}$$

By $\mathfrak{T}^i_j$ in (5.7) and (5.8) we again denote the components of the inverse matrix $\mathfrak{T} = \mathfrak{S}^{-1}$, while $L_{\boldsymbol{\Upsilon}_k}$ is the Lie derivative (see (2.6)). As for $\mathfrak{S}$ and $\mathfrak{T}$, they both are special unitary matrices, i. e. $\mathfrak{S} \in \mathrm{SU}(3)$ and $\mathfrak{T} \in \mathrm{SU}(3)$.

The curvature tensor for a SU(3)-connection $(\Gamma, \mathbb{A}, \bar{\mathbb{A}})$ is given by the general formula (2.11) upon substituting $\mathrm{A}^i_{kj} = \mathbb{A}^i_{kj}$ into this formula:

$$\mathfrak{R}^p_{kij} = L_{\boldsymbol{\Upsilon}_i}(\mathbb{A}^p_{j\,k}) - L_{\boldsymbol{\Upsilon}_j}(\mathbb{A}^p_{i\,k}) + \sum_{h=1}^{3} \left(\mathbb{A}^p_{i\,h}\,\mathbb{A}^h_{j\,k} - \mathbb{A}^p_{j\,h}\,\mathbb{A}^h_{i\,k}\right) - \sum_{h=0}^{3} c^h_{ij}\,\mathbb{A}^p_{hk}. \tag{5.9}$$

The quantities (5.9) are the components of a physical field corresponding to the gauge field with the components $\mathbb{A}^i_{kj}$.

6. Concluding remarks.

Note that the three different bundles UM, SUM, and $S\mathbb{U}M$ are treated above in very similar ways, especially the last two bundles. This means that crucial differences of these bundles are hidden deeper within the Standard Model. They will be studied in a separate paper.

The formulas (3.5) and (3.6) yield a gauge transformation in the case of the bundle UM. Note that this gauge transformation arises as a frame transformation (3.3). The same is true for the gauge transformations (4.6), (4.7) and (5.6), (5.7) in the case of the bundles SUM and $S\mathbb{U}M$. They are initiated by the frame transformations (4.4) and (5.4) respectively. Remember that Lorentz transformations and their spinor companions, including P and T-reflections, were interpreted as frame transformations in [12]. Using the bundles UM, SUM, and $S\mathbb{U}M$, now we do the same for gauge transformations associated with U(1), SU(2) and SU(3) symmetries of the Standard Model in a gravitation field. This is the main result of this paper.

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