

ON THE DIRAC EQUATION IN A GRAVITATION FIELD AND THE SECONDARY QUANTIZATION.

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ABSTRACT. The Dirac equation for massive free electrically neutral spin 1/2 particles in a gravitation field is considered. The secondary quantization procedure is applied to it and the Hilbert space of multiparticle quantum states is constructed.

1. THE DIRAC EQUATION AND ITS CURRENT.

Let M be a *space-time* manifold. It is a four-dimensional orientable manifold equipped with a pseudo-Euclidean Minkowski-type metric \mathbf{g} and with a *polarization*. The polarization of M is responsible for distinguishing the *Future light cone* from the *Past light cone* at each point $p \in M$ (see [1] for more details). Let's denote by DM the bundle of Dirac spinors over M (see [2] and [3] for detailed description of this bundle). In addition to the metric tensor \mathbf{g} inherited from M , the Dirac bundle DM is equipped with four other basic spin-tensorial fields:

Symbol	Name	Spin-tensorial type	
\mathbf{g}	Metric tensor	$(0, 0 0, 0 0, 2)$	
\mathbf{d}	Skew-symmetric metric tensor	$(0, 2 0, 0 0, 0)$	(1.1)
\mathbf{H}	Chirality operator	$(1, 1 0, 0 0, 0)$	
\mathbf{D}	Dirac form	$(0, 1 0, 1 0, 0)$	
γ	Dirac γ -field	$(1, 1 0, 0 1, 0)$	

As we see in the table (1.1), the metric tensor \mathbf{g} is interpreted as a spin-tensorial field of the type $(0, 0|0, 0|0, 2)$. The Dirac bundle is a complex bundle over a real manifold. For this reason spin-tensorial bundles produced from DM are equipped with the involution of complex conjugation τ :

$$D_{\beta}^{\alpha} \bar{D}_{\gamma}^{\nu} T_n^m M \xrightleftharpoons[\tau]{\tau} D_{\gamma}^{\nu} \bar{D}_{\beta}^{\alpha} T_n^m M. \quad (1.2)$$

Note that two fields \mathbf{g} and \mathbf{D} in (1.1) are real fields:

$$\tau(\mathbf{g}) = \mathbf{g}, \quad \tau(\mathbf{D}) = \mathbf{D}.$$

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Other fields \mathbf{d} , \mathbf{H} , and γ in the table (1.1) are not real spin-tensorial fields.

Definition 1.1. A metric connection (Γ, A, \bar{A}) in DM is a spinor connection real in the sense of the involution (1.2) and concordant with \mathbf{d} and γ , i. e.

$$\nabla \mathbf{d} = 0, \quad \nabla \gamma = 0, \quad \text{and} \quad \tau(\nabla \mathbf{X}) = \nabla(\tau(\mathbf{X})), \quad (1.3)$$

where \mathbf{X} is an arbitrary smooth spin-tensorial field of the Dirac bundle.

Theorem 1.1. Any metric connection (Γ, A, \bar{A}) is concordant with all of the basic spin tensorial fields \mathbf{g} , \mathbf{d} , \mathbf{H} , \mathbf{D} , and γ listed in the table (1.1).

The theorem 1.1 means that from (1.3) it follows that

$$\nabla \mathbf{g} = 0, \quad \nabla \mathbf{H} = 0, \quad \nabla \mathbf{D} = 0. \quad (1.4)$$

Applying the last identity (1.3) to (1.4) and to other identities (1.3), we derive

$$\nabla \bar{\gamma} = 0, \quad \nabla \bar{\mathbf{d}} = 0, \quad \nabla \bar{\mathbf{H}} = 0.$$

The general relativity (the Einstein's theory of gravity) is a theory with zero torsion. Exactly for this case we have the following theorem.

Theorem 1.2. There is a unique metric connection (Γ, A, \bar{A}) of the bundle of Dirac spinors DM whose torsion \mathbf{T} is zero.

The metric connection with zero torsion $\mathbf{T} = 0$ is called the *Levi-Civita* connection. The proof of both theorems 1.1 and 1.2 as well as some explicit formulas for the components of the Levi-Civita connection can be found in [3].

A massive spin 1/2 particle is described by a wave-function which is a smooth spinor field ψ . In order to get a coordinate representation of this field we choose two frames $(U, \Upsilon_0, \Upsilon_1, \Upsilon_2, \Upsilon_3)$ and $(U, \Psi_1, \Psi_2, \Psi_3, \Psi_4)$ with common domain U . The first of these two frames is given by four smooth vector fields $\Upsilon_0, \Upsilon_1, \Upsilon_2,$ and Υ_3 linearly independent at each point $p \in U$. The other frame is formed by four spinor fields $\Psi_1, \Psi_2, \Psi_3, \Psi_4$ also linearly independent at each point $p \in U$. Having these two frames and taking their dual and Hermitian conjugate frames, one easily get the coordinate representation for an arbitrary spin-tensorial field. Using the components of the wave function ψ in the frame $(U, \Psi_1, \Psi_2, \Psi_3, \Psi_4)$ we write the following action integral for this field:

$$\begin{aligned} S = i \hbar \int \sum_{a=1}^4 \sum_{\bar{a}=1}^4 \sum_{b=1}^4 \sum_{q=0}^3 D_{a\bar{a}} \gamma_b^{aq} \frac{\bar{\psi}^{\bar{a}} \nabla_q \psi^b - \psi^b \nabla_q \bar{\psi}^{\bar{a}}}{2} dV - \\ - m c \int \sum_{a=1}^4 \sum_{\bar{a}=1}^4 D_{a\bar{a}} \bar{\psi}^{\bar{a}} \psi^a dV. \end{aligned} \quad (1.5)$$

Through \hbar in (1.5) we denote the Planck constant¹, while c is the speed of light:

$$\hbar \approx 1.05457168 \cdot 10^{-27} \text{ erg} \cdot \text{sec}, \quad c \approx 2.99792458 \cdot 10^{10} \text{ cm/sec}.$$

¹ These data are taken from the NIST site <http://physics.nist.gov/cuu/Constants>.

The constant m in (1.5) is the mass of a particle. By dV in (1.5) we denote the 4-dimensional volume element induced by the metric \mathbf{g} . In local coordinates x^0, x^1, x^2, x^3 within the domain $U \subset M$ it is written as follows:

$$dV = \sqrt{-\det \mathbf{g}} d^4x = \sqrt{-\det \mathbf{g}} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3. \quad (1.6)$$

Though $D_{a\bar{a}}, \gamma_b^{an}, \psi^a$, and ψ^b are the components of complex fields, the integral (1.5) is a real quantity. This fact is proved with the use of the identities

$$\overline{D_{a\bar{a}}} = D_{\bar{a}a}, \quad \sum_{a=1}^4 D_{a\bar{a}} \gamma_b^{aq} = \sum_{\bar{s}=1}^4 D_{\bar{b}\bar{s}} \overline{\gamma_{\bar{a}}^{s\bar{q}}}. \quad (1.7)$$

Then remember that the metric connection is a real connection. Therefore, we have

$$\nabla_q \overline{\psi^{\bar{a}}} = \overline{\nabla_q \psi^{\bar{a}}}. \quad (1.8)$$

Taking into account (1.3), (1.4), (1.7), and (1.8), one can easily derive $\overline{S} = S$ for the action integral (1.5).

Applying the extremal action principle to the action integral (1.5), we derive the following differential equation for the components of the spinor wave-function ψ :

$$i \hbar \sum_{b=1}^4 \sum_{q=0}^3 \gamma_b^{aq} \nabla_q \psi^b - m c \psi^a = 0. \quad (1.9)$$

The equation (1.9) is the *Dirac equation* for a spin 1/2 particle with the rest mass m . Conservation laws for relativistic field equations are formulated in terms of currents. The vector-field \mathbf{J} with the components

$$J^q = c \sum_{a=1}^4 \sum_{\bar{a}=1}^4 \sum_{b=1}^4 D_{a\bar{a}} \gamma_b^{aq} \overline{\psi^{\bar{a}}} \psi^b \quad (1.10)$$

is a current for the Dirac equation (1.9). The conservation law is written as

$$\operatorname{div} \mathbf{J} = \sum_{q=0}^3 \nabla_q J^q = 0. \quad (1.11)$$

In the case of the current (1.10) the conservation law (1.11) is derived from the Dirac equation (1.9) with the use of the identities (1.3) and (1.4).

The Dirac current (1.10) is a real vector-field: $\tau(\mathbf{J}) = \mathbf{J}$. Indeed, using the identities (1.7) and applying them to (1.10), one easily derives

$$\overline{J^q} = J^q.$$

Moreover, the vector-field \mathbf{J} is composed by time-like vectors. In order to prove this fact we calculate the following quantity:

$$g(\mathbf{J}, \mathbf{J}) = \sum_{p=0}^3 \sum_{q=0}^3 g_{pq} J^p J^q. \quad (1.12)$$

The quantity $g(\mathbf{J}, \mathbf{J})$ in the left hand side of (1.12) is a scalar invariant of the vector \mathbf{J} . Its value does not depend on a frame choice. Let's assume for a while that $(U, \Upsilon_0, \Upsilon_1, \Upsilon_2, \Upsilon_3)$ and $(U, \Psi_1, \Psi_2, \Psi_3, \Psi_4)$ form a canonically associated frame pair such that $(U, \Upsilon_0, \Upsilon_1, \Upsilon_2, \Upsilon_3)$ is a positively polarized right orthonormal frame in TM and $(U, \Psi_1, \Psi_2, \Psi_3, \Psi_4)$ is a canonically orthonormal chiral frame in DM (see the diagram (5.12) in [3] for more details). In such a frame pair the Dirac γ -field is represented by the following standard Dirac matrices:

$$\begin{aligned} \gamma_b^{a0} &= \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix}, & \gamma_b^{a1} &= \begin{vmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{vmatrix}, \\ \gamma_b^{a2} &= \begin{vmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{vmatrix}, & \gamma_b^{a3} &= \begin{vmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{vmatrix}. \end{aligned} \quad (1.13)$$

Here a stands for a row number, while b is a column number. For the chirality operator and the Dirac form in such a frame pair we have

$$H_j^i = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix}, \quad D_{i\bar{j}} = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix}. \quad (1.14)$$

For the metric tensor \mathbf{g} and the spin-metric tensor \mathbf{d} in such a frame pair we have

$$g_{ij} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix}, \quad d_{ij} = \begin{vmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{vmatrix}. \quad (1.15)$$

A remark. Note that the matrix γ^0 in (1.13) coincides with the matrix D in (1.14). For this reason in many books γ^0 is used instead of D (see § 21 in [4], see section 14.1 in [5], see section 5.4 in [6], see section 2.5 in [7], and see section 6.3 in [8]). This usage contradicts the spin-tensorial nature of the fields γ and \mathbf{D} because the formulas (1.13), (1.14), (1.15) and the equality $\gamma^0 = D$ are highly frame-specific. I think the use of this equality without indicating explicitly its restricted scope is misleading for many generations of readers of the above very famous books.

Returning back to the formula (1.10) and applying the formulas (1.13) and (1.14) to it, we derive the following formulas for the current components:

$$J^0 = c^2(\psi^1 \overline{\psi^1} + \psi^2 \overline{\psi^2} + \psi^3 \overline{\psi^3} + \psi^4 \overline{\psi^4}), \quad (1.16)$$

$$J^1 = c^2(\psi^1 \overline{\psi^2} + \psi^2 \overline{\psi^1} - \psi^3 \overline{\psi^4} - \psi^4 \overline{\psi^3}),$$

$$J^2 = i c^2(\psi^1 \overline{\psi^2} - \psi^2 \overline{\psi^1} - \psi^3 \overline{\psi^4} + \psi^4 \overline{\psi^3}), \quad (1.17)$$

$$J^3 = c^2(\psi^1 \overline{\psi^1} - \psi^2 \overline{\psi^2} - \psi^3 \overline{\psi^3} + \psi^4 \overline{\psi^4}).$$

Substituting (1.16) and (1.17) into (1.12), we derive

$$g(\mathbf{J}, \mathbf{J}) = 4c^2 (\psi^1 \bar{\psi}^1 \psi^3 \bar{\psi}^3 + \psi^2 \bar{\psi}^2 \psi^4 \bar{\psi}^4 + \psi^1 \bar{\psi}^2 \psi^4 \bar{\psi}^3 + \psi^2 \bar{\psi}^1 \psi^3 \bar{\psi}^4). \quad (1.18)$$

Using the well-know inequality $2|xy| \leq |x|^2 + |y|^2$, we get the following inequality:

$$|\psi^1 \bar{\psi}^2 \psi^4 \bar{\psi}^3 + \psi^2 \bar{\psi}^1 \psi^3 \bar{\psi}^4| \leq 2|\psi^1 \psi^2 \psi^3 \psi^4| \leq |\psi^1 \psi^3|^2 + |\psi^2 \psi^4|^2.$$

Applying this inequality to (1.18), we see that $g(\mathbf{J}, \mathbf{J}) \geq 0$. This means that \mathbf{J} is a time-like vector. Moreover, in (1.16) we see that $J^0 \geq 0$. If we remember that for deriving (1.16) and (1.17) we took a positively polarized right orthonormal frame $(U, \Psi_1, \Psi_2, \Psi_3, \Psi_4)$ in SM , then from $J^0 \geq 0$ we conclude that \mathbf{J} is a time-like vector from the interior of a Future light cone. Thus, we have proved the following well-known theorem.

Theorem 1.3. *The Dirac current (1.10) for a massive spin 1/2 particle is a time-like vector-field directed to the Future.*

2. NORMALIZATION CONDITION FOR A SINGLE PARTICLE WAVE FUNCTION.

Let's remember that wave functions in quantum mechanics are usually normalized. In the non-relativistic theory scalar wave functions of bound states are normalized to unity by means of the following integral (see [9]):

$$\int |\psi|^2 d^3x = 1. \quad (2.1)$$

In the case of a Dirac particle in a non-flat space-time M the integral (2.1) is senseless since there is no predefined 3-dimensional submanifold in M . However, using

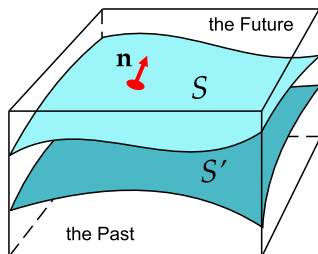


Fig. 2.1

the Dirac current (1.10) one can give a new sense to integrals like (2.1). Let S be some arbitrary space-like hypersurface in M . A part of such a hypersurface S enclosed into a space-time cube is shown on Fig. 2.1. At each point of S there is a unique unit normal vector \mathbf{n} directed to the Future. Then the equality

$$\int_S g(\mathbf{J}, \mathbf{n}) dS = 1 \quad (2.2)$$

is a proper normalization condition for the wave-function of a single massive spin 1/2 particle. By dS in (2.2) we denote the 3-dimensional area element determined by the metric induced from M to S . In local coordinates u^1, u^2, u^3 of S it is given by a formula similar to (1.6):

$$dS = \sqrt{-\det \mathbf{g}} d^3u = \sqrt{-\det \mathbf{g}} du^1 \wedge du^2 \wedge du^3.$$

The quantity $g(\mathbf{J}, \mathbf{n})$ integrated in (2.2) is the scalar product of the Dirac current and the unit normal vector of S calculated in the Minkowski metric \mathbf{g} :

$$g(\mathbf{J}, \mathbf{n}) = \sum_{i=0}^3 \sum_{j=0}^3 g_{ij} J^i n^j. \quad (2.3)$$

Since both \mathbf{J} and \mathbf{n} are time-like vectors directed to the Future, the scalar product (2.3) is a positive quantity.

Assume that $\psi^{\bar{a}}$ and ψ^b in (1.10) are the components of the wave function satisfying the Dirac equation (1.9). Then the components of the current \mathbf{J} satisfy the differential equation (1.11). Assume that S' is some other space-like hypersurface in M such that $S - S'$ is the boundary for some domain Ω (see Fig. 2.1):

$$S - S' = \partial\Omega. \quad (2.4)$$

In this case from (1.11) and (2.4) we derive

$$\int_S g(\mathbf{J}, \mathbf{n}) dS - \int_{S'} g(\mathbf{J}, \mathbf{n}) dS = \int_{\Omega} \operatorname{div} \mathbf{J} dV = 0. \quad (2.5)$$

The equality (2.5) means that the choice of the hypersurface S in the normalization condition (2.2) is inessential. This normalization condition is preserved in time dynamics given by the Dirac equation (1.9).

Let φ and ψ be two different wave functions corresponding to different quantum states of a massive spin 1/2 particle. They both satisfy the Dirac equation (1.9). By analogy to (1.10) we define the current $\mathbf{J}(\varphi, \psi)$ with the following components:

$$J^a(\varphi, \psi) = c \sum_{\bar{a}=1}^4 \sum_{\bar{b}=1}^4 \sum_{b=1}^4 D_{\bar{a}\bar{b}} \gamma_b^{a\bar{a}} \overline{\varphi^{\bar{a}}} \psi^b. \quad (2.6)$$

Like the initial Dirac current (1.10), this current (2.6) satisfies the differential equation (1.11). Therefore, relying on (2.4) and (2.5), we define the pairing

$$\langle \varphi | \psi \rangle = \int_S g(\mathbf{J}(\varphi, \psi), \mathbf{n}) dS. \quad (2.7)$$

The pairing (2.7) is a Hermitian pairing in the sense of the following equality:

$$\langle \varphi | \psi \rangle = \overline{\langle \psi | \varphi \rangle}. \quad (2.8)$$

Due to the theorem 1.3 it is a positive pairing:

$$\langle \psi | \psi \rangle = \|\psi\|^2 \geq 0. \quad (2.9)$$

Moreover, if $\|\psi\| = 0$, then $\psi = 0$ almost everywhere on the hypersurface S in the sense of the 3-dimensional Lebesgue measure on S .

The pairing (2.7) is preserved in time dynamics determined by the Dirac equation (1.9). Due to (2.8) and (2.9) it defines the Hilbert space of quantum states of a single massive spin 1/2 particle. We denote it \mathcal{H}_1 .

3. MULTIPARTICLE WAVE-FUNCTIONS.

Let $\psi_{[0]}(p)$, $\psi_{[1]}(p)$, $\psi_{[2]}(p)$, $\psi_{[3]}(p)$, \dots , where $p \in M$, be a series of single particle wave-functions forming an orthonormal basis in the Hilbert space \mathcal{H}_1 :

$$\langle \psi_{[i]} | \psi_{[j]} \rangle = \delta_{[ij]}. \quad (3.1)$$

Note that in (3.1) the indices are enclosed into the square brackets. This is done in order to distinguish them from tensorial and spin-tensorial indices enumerating the components of wave-functions. Let $p_{[1]}, \dots, p_{[n]}$ be n points of the space-time symbolizing the positions of n particles. Then a multiparticle wave-function can be constructed as a product of single particle wave-functions:

$$\psi_{[i_1]}(p_{[1]}) \otimes \dots \otimes \psi_{[i_n]}(p_{[n]}). \quad (3.2)$$

In a coordinate form, i. e. upon choosing some frame pair $(U, \Upsilon_0, \Upsilon_1, \Upsilon_2, \Upsilon_3)$ and $(U, \Psi_1, \Psi_2, \Psi_3, \Psi_4)$, the wave-function (3.2) is represented as

$$\psi_{[i_1]}^{b_1}(p_{[1]}) \cdot \dots \cdot \psi_{[i_n]}^{b_n}(p_{[n]}).$$

Though the wave-function (3.2) is a tensor product of n spin-tensorial fields of the type $(1, 0|0, 0|0, 0)$, it is not a spin-tensorial field itself since the multiplicands are spin-tensors at n different points $p_{[1]}, \dots, p_{[n]}$. The wave-function (3.2) satisfies the Dirac equation (1.9) with respect to each its argument p_s :

$$\sum_{b_s=1}^4 \left(i \hbar \sum_{q=0}^3 \gamma_{b_s}^{aq}(p_s) \nabla_q^{[s]} - m c \delta_{b_s}^a \right) \psi_{[i_1]}^{b_1}(p_{[1]}) \cdot \dots \cdot \psi_{[i_s]}^{b_s}(p_{[s]}) \cdot \dots \cdot \psi_{[i_n]}^{b_n}(p_{[n]}) = 0.$$

Lets consider some other multiparticle wave-function of the form (3.2):

$$\psi_{[j_1]}(p_{[1]}) \otimes \dots \otimes \psi_{[j_n]}(p_{[n]}). \quad (3.3)$$

Using the wave-functions (3.2) and (3.3) and applying the formula (2.6) to them, we can define the following multicurrent:

$$\mathbf{J}_{[i_1 j_1]}(p_{[1]}) \otimes \dots \otimes \mathbf{J}_{[i_n j_n]}(p_{[n]}). \quad (3.4)$$

Here $\mathbf{J}_{[i_s j_s]} = \mathbf{J}(\psi_{[i_s]}, \psi_{[j_s]})$ for $s = 1, \dots, n$. One can integrate the multicurrent (3.4) over the Cartesian product of n copies of the hypersurface S thus defining a pairing for multiparticle wave functions:

$$\langle \psi_{[i_1]} \otimes \dots \otimes \psi_{[i_n]} | \psi_{[j_1]} \otimes \dots \otimes \psi_{[j_n]} \rangle = \prod_{s=1}^n \delta_{[i_s j_s]}. \quad (3.5)$$

The formula (3.5) shows that the wave-functions of the form (3.2) constitute an orthonormal basis in a Hilbert space defined by the multicurrent (3.4). This Hilbert space is denoted \mathcal{H}_n . It is the tensor product of n copies of \mathcal{H}_1 :

$$\mathcal{H}_n = \underbrace{\mathcal{H}_1 \otimes \dots \otimes \mathcal{H}_1}_{n \text{ times}}. \quad (3.6)$$

Note that the Hilbert space (3.6) is not a space of wave-functions for actual quantum states of n particles. Wave-functions of actual states should be symmetric for bosons and skew-symmetric for fermions. In our case of spin 1/2 particles they are fermions. For this reason we construct an actual wave-function as follows:

$$\psi_{[i_1 \dots i_n]} = \sum_{\sigma \in \mathfrak{S}_n} \frac{(-1)^\sigma}{\sqrt{n!}} \psi_{[i_{\sigma 1}]}(p_{[1]}) \otimes \dots \otimes \psi_{[i_{\sigma n}]}(p_{[n]}). \quad (3.7)$$

In a coordinate form, i. e. upon choosing some frame pair $(U, \Upsilon_0, \Upsilon_1, \Upsilon_2, \Upsilon_3)$ and $(U, \Psi_1, \Psi_2, \Psi_3, \Psi_4)$, the wave-function (3.7) is represented as

$$\psi_{[i_1 \dots i_n]}^{b_1 \dots b_n} = \sum_{\sigma \in \mathfrak{S}_n} \frac{(-1)^\sigma}{\sqrt{n!}} \psi_{[i_{\sigma 1}]}^{b_1}(p_{[1]}) \otimes \dots \otimes \psi_{[i_{\sigma n}]}^{b_n}(p_{[n]}). \quad (3.8)$$

By σ in (3.7) and (3.8) we denote a transposition from the n -th symmetric group \mathfrak{S}_n . Like the wave-function (3.7), the wave-function (3.7) satisfies the Dirac equation (1.9) with respect to each its argument p_s :

$$\sum_{b_s=1}^4 \left(i \hbar \sum_{q=0}^3 \gamma_{b_s}^{aq}(p_s) \nabla_q^{[s]} - m c \delta_{b_s}^a \right) \psi_{[i_1 \dots i_n]}^{b_1 \dots b_n}(p_{[1]}, \dots, p_{[n]}) = 0. \quad (3.9)$$

Due to (3.9) one can apply the pairing (3.5) defined by means of the multicurrent (3.4) to functions of the form (3.7). As a result we get

$$\langle \psi_{[i_1 \dots i_n]} | \psi_{[j_1 \dots j_n]} \rangle = \prod_{s=1}^n \delta_{[i_s j_s]}. \quad (3.10)$$

The formula (3.10) means that the wave-functions of the form (3.7) constitute an orthonormal basis in a subspace of the Hilbert space (3.6). We denote this subspace through $\mathcal{H}_n^{\text{skew}}$. Note that $\mathcal{H}_1^{\text{skew}} = \mathcal{H}_1$. Let's denote $\mathcal{H}_0 = \mathbb{C}$ and $|\Phi_0| = 1$. Then we consider the following direct sum of Hilbert spaces:

$$\mathcal{H} = \mathcal{H}_0 \oplus \bigoplus_{n=1}^{\infty} \mathcal{H}_n^{\text{skew}}. \quad (3.11)$$

The pairing (3.10) can be extended to (3.11) so that

$$\begin{aligned} \mathcal{H}_n^{\text{skew}} &\perp \mathcal{H}_m^{\text{skew}} \quad \text{for all } n \neq m, \\ \mathcal{H}_n^{\text{skew}} &\perp \mathcal{H}_0 \quad \text{for all } n, \\ |\Phi_0| &= 1. \end{aligned} \quad (3.12)$$

Then due to (3.12) the direct sum \mathcal{H} gains the structure of a Hilbert space. This is the Hilbert space of all multiparticle quantum states of massive spin 1/2 particles described by the Dirac equation (1.9). The vector $\Phi_0 \in \mathcal{H}$ is called the *vacuum vector* in the secondary quantization scheme (see [10]). According to this scheme the creation operators are introduced as follows:

$$a_{[i_s]}^+ \Phi_0 = \psi_{[i_s]}, \quad (3.13)$$

$$a_{[i_s]}^+ \psi_{[i_s]} = 0, \quad (3.14)$$

$$a_{[i_s]}^+ \psi_{[i_1 \dots i_{s-1} i_{s+1} \dots i_n]} = \psi_{[i_1 \dots i_{s-1} i_s i_{s+1} \dots i_n]}, \quad (3.15)$$

$$a_{[i_s]}^+ \psi_{[i_1 \dots i_{s-1} i_s i_{s+1} \dots i_n]} = 0, \quad (3.16)$$

where $i_1 < \dots < i_s < \dots < i_n$. The annihilation operators are defined as Hermitian conjugates for creation operators with respect to the pairing (3.10) extended to the Hilbert space (3.11). From (3.13), (3.14), (3.15), and (3.16) one easily derives

$$a_{[i_s]} \psi_{[i_s]} = \Phi_0, \quad (3.17)$$

$$a_{[i_s]} \Phi_0 = 0, \quad (3.18)$$

$$a_{[i_s]} \psi_{[i_1 \dots i_{s-1} i_s i_{s+1} \dots i_n]} = \psi_{[i_1 \dots i_{s-1} i_{s+1} \dots i_n]}, \quad (3.19)$$

$$a_{[i_s]} \psi_{[i_1 \dots i_{s-1} i_s i_{s+1} \dots i_n]} = 0. \quad (3.20)$$

Note that the creation and annihilation operators introduced by the formulas (3.13), (3.14), (3.15), (3.16), (3.17), (3.18), (3.19), and (3.20) are constant operators in a constant Hilbert space \mathcal{H} . This fact is not surprising. In the absence of interaction terms in the action integral (1.5) the quantum states of Dirac particles remain unchanged in dynamics regardless to the number of particles we have.

4. SOME CONCLUSIONS.

All results of this paper are known. In the case of the flat Minkowski space they are broadly known. The main goal of the present paper is not to claim a new result, but to emphasize the existence of five basic spin-tensorial fields (1.1) in the theory of Dirac particles and to fix the novel notations for these basic fields (see the remark below the formula (1.15)).

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¹ VINITI is the Russian Institute for Scientific and Technological Information (Всероссийский Институт Научной и Технической Информации). It publishes serial books with selected and invited articles in various areas of science and technology.

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