A NOTE ON METRIC CONNECTIONS FOR CHIRAL AND DIRAC SPINORS.

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Abstract. It is known that the bundle of Dirac spinors is produced as a direct sum of two bundles — the bundle of chiral spinors and its Hermitian conjugate bundle. In this paper some aspects of metric connections for chiral and Dirac spinors are resumed and their relation is studied.

1. Chiral spinors.

The construction of two-component Weyl spinors, they are also called chiral spinors, is based on the following well-known group homomorphism

$$\varphi\colon \mathrm{SL}(2,\mathbb{C}) \to \mathrm{SO}^{+}(1,3,\mathbb{R}). \tag{1.1}$$

The homomorphism (1.1) is defined through the formula

$$\mathfrak{S}\cdot\boldsymbol\sigma_m\cdot\mathfrak{S}^{\dagger}=\sum_{k=0}^{3} S^k_m\,\boldsymbol\sigma_k, \tag{1.2}$$

where $\boldsymbol\sigma_1$, $\boldsymbol\sigma_2$, $\boldsymbol\sigma_3$ are Pauli matrices complemented with the unit matrix $\boldsymbol\sigma_0$:

$$\begin{aligned} &\boldsymbol\sigma_0=\begin{Vmatrix}1&0\\0&1\end{Vmatrix}, &&\boldsymbol\sigma_2=\begin{Vmatrix}0&-i\\i&0\end{Vmatrix},\\ &\boldsymbol\sigma_1=\begin{Vmatrix}0&1\\1&0\end{Vmatrix}, &&\boldsymbol\sigma_3=\begin{Vmatrix}1&0\\0&-1\end{Vmatrix}. \end{aligned} \tag{1.3}$$

By means of (1.2) and (1.3) each matrix $\mathfrak{S}\in\mathrm{SL}(2,\mathbb{C})$ is associated with some matrix $S\in\mathrm{SO}^{+}(1,3,\mathbb{R})$ so that we can write $S=\varphi(\mathfrak{S})$, see [1], [2], and [3] for detailed description of this construction.

Let M be a *space-time* manifold, i. e. a four-dimensional orientable manifold equipped with a pseudo-Euclidean Minkowski-type metric $\mathbf{g}$ and carrying a special smooth geometric structure which is called a *polarization.* Once some polarization is fixed, one can distinguish the *Future light cone* from the *Past light cone* at each point $p\in M$ (see [4] for more details). A moving frame $(U,\,\boldsymbol\Upsilon_0,\,\boldsymbol\Upsilon_1,\,\boldsymbol\Upsilon_2,\,\boldsymbol\Upsilon_3)$ of the tangent bundle TM is an ordered set of four smooth vector fields $\boldsymbol\Upsilon_0$, $\boldsymbol\Upsilon_1$, $\boldsymbol\Upsilon_2$, $\boldsymbol\Upsilon_3$ which are defined and $\mathbb{R}$-linearly independent at each point p of some open subset

2000 *Mathematics Subject Classification.* 53B30, 53C27, 83C60.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

$U \subset M$. This moving frame is called a *positively polarized right orthonormal frame* if the following conditions are fulfilled:

(1) the value of the first vector filed $\boldsymbol{\Upsilon}_0$ at each point $p \in U$ belongs to the interior of the Future light cone determined by the polarization of M;
(2) it is a right frame in the sense of the orientation of M;
(3) the metric tensor $\mathbf{g}$ is given by the standard Minkowski matrix in this frame:

$$g_{ij} = g(\boldsymbol{\Upsilon}_i, \boldsymbol{\Upsilon}_j) = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{Vmatrix}.$$

Apart from positively polarized right orthonormal frames, below we shall consider the following three special types of frames in TM:

— positively polarized left orthonormal frames;
— negatively polarized right orthonormal frames;
— negatively polarized left orthonormal frames.

The definitions of these types of frames are easily obtained by alternating the above condition (1) and (2) with the opposite ones.

Let $(U,\, \boldsymbol{\Upsilon}_0,\, \boldsymbol{\Upsilon}_1,\, \boldsymbol{\Upsilon}_2,\, \boldsymbol{\Upsilon}_3)$ and $(\tilde{U},\, \tilde{\boldsymbol{\Upsilon}}_0,\, \tilde{\boldsymbol{\Upsilon}}_1,\, \tilde{\boldsymbol{\Upsilon}}_2,\, \tilde{\boldsymbol{\Upsilon}}_3)$ be arbitrary two frames of the tangent bundle TM such that $U \cap \tilde{U} \neq \varnothing$. Then at each point $p \in U \cap \tilde{U}$ we can write the following relationships for their frame vectors:

$$\tilde{\boldsymbol{\Upsilon}}_i = \sum_{j=0}^{3} S_i^j\, \boldsymbol{\Upsilon}_j, \qquad\qquad \boldsymbol{\Upsilon}_i = \sum_{j=0}^{3} T_i^j\, \tilde{\boldsymbol{\Upsilon}}_j. \tag{1.4}$$

The relationships (1.4) are called *transition formulas*, while the coefficients S_i^j and T_i^j in them are the components of two mutually inverse transition matrices S and $T = S^{-1}$. If both frames $(U,\, \boldsymbol{\Upsilon}_0,\, \boldsymbol{\Upsilon}_1,\, \boldsymbol{\Upsilon}_2,\, \boldsymbol{\Upsilon}_3)$ and $(\tilde{U},\, \tilde{\boldsymbol{\Upsilon}}_0,\, \tilde{\boldsymbol{\Upsilon}}_1,\, \tilde{\boldsymbol{\Upsilon}}_2,\, \tilde{\boldsymbol{\Upsilon}}_3)$ are positively polarized right orthonormal frames, then the transition matrices S and T both are orthochronous Lorentzian matrices with $\det S = 1$ and $\det T = 1$. Such matrices form the *special orthochronous matrix Lorentz group* $\mathrm{SO}^+(1, 3, \mathbb{R})$.

Assume that SM is a two-dimensional smooth complex vector bundle over the space-time M equipped with a non-vanishing skew-symmetric bilinear form $\mathbf{d}$ at each point $p \in M$. This bilinear form $\mathbf{d}$ is called the *spin-metric tensor*. A moving frame $(U,\, \boldsymbol{\Psi}_1,\, \boldsymbol{\Psi}_2)$ of SM is an ordered set of two smooth sections $\boldsymbol{\Psi}_1$ and $\boldsymbol{\Psi}_2$ over some open subset $U \subset M$ which are $\mathbb{C}$-linearly independent at each point $p \in U$. A moving frame $(U,\, \boldsymbol{\Psi}_1,\, \boldsymbol{\Psi}_2)$ is called an *orthonormal frame* if

$$d_{ij} = d(\boldsymbol{\Psi}_i, \boldsymbol{\Psi}_j) = \begin{Vmatrix} 0 & 1 \\ -1 & 0 \end{Vmatrix}, \tag{1.5}$$

i. e. if the spin-metric tensor $\mathbf{d}$ is given by the skew-symmetric matrix (1.5) in this frame. For two arbitrary frames $(U,\, \boldsymbol{\Psi}_1,\, \boldsymbol{\Psi}_2)$ and $(\tilde{U},\, \tilde{\boldsymbol{\Psi}}_1,\, \tilde{\boldsymbol{\Psi}}_2)$ of the bundle SM with overlapping domains $U \cap \tilde{U} \neq \varnothing$ we can write the following transition formulas:

$$\tilde{\boldsymbol{\Psi}}_i = \sum_{j=1}^{2} \mathfrak{S}_i^j\, \boldsymbol{\Psi}_j, \qquad\qquad \boldsymbol{\Psi}_i = \sum_{j=1}^{2} \mathfrak{T}_i^j\, \tilde{\boldsymbol{\Psi}}_j. \tag{1.6}$$

Like in (1.4), the coefficients $\mathfrak{S}_i^j$ and $\mathfrak{T}_i^j$ in (1.6) are the components of two mutually inverse transition matrices $\mathfrak{S}$ and $\mathfrak{T} = \mathfrak{S}^{-1}$. If $(U,\, \mathbf{\Psi}_1,\, \mathbf{\Psi}_2)$ and $(\tilde{U},\, \tilde{\mathbf{\Psi}}_1,\, \tilde{\mathbf{\Psi}}_2)$ are two orthonormal frames, then both matrices $\mathfrak{S}$ and $\mathfrak{T}$ in (1.6) belong to the special linear matrix group $\mathrm{SL}(2, \mathbb{C})$.

Definition 1.1. A two-dimensional complex vector bundle SM over the space-time manifold M equipped with a nonzero spin-metric $\mathbf{d}$ is called a *spinor bundle* if each orthonormal frame $(U,\, \mathbf{\Psi}_1,\, \mathbf{\Psi}_2)$ of SM is associated with some positively polarized right orthonormal frame $(U,\, \mathbf{\Upsilon}_0,\, \mathbf{\Upsilon}_1,\, \mathbf{\Upsilon}_2,\, \mathbf{\Upsilon}_3)$ of the tangent bundle TM such that for any two orthonormal frames $(U,\, \mathbf{\Psi}_1,\, \mathbf{\Psi}_2)$ and $(\tilde{U},\, \tilde{\mathbf{\Psi}}_1,\, \tilde{\mathbf{\Psi}}_2)$ with overlapping domains $U \cap \tilde{U} \neq \varnothing$ the associated tangent frames $(U,\, \mathbf{\Upsilon}_0,\, \mathbf{\Upsilon}_1,\, \mathbf{\Upsilon}_2,\, \mathbf{\Upsilon}_3)$ and $(\tilde{U},\, \tilde{\mathbf{\Upsilon}}_0,\, \tilde{\mathbf{\Upsilon}}_1,\, \tilde{\mathbf{\Upsilon}}_2,\, \tilde{\mathbf{\Upsilon}}_3)$ are related to each other by means of the formulas (1.4), where the transition matrices S and T are obtained from the transition matrices $\mathfrak{S}$ and $\mathfrak{T}$ in (1.6) by applying the homomorphism (1.1), i. e. $S = \varphi(\mathfrak{S})$ and $T = \varphi(\mathfrak{T})$.

The definition 1.1 reflects the basic feature of all spinor bundles. They are closely related to tangent bundle and this relation is implemented through associated frame pairs of some definite types.

2. Tensorial and spin-tensorial fields.

Tensorial an spin-tensorial fields are introduced in a standard way as described in [3], [5], [6], and many other papers. First of all we introduce the complexified tangent and cotangent bundles $\mathbb{C}TM$ and $\mathbb{C}T^*M$:

$$\mathbb{C}T_p(M) = \mathbb{C} \otimes T_p(M), \qquad \mathbb{C}T_p^*(M) = \mathbb{C} \otimes T_p^*(M). \tag{2.1}$$

The complex bundles (2.1) are obviously dual to each other. Then we introduce the conjugate and Hermitian conjugate bundles for the spinor bundle SM:

$$S^*M, \qquad S^\dagger M, \qquad S^{*\dagger}M = S^{\dagger *}M. \tag{2.2}$$

Using (2.1) and (2.2), we define the following tensor products:

$$\mathbb{C}T_n^mM = \overbrace{\mathbb{C}TM \otimes \ldots \otimes \mathbb{C}TM}^{m\ \text{times}} \otimes \underbrace{\mathbb{C}T^*M \otimes \ldots \otimes \mathbb{C}T^*M}_{n\ \text{times}}, \tag{2.3}$$

$$S_\beta^\alpha M = \overbrace{SM \otimes \ldots \otimes SM}^{\alpha\ \text{times}} \otimes \underbrace{S^*M \otimes \ldots \otimes S^*M}_{\beta\ \text{times}}, \tag{2.4}$$

$$\bar{S}_\gamma^\nu M = \overbrace{S^{\dagger *}M \otimes \ldots \otimes S^{\dagger *}M}^{\nu\ \text{times}} \otimes \underbrace{S^\dagger M) \otimes \ldots \otimes S^\dagger M}_{\gamma\ \text{times}}. \tag{2.5}$$

Note that (2.3) is the *complexified tensor bundle of the type* (m, n), (2.4) is the *spin-tensorial bundle of the type* (α, β), and (2.5) is the *barred spin-tensorial bundle of the type* (ν, γ). Combining these three bundles, we define the following spin-tensorial bundle of the mixed type $(\alpha, \beta|\nu, \gamma|m, n)$:

$$S_\beta^\alpha \bar{S}_\gamma^\nu T_n^m M = S_\beta^\alpha M \otimes \bar{S}_\gamma^\nu M \otimes \mathbb{C}T_n^m M. \tag{2.6}$$

Smooth sections of the bundle (2.6) are called *spin-tensorial* fields of the type $(\alpha,\beta|\nu,\gamma|m,n)$. The metric tensor $\mathbf{g}$ of the base space-time manifold M now is interpreted as a spin-tensorial field of the type $(0,0|0,0|0,2)$, while the spin-metric tensor $\mathbf{d}$ is a spin-tensorial field of the type $(0,2|0,0|0,0)$.

Note that an arbitrary spin-tensorial field of the type $(0,0|0,0|0,2)$ is a complex field, while $\mathbf{g}$ is a real field. Therefore, we have

$$\mathbf{g}=\tau(\mathbf{g}), \tag{2.7}$$

where τ is the semilinear involution of complex conjugation:

$$S^{\alpha}_{\beta}\bar{S}^{\nu}_{\gamma}T^{m}_{n}M \;\underset{\tau}{\overset{\tau}{\rightleftarrows}}\; S^{\nu}_{\gamma}\bar{S}^{\alpha}_{\beta}T^{m}_{n}M. \tag{2.8}$$

Both mappings (2.8) are denoted by the same symbol, hence, formally we have the involution identity $\tau^2=\tau\circ\tau=\mathbf{id}$. More detailed description of the involution τ can be found in [3] and [6].

A coordinate description of spin-tensorial fields is obtained in terms of frame pairs. Let $(U,\,\boldsymbol{\Upsilon}_0,\,\boldsymbol{\Upsilon}_1,\,\boldsymbol{\Upsilon}_2,\,\boldsymbol{\Upsilon}_3)$ be an arbitrary frame of the tangent bundle TM and let $(U,\,\boldsymbol{\Psi}_1,\,\boldsymbol{\Psi}_2)$ be an arbitrary frame of the spinor bundle SM. Denote by $(U,\,\boldsymbol{\eta}^0,\,\boldsymbol{\eta}^1,\,\boldsymbol{\eta}^2,\,\boldsymbol{\eta}^3)$ and $(U,\,\boldsymbol{\vartheta}^1,\,\boldsymbol{\vartheta}^2)$ the dual frames for $(U,\,\boldsymbol{\Upsilon}_0,\,\boldsymbol{\Upsilon}_1,\,\boldsymbol{\Upsilon}_2,\,\boldsymbol{\Upsilon}_3)$ and $(U,\,\boldsymbol{\Psi}_1,\,\boldsymbol{\Psi}_2)$ respectively. Then let's denote

$$\overline{\boldsymbol{\Psi}}_i=\tau(\boldsymbol{\Psi}_i), \qquad\qquad \overline{\boldsymbol{\vartheta}}^{\,i}=\tau(\boldsymbol{\vartheta}^{\,i}). \tag{2.9}$$

The barred spinor fields in (2.9) compose two frames $(U,\,\overline{\boldsymbol{\Psi}}_1,\,\overline{\boldsymbol{\Psi}}_2)$ and $(U,\,\overline{\boldsymbol{\vartheta}}^{\,1},\,\overline{\boldsymbol{\vartheta}}^{\,2})$ in $S^{\dagger}M$ and $S^{*\dagger}M$ respectively. Now, according to the formulas (2.3), (2.4), and (2.5), we define the following tensor products:

$$\boldsymbol{\Upsilon}^{k_1\ldots\,k_n}_{h_1\ldots\,h_m}=\boldsymbol{\Upsilon}_{h_1}\otimes\ldots\otimes\boldsymbol{\Upsilon}_{h_m}\otimes\boldsymbol{\eta}^{k_1}\otimes\ldots\otimes\boldsymbol{\eta}^{k_n}, \tag{2.10}$$

$$\boldsymbol{\Psi}^{j_1\ldots\,j_\beta}_{i_1\ldots\,i_\alpha}=\boldsymbol{\Psi}_{i_1}\otimes\ldots\otimes\boldsymbol{\Psi}_{i_\alpha}\otimes\boldsymbol{\vartheta}^{\,j_1}\otimes\ldots\otimes\boldsymbol{\vartheta}^{\,j_\beta}, \tag{2.11}$$

$$\overline{\boldsymbol{\Psi}}^{\bar{j}_1\ldots\,\bar{j}_\gamma}_{\bar{i}_1\ldots\,\bar{i}_\nu}=\overline{\boldsymbol{\Psi}}_{\bar{i}_1}\otimes\ldots\otimes\overline{\boldsymbol{\Psi}}_{\bar{i}_\nu}\otimes\overline{\boldsymbol{\vartheta}}^{\,j_1}\otimes\ldots\otimes\overline{\boldsymbol{\vartheta}}^{\,j_\gamma}. \tag{2.12}$$

And finally, according to (2.6), from (2.10), (2.11), and (2.12) we produce

$$\boldsymbol{\Psi}^{j_1\ldots\,j_\beta\,\bar{j}_1\ldots\,\bar{j}_\gamma\,k_1\ldots\,k_n}_{i_1\ldots\,i_\alpha\,\bar{i}_1\ldots\,\bar{i}_\nu\,h_1\ldots\,h_m}=\boldsymbol{\Psi}^{j_1\ldots\,j_\beta}_{i_1\ldots\,i_\alpha}\otimes\overline{\boldsymbol{\Psi}}^{\bar{j}_1\ldots\,\bar{j}_\gamma}_{\bar{i}_1\ldots\,\bar{i}_\nu}\otimes\boldsymbol{\Upsilon}^{k_1\ldots\,k_n}_{h_1\ldots\,h_m}. \tag{2.13}$$

Using (2.13), for any spin-tensorial field of the type $(\alpha,\beta|\nu,\gamma|m,n)$ we write

$$\mathbf{X}=\sum^{2}_{\substack{i_1,\,\ldots,\,i_\alpha\\ j_1,\,\ldots,\,j_\beta}}\!\!\ldots\!\!\sum^{2}\;\sum^{2}_{\substack{\bar{i}_1,\,\ldots,\,\bar{i}_\nu\\ \bar{j}_1,\,\ldots,\,\bar{j}_\gamma}}\!\!\ldots\!\!\sum^{2}\;\sum^{3}_{\substack{h_1,\,\ldots,\,h_m\\ k_1,\,\ldots,\,k_n}}\!\!\ldots\!\!\sum^{3}X^{i_1\ldots\,i_\alpha\,\bar{i}_1\ldots\,\bar{i}_\nu\,h_1\ldots\,h_m}_{j_1\ldots\,j_\beta\,\bar{j}_1\ldots\,\bar{j}_\gamma\,k_1\ldots\,k_n}\;\boldsymbol{\Psi}^{j_1\ldots\,j_\beta\,\bar{j}_1\ldots\,\bar{j}_\gamma\,k_1\ldots\,k_n}_{i_1\ldots\,i_\alpha\,\bar{i}_1\ldots\,\bar{i}_\nu\,h_1\ldots\,h_m}. \tag{2.14}$$

Since all of the above frames $(U,\,\boldsymbol{\eta}^0,\,\boldsymbol{\eta}^1,\,\boldsymbol{\eta}^2,\,\boldsymbol{\eta}^3)$, $(U,\,\boldsymbol{\vartheta}^1,\,\boldsymbol{\vartheta}^2)$, $(U,\,\overline{\boldsymbol{\Psi}}_1,\,\overline{\boldsymbol{\Psi}}_2)$, and $(U,\,\overline{\boldsymbol{\vartheta}}^{\,1},\,\overline{\boldsymbol{\vartheta}}^{\,2})$ and their tensor products (2.10), (2.11), (2.12), and (2.13) are produced from two initial frames $(U,\,\boldsymbol{\Upsilon}_0,\,\boldsymbol{\Upsilon}_1,\,\boldsymbol{\Upsilon}_2,\,\boldsymbol{\Upsilon}_3)$ and $(U,\,\boldsymbol{\Psi}_1,\,\boldsymbol{\Psi}_2)$, the coefficients $X^{i_1\ldots\,i_\alpha\,\bar{i}_1\ldots\,\bar{i}_\nu\,h_1\ldots\,h_m}_{j_1\ldots\,j_\beta\,\bar{j}_1\ldots\,\bar{j}_\gamma\,k_1\ldots\,k_n}$ in (2.14) are called the *coordinate representation* of the field $\mathbf{X}$ in the frame pair $(U,\,\boldsymbol{\Upsilon}_0,\,\boldsymbol{\Upsilon}_1,\,\boldsymbol{\Upsilon}_2,\,\boldsymbol{\Upsilon}_3)$ and $(U,\,\boldsymbol{\Psi}_1,\,\boldsymbol{\Psi}_2)$. When passing from this frame pair to another frame pair $(\tilde{U},\,\tilde{\boldsymbol{\Upsilon}}_0,\,\tilde{\boldsymbol{\Upsilon}}_1,\,\tilde{\boldsymbol{\Upsilon}}_2,\,\tilde{\boldsymbol{\Upsilon}}_3)$ and $(\tilde{U},\,\tilde{\boldsymbol{\Psi}}_1,\,\tilde{\boldsymbol{\Psi}}_2)$ these coefficients are transformed as follows:

$$
\begin{aligned}
\tilde{X}^{i_1\ldots\,i_\alpha\,\bar{i}_1\ldots\,\bar{i}_\nu\,h_1\ldots\,h_m}_{j_1\ldots\,j_\beta\,\bar{j}_1\ldots\,\bar{j}_\gamma\,k_1\ldots\,k_n}
&=\sum^{4}_{\substack{a_1,\,\ldots\,,\,a_\alpha\\ b_1,\,\ldots\,,\,b_\beta}}\!\!\ldots\sum^{4}\ \sum^{4}_{\substack{\bar{a}_1,\,\ldots\,,\,\bar{a}_\nu\\ \bar{b}_1,\,\ldots\,,\,\bar{b}_\gamma}}\!\!\ldots\sum^{4}\ \sum^{3}_{\substack{c_1,\,\ldots\,,\,c_m\\ d_1,\,\ldots\,,\,d_n}}\!\!\ldots\sum^{3}\hat{\mathfrak{T}}^{i_1}_{a_1}\ldots\,\hat{\mathfrak{T}}^{i_\alpha}_{a_\alpha}\,\times\\
&\times\,\hat{\mathfrak{S}}^{b_1}_{j_1}\ldots\,\hat{\mathfrak{S}}^{b_\beta}_{j_\beta}\ \overline{\hat{\mathfrak{T}}^{\bar{i}_1}_{\bar{a}_1}}\ldots\,\overline{\hat{\mathfrak{T}}^{\bar{i}_\nu}_{\bar{a}_\nu}}\ \overline{\hat{\mathfrak{S}}^{\bar{b}_1}_{\bar{j}_1}}\ldots\,\overline{\hat{\mathfrak{S}}^{\bar{b}_\gamma}_{\bar{j}_\gamma}}\ T^{h_1}_{c_1}\ldots\,T^{h_m}_{c_m}\,\times\\
&\times\,S^{d_1}_{k_1}\ldots\,S^{d_n}_{k_n}\ X^{a_1\ldots\,a_\alpha\,\bar{a}_1\ldots\,\bar{a}_\nu\,c_1\ldots\,c_m}_{b_1\ldots\,b_\beta\,\bar{b}_1\ldots\,\bar{b}_\gamma\,d_1\ldots\,d_n},
\end{aligned}
\tag{2.15}
$$

$$
\begin{aligned}
X^{i_1\ldots\,i_\alpha\,\bar{i}_1\ldots\,\bar{i}_\nu\,h_1\ldots\,h_m}_{j_1\ldots\,j_\beta\,\bar{j}_1\ldots\,\bar{j}_\gamma\,k_1\ldots\,k_n}
&=\sum^{4}_{\substack{a_1,\,\ldots\,,\,a_\alpha\\ b_1,\,\ldots\,,\,b_\beta}}\!\!\ldots\sum^{4}\ \sum^{4}_{\substack{\bar{a}_1,\,\ldots\,,\,\bar{a}_\nu\\ \bar{b}_1,\,\ldots\,,\,\bar{b}_\gamma}}\!\!\ldots\sum^{4}\ \sum^{3}_{\substack{c_1,\,\ldots\,,\,c_m\\ d_1,\,\ldots\,,\,d_n}}\!\!\ldots\sum^{3}\hat{\mathfrak{S}}^{i_1}_{a_1}\ldots\,\hat{\mathfrak{S}}^{i_\alpha}_{a_\alpha}\,\times\\
&\times\,\hat{\mathfrak{T}}^{b_1}_{j_1}\ldots\,\hat{\mathfrak{T}}^{b_\beta}_{j_\beta}\ \overline{\hat{\mathfrak{S}}^{\bar{i}_1}_{\bar{a}_1}}\ldots\,\overline{\hat{\mathfrak{S}}^{\bar{i}_\nu}_{\bar{a}_\nu}}\ \overline{\hat{\mathfrak{T}}^{\bar{b}_1}_{\bar{j}_1}}\ldots\,\overline{\hat{\mathfrak{T}}^{\bar{b}_\gamma}_{\bar{j}_\gamma}}\ S^{h_1}_{c_1}\ldots\,S^{h_m}_{c_m}\,\times\\
&\times\,T^{d_1}_{k_1}\ldots\,T^{d_n}_{k_n}\ \tilde{X}^{a_1\ldots\,a_\alpha\,\bar{a}_1\ldots\,\bar{a}_\nu\,c_1\ldots\,c_m}_{b_1\ldots\,b_\beta\,\bar{b}_1\ldots\,\bar{b}_\gamma\,d_1\ldots\,d_n}.
\end{aligned}
\tag{2.16}
$$

The formulas (2.15) and (2.16) express the general transformation rules for the components of a chiral spin-tensorial field under a change of frame pairs. The matrices $\mathfrak{S}$, $\mathfrak{T}$, S, and T in them are taken from (1.4) and (1.6).

Note that $(U,\,\boldsymbol{\Upsilon}_0,\,\boldsymbol{\Upsilon}_1,\,\boldsymbol{\Upsilon}_2,\,\boldsymbol{\Upsilon}_3)$ and $(U,\,\boldsymbol{\eta}^0,\,\boldsymbol{\eta}^1,\,\boldsymbol{\eta}^2,\,\boldsymbol{\eta}^3)$ are real frames. They are invariant under the action of the semilinear involution τ:

$$
\tau(\boldsymbol{\Upsilon}_i)=\boldsymbol{\Upsilon}_i, \qquad\qquad \tau(\boldsymbol{\eta}_i)=\boldsymbol{\eta}_i.
\tag{2.17}
$$

Applying τ to (2.14) and taking into account (2.9) and (2.17), we obtain

$$
\tau(\mathbf{X})=\sum^{2}_{\substack{i_1,\,\ldots\,,\,i_\nu\\ j_1,\,\ldots\,,\,j_\gamma\\ \bar{i}_1,\,\ldots\,,\,\bar{i}_\alpha\\ \bar{j}_1,\,\ldots\,,\,\bar{j}_\beta}}\!\!\ldots\sum^{2}\ \sum^{3}_{\substack{h_1,\,\ldots\,,\,h_m\\ k_1,\,\ldots\,,\,k_n}}\!\!\ldots\sum^{3}\tau X^{i_1\ldots\,i_\nu\,\bar{i}_1\ldots\,\bar{i}_\alpha\,h_1\ldots\,h_m}_{j_1\ldots\,j_\gamma\,\bar{j}_1\ldots\,\bar{j}_\beta\,k_1\ldots\,k_n}\ \boldsymbol{\Psi}^{j_1\ldots\,j_\gamma\,\bar{j}_1\ldots\,\bar{j}_\beta\,k_1\ldots\,k_n}_{i_1\ldots\,i_\nu\,\bar{i}_1\ldots\,\bar{i}_\alpha\,h_1\ldots\,h_m},
\tag{2.18}
$$

where

$$
\tau X^{i_1\ldots\,i_\nu\,\bar{i}_1\ldots\,\bar{i}_\alpha\,h_1\ldots\,h_m}_{j_1\ldots\,j_\gamma\,\bar{j}_1\ldots\,\bar{j}_\beta\,k_1\ldots\,k_n}=\overline{X^{\bar{i}_1\ldots\,\bar{i}_\alpha\,i_1\ldots\,i_\nu\,h_1\ldots\,h_m}_{\bar{j}_1\ldots\,\bar{j}_\beta\,j_1\ldots\,j_\gamma\,k_1\ldots\,k_n}}.
\tag{2.19}
$$

Note that the formulas (2.18) and (2.19) are in agreement with (2.8). They mean that the involution τ acts upon the components of spin-tensorial fields as the complex conjugation exchanging barred and non-barred spinor indices.

3. Basic spin-tensorial fields of chiral spinors.

The metric tensor $\mathbf{g}$ is the basic field for both chiral and Dirac spinors. As it was already mentioned above, $\mathbf{g}$ is interpreted as a spin-tensorial field of the type $(0,0|0,0|0,2)$ satisfying the reality condition (2.7). Apart from $\mathbf{g}$, in the theory of chiral spinors there are two other basic spin-tensorial fields. The first of them is the spin-metric tensor $\mathbf{d}$. It is a field of the type $(0,2|0,0|0,0)$. In canonically associated frame pairs (see definition 1.1) its components are given by the matrix (1.5). The second basic spin tensorial field in the theory of chiral spinors is denoted by $\mathbf{G}$. It is called the *Infeld-van der Waerden field*, its components $G^{i\bar{i}}_q$ are called the *Infeld-van der Waerden symbols*. In canonically associated frame pairs the Infeld-van der Waerden symbols are given explicitly by the formulas

$$\begin{aligned}
&G^{11}_0=1, && G^{11}_1=0, && G^{11}_2=0, && G^{11}_3=1,\\
&G^{12}_0=0, && G^{12}_1=1, && G^{12}_2=-i, && G^{12}_3=0,\\
&G^{21}_0=0, && G^{21}_1=1, && G^{21}_2=i, && G^{21}_3=0,\\
&G^{22}_0=1, && G^{22}_1=0, && G^{22}_2=0, && G^{22}_3=-1.
\end{aligned}\tag{3.1}$$

These formulas (3.1) are derived from (1.3) due to the formula (1.2). The Infeld-van der Waerden field is a spin-tensorial field of the type $(1,0|1,0|0,1)$.

Applying the index lowering and index raising procedures to the Infeld-van der Waerden symbols (3.1) we get the inverse[1] Infeld-van der Waerden symbols:

$$G^q_{i\bar{i}}=\sum^2_{j=1}\sum^2_{\bar{j}=1}\sum^3_{k=0}G^{j\bar{j}}_k\ g^{kq}\,d_{ji}\ \bar{d}_{\bar{j}\bar{i}}.\tag{3.2}$$

The inverse Infeld-van der Waerden given by the formula (3.2) are the components of a spin-tensorial field of the type $(0,1|0,1|1,0)$, it is denoted by the same symbol $\mathbf{G}$ as the initial Infeld-van der Waerden field. Here are the numeric values of the inverse Infeld-van der Waerden symbols:

$$\begin{aligned}
&G^0_{11}=1, && G^0_{12}=0, && G^0_{21}=0, && G^0_{22}=1,\\
&G^1_{11}=0, && G^1_{12}=1, && G^1_{21}=1, && G^1_{22}=0,\\
&G^2_{11}=0, && G^2_{12}=i, && G^2_{21}=-i, && G^2_{22}=0,\\
&G^3_{11}=1, && G^3_{12}=0, && G^3_{21}=0, && G^3_{22}=-1.
\end{aligned}\tag{3.3}$$

The barred spin-metric tensor $\bar{\mathbf{d}}$ is derived from $\mathbf{d}$ by applying τ:

$$\bar{\mathbf{d}}=\tau(\mathbf{d}).\tag{3.4}$$

It is a spin-tensorial field of the type $(0,0|0,2|0,0)$. The components of the field (3.4) in (3.2) are derived by means of the formula

$$\bar{d}_{\bar{j}\bar{i}}=\overline{d_{\bar{j}\bar{i}}}.\tag{3.5}$$

This formula (3.5) is a special case of the general formula (2.19). In the case of the

[1] Note that the definition (3.2) of the inverse Infeld-van der Waerden symbols is different from that of [3]. Their numeric values (3.3) are also different from (7.9) in [3]. However, they differ only by the numeric factor 1/2.

Infeld-van der Waerden field the reality condition for it is given by the formula:

$$\tau(\mathbf{G}) = \mathbf{G}. \tag{3.6}$$

Applying (2.19) to (3.6), we get the following equalities:

$$G^{i\bar{i}}_{q} = \overline{G^{\bar{i}i}_{q}}, \qquad\qquad G^{\,q}_{i\bar{i}} = \overline{G^{\,q}_{\bar{i}i}}. \tag{3.7}$$

These equalities (3.7) express the reality condition (3.6) in a coordinate form.

The Infeld-van der Waerden symbols satisfy various identities relating these symbols with the metric and spin-metric tensors:

$$\sum_{p=0}^{3}\sum_{q=0}^{3} g_{p\,q}\; G^{\,p}_{i\bar{i}}\; G^{\,q}_{j\bar{j}} = 2\,d_{ij}\;\bar{d}_{\bar{i}\bar{j}}, \tag{3.8}$$

$$\sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{\bar{i}=1}^{2}\sum_{\bar{j}=1}^{2} d_{ij}\;\bar{d}_{\bar{i}\bar{j}}\; G^{i\bar{i}}_{p}\; G^{j\bar{j}}_{q} = 2\,g_{p\,q}, \tag{3.9}$$

$$\sum_{p=0}^{3}\sum_{q=0}^{3} g^{p\,q}\; G^{i\bar{i}}_{p}\; G^{j\bar{j}}_{q} = 2\,d^{\,ij}\;\bar{d}^{\,\bar{i}\bar{j}}, \tag{3.10}$$

$$\sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{\bar{i}=1}^{2}\sum_{\bar{j}=1}^{2} d^{\,ij}\;\bar{d}^{\,\bar{i}\bar{j}}\, G^{\,p}_{i\bar{i}}\; G^{\,q}_{j\bar{j}} = 2\,g^{p\,q}. \tag{3.11}$$

The following equalities are easily derived from (3.8), (3.9), (3.10), and (3.11):

$$\sum_{i=1}^{2}\sum_{\bar{i}=1}^{2} G^{i\bar{i}}_{p}\; G^{\,q}_{i\bar{i}} = 2\,\delta^{q}_{p}, \qquad\qquad \sum_{q=0}^{3} G^{i\bar{i}}_{q}\; G^{\,q}_{j\bar{j}} = 2\,\delta^{i}_{j}\,\delta^{\bar{i}}_{\bar{j}}. \tag{3.12}$$

The equalities (3.8), (3.9), (3.10), (3.11), and (3.12) here are slightly different from (7.14), (7.15), (7.16), (7.17), and (7.13) in [3] since the quantities $G^{\,q}_{i\bar{i}}$ here differ from those of [3] by the numeric factor 1/2. The components of the dual spin-metric tensors in (3.10) and (3.11) are given by the matrices inverse to d_{ij} and $\bar{d}_{\bar{i}\bar{j}}$:

$$\sum_{q=1}^{2} d_{i\,q}\,d^{\,qj} = \delta^{j}_{i}, \qquad\qquad \sum_{\bar{q}=1}^{2} \bar{d}_{\bar{i}\bar{q}}\;\bar{d}^{\,\bar{q}\bar{j}} = \delta^{\bar{j}}_{\bar{i}}.$$

The identities (3.8), (3.9), (3.10), (3.11) are easily derived in canonically associated frame pairs (see definition 1.1). However, due to the spin-tensorial nature of the quantities in them they remain valid for arbitrary two frames of TM and SM.

4. Metric connections for chiral spinors.

Let $(U,\,\boldsymbol{\Upsilon}_0,\,\boldsymbol{\Upsilon}_1,\,\boldsymbol{\Upsilon}_2,\,\boldsymbol{\Upsilon}_3)$ and $(U,\,\boldsymbol{\Psi}_1,\,\boldsymbol{\Psi}_2)$ be two frames with a common domain U of the bundles TM and SM respectively. Let $(\tilde{U},\,\tilde{\boldsymbol{\Upsilon}}_0,\,\tilde{\boldsymbol{\Upsilon}}_1,\,\tilde{\boldsymbol{\Upsilon}}_2,\,\tilde{\boldsymbol{\Upsilon}}_3)$ and $(\tilde{U},\,\tilde{\boldsymbol{\Psi}}_1,\,\tilde{\boldsymbol{\Psi}}_2)$ be other two such frames. Assume that $U\cap\tilde{U}\neq\varnothing$. Then at each point p of the intersection $U\cap\tilde{U}$ one can write the transition formulas (1.4) and (1.6).

Assume that the domains U and $\tilde{U}$ are small enough so that one can introduce local coordinates x^0, x^1, x^2, x^3 and $\tilde{x}^0$, $\tilde{x}^1$, $\tilde{x}^2$, $\tilde{x}^3$ in them. Then, apart from the frames $(U,\,\boldsymbol{\Upsilon}_0,\,\boldsymbol{\Upsilon}_1,\,\boldsymbol{\Upsilon}_2,\,\boldsymbol{\Upsilon}_3)$ and $(\tilde{U},\,\tilde{\boldsymbol{\Upsilon}}_0,\,\tilde{\boldsymbol{\Upsilon}}_1,\,\tilde{\boldsymbol{\Upsilon}}_2,\,\tilde{\boldsymbol{\Upsilon}}_3)$, which are non-holonomic in general case, we have two holonomic coordinate frames $(U,\,\mathbf{E}_0,\,\mathbf{E}_1,\,\mathbf{E}_2,\,\mathbf{E}_3)$ and $(U,\,\tilde{\mathbf{E}}_0,\,\tilde{\mathbf{E}}_1,\,\tilde{\mathbf{E}}_2,\,\tilde{\mathbf{E}}_3)$ composed by the vector fields

$$\mathbf{E}_i=\frac{\partial}{\partial x^i}, \qquad\qquad \tilde{\mathbf{E}}_i=\frac{\partial}{\partial \tilde{x}^i}. \tag{4.1}$$

Taking the expansions of $\boldsymbol{\Upsilon}_i$ and $\tilde{\boldsymbol{\Upsilon}}_i$ in these holonomic frames

$$\boldsymbol{\Upsilon}_i=\sum_{j=0}^{3}\Upsilon_i^j\,\mathbf{E}_j, \qquad\qquad \tilde{\boldsymbol{\Upsilon}}_i=\sum_{j=0}^{3}\tilde{\Upsilon}_i^j\,\tilde{\mathbf{E}}_j, \tag{4.2}$$

due to (4.1) and (4.2) we can represent $\boldsymbol{\Upsilon}_i$ and $\tilde{\boldsymbol{\Upsilon}}_i$ as linear differential operators

$$\boldsymbol{\Upsilon}_i=\Upsilon_i^0\,\frac{\partial}{\partial x^0}+\Upsilon_i^1\,\frac{\partial}{\partial x^1}+\Upsilon_i^2\,\frac{\partial}{\partial x^2}+\Upsilon_i^3\,\frac{\partial}{\partial x^3}, \tag{4.3}$$

$$\tilde{\boldsymbol{\Upsilon}}_i=\tilde{\Upsilon}_i^0\,\frac{\partial}{\partial \tilde{x}^0}+\tilde{\Upsilon}_i^1\,\frac{\partial}{\partial \tilde{x}^1}+\tilde{\Upsilon}_i^2\,\frac{\partial}{\partial \tilde{x}^2}+\tilde{\Upsilon}_i^3\,\frac{\partial}{\partial \tilde{x}^3}. \tag{4.4}$$

Applying the differential operators (4.3) and (4.4) to a smooth scalar function f, we denote the resulting functions as the Lie derivatives:

$$L_{\boldsymbol{\Upsilon}_i}(f)=\sum_{j=0}^{3}\Upsilon_i^j\,\frac{\partial f}{\partial x^i}, \qquad\qquad L_{\tilde{\boldsymbol{\Upsilon}}_i}(f)=\sum_{j=0}^{3}\tilde{\Upsilon}_i^j\,\frac{\partial f}{\partial \tilde{x}^i}. \tag{4.5}$$

Note that the components of the transition matrices S, T, $\mathfrak{S}$, $\mathfrak{T}$ from (1.4) and (1.6) are smooth functions within the intersection domain $U\cap\tilde{U}$. Therefore, one can substitute them for f into (4.5). As a result we can define the following functions:

$$\tilde{\theta}_{ij}^k=\sum_{a=0}^{3}T_a^k\,L_{\tilde{\boldsymbol{\Upsilon}}_i}(S_j^a)=-\sum_{a=0}^{3}L_{\tilde{\boldsymbol{\Upsilon}}_i}(T_a^k)\,S_j^a, \tag{4.6}$$

$$\tilde{\vartheta}_{ij}^k=\sum_{a=1}^{2}\mathfrak{T}_a^k\,L_{\tilde{\boldsymbol{\Upsilon}}_i}(\mathfrak{S}_j^a)=-\sum_{a=1}^{2}L_{\tilde{\boldsymbol{\Upsilon}}_i}(\mathfrak{T}_a^k)\,\mathfrak{S}_j^a, \tag{4.7}$$

$$\theta_{ij}^k=\sum_{a=0}^{3}S_a^k\,L_{\boldsymbol{\Upsilon}_i}(T_j^a)=-\sum_{a=0}^{3}L_{\boldsymbol{\Upsilon}_i}(S_a^k)\,T_j^a, \tag{4.8}$$

$$\vartheta_{ij}^k=\sum_{a=1}^{2}\mathfrak{S}_a^k\,L_{\boldsymbol{\Upsilon}_i}(\mathfrak{T}_j^a)=-\sum_{a=1}^{2}L_{\boldsymbol{\Upsilon}_i}(\mathfrak{S}_a^k)\,\mathfrak{T}_j^a. \tag{4.9}$$

The θ-parameters with and without tilde introduced by the above formulas (4.6), (4.7), (4.8), (4.9) are related to each other through the following formulas:

$$\theta_{ij}^k=-\sum_{a=0}^{3}\sum_{b=0}^{3}\sum_{c=0}^{3}T_i^a\,\tilde{\theta}_{ab}^c\,S_c^k\,T_j^b, \tag{4.10}$$

$$\tilde{\theta}^k_{ij} = -\sum_{a=0}^{3}\sum_{b=0}^{3}\sum_{c=0}^{3} S^a_i\,\theta^{\,c}_{ab}\;T^k_c\,S^b_j, \tag{4.11}$$

$$\vartheta^k_{ij} = -\sum_{a=0}^{3}\sum_{b=1}^{2}\sum_{c=1}^{2} T^a_i\,\tilde{\vartheta}^{\,c}_{ab}\;\mathfrak{S}^k_c\,\mathfrak{T}^b_j, \tag{4.12}$$

$$\tilde{\vartheta}^k_{ij} = -\sum_{a=0}^{3}\sum_{b=1}^{2}\sum_{c=1}^{2} S^a_i\,\vartheta^{\,c}_{ab}\;\mathfrak{T}^k_c\,\mathfrak{S}^b_j. \tag{4.13}$$

In general case θ^k_{ij} and $\tilde{\theta}^k_{ij}$ are asymmetric in their lower indices and the extent of this asymmetry is characterized by the formulas

$$\theta^k_{ij} - \theta^k_{j\,i} = c^k_{ij}, \qquad\qquad \tilde{\theta}^k_{ij} - \tilde{\theta}^k_{j\,i} = \tilde{c}^k_{ij}, \tag{4.14}$$

where c^k_{ij} and $\tilde{c}^k_{ij}$ are defined by the following commutator relationships:

$$[\boldsymbol{\Upsilon}_i,\,\boldsymbol{\Upsilon}_j] = \sum_{k=0}^{3} c^k_{ij}\,\boldsymbol{\Upsilon}_k, \qquad\qquad [\tilde{\boldsymbol{\Upsilon}}_i,\,\tilde{\boldsymbol{\Upsilon}}_j] = \sum_{k=0}^{3} \tilde{c}^k_{ij}\,\tilde{\boldsymbol{\Upsilon}}_k. \tag{4.15}$$

The quantities c^k_{ij} and $\tilde{c}^k_{ij}$ in (4.14) and (4.15) are similar to structural constants of Lie algebras. For this reason they are called the *structural constants* of the frames $(U,\,\boldsymbol{\Upsilon}_0,\,\boldsymbol{\Upsilon}_1,\,\boldsymbol{\Upsilon}_2,\,\boldsymbol{\Upsilon}_3)$ and $(\tilde{U},\,\tilde{\boldsymbol{\Upsilon}}_0,\,\tilde{\boldsymbol{\Upsilon}}_1,\,\tilde{\boldsymbol{\Upsilon}}_2,\,\tilde{\boldsymbol{\Upsilon}}_3)$, though actually they are not constants, but smooth real-valued functions within the domains U and $\tilde{U}$ respectively. As for the identities (4.10), (4.11), (4.12), and (4.13), they are easily derived from (4.6), (4.7), (4.8), and (4.9) due to (4.5).

Definition 4.1. A *spinor connection* of the bundle of chiral spinors SM is a geometric object such that in each frame pair $(U,\,\boldsymbol{\Upsilon}_0,\,\boldsymbol{\Upsilon}_1,\,\boldsymbol{\Upsilon}_2,\,\boldsymbol{\Upsilon}_3)$ and $(U,\,\boldsymbol{\Psi}_1,\,\boldsymbol{\Psi}_2)$ of TM and SM it is given by three arrays of smooth complex-valued functions

$$\begin{aligned}
\Gamma^k_{ij} &= \Gamma^k_{ij}(p), \quad i,j,k = 0,\,\ldots\,,3,\\
\mathrm{A}^k_{ij} &= \mathrm{A}^k_{ij}(p), \quad i = 0,\,\ldots\,,3, \quad j,k = 1,2,\\
\bar{\mathrm{A}}^k_{ij} &= \bar{\mathrm{A}}^k_{ij}(p), \quad i = 0,\,\ldots\,,3, \quad j,k = 1,2,
\end{aligned}$$

where $p \in U$, such that when passing from $(U,\,\boldsymbol{\Upsilon}_0,\,\boldsymbol{\Upsilon}_1,\,\boldsymbol{\Upsilon}_2,\,\boldsymbol{\Upsilon}_3)$ and $(U,\,\boldsymbol{\Psi}_1,\,\boldsymbol{\Psi}_2)$ to some other frame pair $(\tilde{U},\,\tilde{\boldsymbol{\Upsilon}}_0,\,\tilde{\boldsymbol{\Upsilon}}_1,\,\tilde{\boldsymbol{\Upsilon}}_2,\,\tilde{\boldsymbol{\Upsilon}}_3)$ and $(\tilde{U},\,\tilde{\boldsymbol{\Psi}}_1,\,\tilde{\boldsymbol{\Psi}}_2)$ with $U \cap \tilde{U} \neq \varnothing$ these functions are transformed as follows:

$$\Gamma^k_{ij} = \sum_{b=0}^{3}\sum_{a=0}^{3}\sum_{c=0}^{3} S^k_a\,T^b_j\,T^c_i\;\tilde{\Gamma}^a_{c\,b} + \theta^k_{ij}, \tag{4.16}$$

$$\mathrm{A}^k_{ij} = \sum_{b=1}^{2}\sum_{a=1}^{2}\sum_{c=0}^{3} \mathfrak{S}^k_a\,\mathfrak{T}^b_j\,T^c_i\;\tilde{\mathrm{A}}^a_{c\,b} + \vartheta^k_{ij}, \tag{4.17}$$

$$\bar{\mathrm{A}}^k_{ij} = \sum_{b=1}^{2}\sum_{a=1}^{2}\sum_{c=0}^{3} \overline{\mathfrak{S}^k_a}\;\overline{\mathfrak{T}^b_j}\,T^c_i\;\tilde{\bar{\mathrm{A}}}^a_{c\,b} + \overline{\vartheta^k_{ij}}. \tag{4.18}$$

The components of transition matrices S, T, $\mathfrak{S}$, and $\mathfrak{T}$ in (4.16), (4.17), and (4.18) are taken from (1.4) and (1.6), while the quantities θ^k_{ij} and ϑ^k_{ij} are defined in (4.8) and (4.9). Spinor connections introduced by the definition 4.1 are used in order to define covariant differentiations acting upon spin-tensorial fields and producing other spin-tensorial fields from them. The covariant differential ∇ associated with the spinor connection $(\Gamma, \mathrm{A}, \bar{\mathrm{A}})$ is a differential operator

$$\nabla\colon\ S^\alpha_\beta \bar{S}^\nu_\gamma T^m_n M \to S^\alpha_\beta \bar{S}^\nu_\gamma T^m_{n+1} M. \tag{4.19}$$

In a frame pair $(U,\, \mathbf{\Upsilon}_0,\, \mathbf{\Upsilon}_1,\, \mathbf{\Upsilon}_2,\, \mathbf{\Upsilon}_3)$ and $(U,\, \mathbf{\Psi}_1,\, \mathbf{\Psi}_2)$, i.e. in a coordinate form, the operator (4.19) is represented by the corresponding covariant derivative

$$\begin{aligned}
&\nabla_{k_{n+1}} X^{i_1\ldots\, i_\alpha\, \bar{i}_1\ldots\, \bar{i}_\nu\, h_1\ldots\, h_m}_{j_1\ldots\, j_\beta\, \bar{j}_1\ldots\, \bar{j}_\gamma\, k_1\ldots\, k_n} = L_{\mathbf{\Upsilon}_{k_{n+1}}}\bigl(X^{i_1\ldots\, i_\alpha\, \bar{i}_1\ldots\, \bar{i}_\nu\, h_1\ldots\, h_m}_{j_1\ldots\, j_\beta\, \bar{j}_1\ldots\, \bar{j}_\gamma\, k_1\ldots\, k_n}\bigr) - \\
&+\sum^{\alpha}_{\mu=1}\sum^{2}_{v_\mu=1} \mathrm{A}^{i_\mu}_{k_{n+1}\, v_\mu}\ X^{i_1\ldots\, v_\mu\, \ldots\, i_\varepsilon\, \bar{i}_1\ldots\, \bar{i}_\nu\, h_1\ldots\, h_m}_{j_1\ldots\,\ldots\,\ldots\, j_\beta\, \bar{j}_1\ldots\, \bar{j}_\gamma\, k_1\ldots\, k_n} - \\
&\qquad -\sum^{\beta}_{\mu=1}\sum^{2}_{w_\mu=1} \mathrm{A}^{w_\mu}_{k_{n+1}\, j_\mu}\ X^{i_1\ldots\,\ldots\,\ldots\, i_\alpha\, \bar{i}_1\ldots\, \bar{i}_\nu h_1\ldots\, h_m}_{j_1\ldots\, w_\mu\, \ldots\, j_\beta\, \bar{j}_1\ldots\, \bar{j}_\gamma k_1\ldots\, k_n} + \\
&+\sum^{\nu}_{\mu=1}\sum^{2}_{v_\mu=1} \bar{\mathrm{A}}^{\bar{i}_\mu}_{k_{n+1}\, v_\mu}\ X^{i_1\ldots\, i_\alpha\, \bar{i}_1\ldots\, v_\mu\, \ldots\, \bar{i}_\nu\, h_1\ldots\, h_m}_{j_1\ldots\, j_\beta\, \bar{j}_1\ldots\,\ldots\,\ldots\, \bar{j}_\gamma\, k_1\ldots\, k_n} - \\
&\qquad -\sum^{\gamma}_{\mu=1}\sum^{2}_{w_\mu=1} \bar{\mathrm{A}}^{w_\mu}_{k_{n+1}\, \bar{j}_\mu}\ X^{i_1\ldots\, i_\alpha\, \bar{i}_1\ldots\,\ldots\,\ldots\, \bar{i}_\nu\, h_1\ldots\, h_m}_{j_1\ldots\, j_\beta\, \bar{j}_1\ldots\, w_\mu\, \ldots\, \bar{j}_\gamma\, k_1\ldots\, k_n} + \\
&+\sum^{m}_{\mu=1}\sum^{3}_{v_\mu=0} \Gamma^{h_\mu}_{k_{n+1}\, v_\mu}\ X^{i_1\ldots\, i_\alpha\, \bar{i}_1\ldots\, \bar{i}_\nu\, h_1\ldots\, v_\mu\, \ldots\, h_m}_{j_1\ldots\, j_\beta\, \bar{j}_1\ldots\, \bar{j}_\gamma\, k_1\ldots\,\ldots\,\ldots\, k_n} - \\
&\qquad -\sum^{n}_{\mu=1}\sum^{3}_{w_\mu=0} \Gamma^{w_\mu}_{k_{n+1}\, k_\mu}\ X^{i_1\ldots\, i_\alpha\, \bar{i}_1\ldots\, \bar{i}_\nu\, h_1\ldots\,\ldots\,\ldots\, h_m}_{j_1\ldots\, j_\beta\, \bar{j}_1\ldots\, \bar{j}_\gamma\, k_1\ldots\, w_\mu\, \ldots\, k_n}.
\end{aligned} \tag{4.20}$$

The formula (4.20) should be understood in the following way. If $\mathbf{X}$ is a spin-tensorial field of the type $(\alpha, \beta|\nu, \gamma|m, n)$ and $X^{i_1\ldots\, i_\alpha\, \bar{i}_1\ldots\, \bar{i}_\nu\, h_1\ldots\, h_m}_{j_1\ldots\, j_\beta\, \bar{j}_1\ldots\, \bar{j}_\gamma\, k_1\ldots\, k_n}$ is its coordinate representation in the expansion (2.14), then for the spin-tensorial field $\mathbf{Y} = \nabla\mathbf{X}$ its coordinate representation is given by the formula

$$Y^{i_1\ldots\, i_\alpha\, \bar{i}_1\ldots\, \bar{i}_\nu\, h_1\ldots\, h_m}_{j_1\ldots\, j_\beta\, \bar{j}_1\ldots\, \bar{j}_\gamma\, k_1\ldots\, k_{n+1}} = \nabla_{k_{n+1}} X^{i_1\ldots\, i_\alpha\, \bar{i}_1\ldots\, \bar{i}_\nu\, h_1\ldots\, h_m}_{j_1\ldots\, j_\beta\, \bar{j}_1\ldots\, \bar{j}_\gamma\, k_1\ldots\, k_n}.$$

Definition 4.2. A spinor connection $(\Gamma, \mathrm{A}, \bar{\mathrm{A}})$ of the bundle of chiral spinors SM is called *concordant with the complex conjugation* if the corresponding covariant differential (4.19) commute with the involution τ, i. e. if $\nabla(\tau(\mathbf{X})) = \tau(\nabla\mathbf{X})$ for any spin-tensorial field $\mathbf{X}$.

Spinor connections concordant with the complex conjugation τ in the sense of the above definition 4.2 are also called *real connections*.

Theorem 4.1. *A spinor connection* $(\Gamma, \mathrm{A}, \bar{\mathrm{A}})$ *of the bundle of chiral spinors* SM *is concordant with the complex conjugation* τ *if and only if*

$$\Gamma^k_{ij} = \overline{\Gamma^k_{ij}}, \qquad\qquad \bar{\mathrm{A}}^k_{ij} = \overline{\mathrm{A}^k_{ij}}. \tag{4.21}$$

The theorem 4.1 is proved by direct calculations on the base of the formula (4.20). The first relationship in (4.21) means that Γ-components of a real spinor connection are real functions. They obey the transformation rules (4.16) coinciding with the transformation rules for the components of an affine connection.

Corollary 4.1. *Any real spinor connection* $(\Gamma, \mathrm{A}, \bar{\mathrm{A}})$ *of the bundle of chiral spinors* SM *comprises some affine connection* Γ *as its constituent part.*

Definition 4.3. A spinor connection $(\Gamma, \mathrm{A}, \bar{\mathrm{A}})$ of the bundle of chiral spinors SM is called *concordant with the Infeld-van der Waerden field* if $\nabla\mathbf{G} = 0$.

Definition 4.4. A spinor connection $(\Gamma, \mathrm{A}, \bar{\mathrm{A}})$ of the bundle of chiral spinors SM is called *concordant with the spin-metric tensor* if $\nabla\mathbf{d} = 0$.

Definition 4.5. A spinor connection $(\Gamma, \mathrm{A}, \bar{\mathrm{A}})$ of the bundle of chiral spinors SM is called *concordant with the metric tensor* if $\nabla\mathbf{g} = 0$.

Theorem 4.2. *Any real spinor connection* $(\Gamma, \mathrm{A}, \bar{\mathrm{A}})$ *of the bundle of chiral spinors* SM *concordant with the Infeld-van der Waerden field* $\mathbf{G}$ *and with the spin-metric tensor* $\mathbf{d}$ *is concordant with the metric tensor* $\mathbf{g}$ *too, i. e. for a real spinor connection* $\nabla\mathbf{G} = 0$ *and* $\nabla\mathbf{d} = 0$ *imply* $\nabla\mathbf{g} = 0$.

Proof. Since $(\Gamma, \mathrm{A}, \bar{\mathrm{A}})$ is real, from $\nabla\mathbf{d} = 0$ we easily derive that $\nabla\bar{\mathbf{d}} = 0$:

$$\nabla\bar{\mathbf{d}} = \nabla(\tau(\mathbf{d})) = \tau(\nabla\mathbf{d}) = 0.$$

Then we apply ∇_r to the identity (3.9). As a result we get

$$\begin{aligned} 2\,\nabla_r g_{p\,q} = \sum^2_{i=1}\sum^2_{j=1}\sum^2_{\bar{i}=1}\sum^2_{\bar{j}=1}\Bigl(\nabla_r d_{ij}\ \bar{d}_{\bar{i}\bar{j}}\ G^{i\bar{i}}_p\ G^{j\bar{j}}_q + d_{ij}\ \nabla_r\bar{d}_{\bar{i}\bar{j}}\ \times\\ \times\, G^{i\bar{i}}_p\ G^{j\bar{j}}_q + d_{ij}\ \bar{d}_{\bar{i}\bar{j}}\ \nabla_r G^{i\bar{i}}_p\ G^{j\bar{j}}_q + d_{ij}\ \bar{d}_{\bar{i}\bar{j}}\ G^{i\bar{i}}_p\ \nabla_r G^{j\bar{j}}_q\Bigr) = 0. \end{aligned} \tag{4.22}$$

The identity (4.22) means that $\nabla\mathbf{g} = 0$. Thus, the theorem 4.2 is proved. □

The concordance condition $\nabla\mathbf{g} = 0$ is well-known. Since $\mathbf{g}$ is a spin-tensorial field of the type $(0,0|0,0|0,2)$, this condition is written in terms of the Γ-components of a spinor connection only. Applying the formula (4.20) to $\nabla_r g_{ij}$, we get

$$L_{\boldsymbol{\Upsilon}_r}(g_{ij}) - \sum^3_{s=0}\Gamma^s_{ri}\ g_{sj} - \sum^3_{s=0}\Gamma^s_{rj}\ g_{is} = 0. \tag{4.23}$$

In general non-holonomic frame $(U,\ \boldsymbol{\Upsilon}_0,\ \boldsymbol{\Upsilon}_1,\ \boldsymbol{\Upsilon}_2,\ \boldsymbol{\Upsilon}_3)$ the Γ-components of a spinor connection are not symmetric. Therefore we subdivide Γ^k_{rs} into symmetric a skew-symmetric parts $\hat{\Gamma}^k_{rs}$ and $\check{\Gamma}^k_{rs}$ respectively:

$$\Gamma^k_{rs} = \hat{\Gamma}^k_{rs} + \check{\Gamma}^k_{rs}. \tag{4.24}$$

Lowering the upper index k of Γ^k_{rs}, $\hat{\Gamma}^k_{rs}$, and $\check{\Gamma}^k_{rs}$, we define the following quantities:

$$\Gamma_{rsq}=\sum_{k=0}^{3}\Gamma^k_{rs}\;g_{kq},\qquad\hat{\Gamma}_{rsq}=\sum_{k=0}^{3}\hat{\Gamma}^k_{rs}\;g_{kq},\qquad\check{\Gamma}_{rsq}=\sum_{k=0}^{3}\check{\Gamma}^k_{rs}\;g_{kq}.\tag{4.25}$$

From (4.24) we derive the analogous expansion of Γ_{rsq} into two parts

$$\Gamma_{rsq}=\hat{\Gamma}_{rsq}+\check{\Gamma}_{rsq},\tag{4.26}$$

where $\hat{\Gamma}_{rsq}=\hat{\Gamma}_{srq}$ and $\check{\Gamma}_{rsq}=-\check{\Gamma}_{srq}$. Applying (4.25) and (4.26) to (4.23), we get

$$\hat{\Gamma}_{rij}+\hat{\Gamma}_{rji}=L_{\boldsymbol{\Upsilon}_r}(g_{ij})-\check{\Gamma}_{rij}-\check{\Gamma}_{rji}.\tag{4.27}$$

By means of the cyclic transposition of indices from (4.27) we derive

$$\hat{\Gamma}_{ijr}+\hat{\Gamma}_{irj}=L_{\boldsymbol{\Upsilon}_i}(g_{jr})-\check{\Gamma}_{ijr}-\check{\Gamma}_{irj},\tag{4.28}$$

$$\hat{\Gamma}_{jri}+\hat{\Gamma}_{jir}=L_{\boldsymbol{\Upsilon}_j}(g_{ri})-\check{\Gamma}_{jri}-\check{\Gamma}_{jir}.\tag{4.29}$$

Now let's add (4.28) and (4.29), then subtract (4.27) from the sum. As a result, taking into account the symmetry of $\hat{\Gamma}$ and the skew-symmetry of $\check{\Gamma}$, we get

$$\hat{\Gamma}_{ijr}=\frac{L_{\boldsymbol{\Upsilon}_i}(g_{jr})+L_{\boldsymbol{\Upsilon}_j}(g_{ri})-L_{\boldsymbol{\Upsilon}_r}(g_{ij})}{2}-\check{\Gamma}_{irj}-\check{\Gamma}_{jri}.\tag{4.30}$$

Now, raising the lower index r in (4.30), we obtain the explicit formula for the symmetric part of the Γ-symbols expressing them through the Lie derivatives $L_{\boldsymbol{\Upsilon}_i}(g_{jr})$, $L_{\boldsymbol{\Upsilon}_j}(g_{ri})$, $L_{\boldsymbol{\Upsilon}_r}(g_{ij})$ and through the skew-symmetric part of these Γ-symbols:

$$\begin{aligned}\hat{\Gamma}^k_{ij}=\sum_{r=0}^{3}\frac{g^{kr}}{2}\left(L_{\boldsymbol{\Upsilon}_i}(g_{jr})+L_{\boldsymbol{\Upsilon}_j}(g_{ri})-L_{\boldsymbol{\Upsilon}_r}(g_{ij})\right)\,-\\-\sum_{r=0}^{3}\sum_{s=0}^{3}\check{\Gamma}^s_{ir}\,g^{kr}\,g_{sj}-\sum_{r=0}^{3}\sum_{s=0}^{3}\check{\Gamma}^s_{jr}\,g^{kr}\,g_{si}.\end{aligned}\tag{4.31}$$

From (4.31) and (4.24) for the Γ-symbols themselves we derive

$$\begin{aligned}\Gamma^k_{ij}=\sum_{r=0}^{3}\frac{g^{kr}}{2}\left(L_{\boldsymbol{\Upsilon}_i}(g_{jr})+L_{\boldsymbol{\Upsilon}_j}(g_{ri})-L_{\boldsymbol{\Upsilon}_r}(g_{ij})\right)+\\+\check{\Gamma}^k_{ij}-\sum_{r=0}^{3}\sum_{s=0}^{3}\check{\Gamma}^s_{ir}\,g^{kr}\,g_{sj}-\sum_{r=0}^{3}\sum_{s=0}^{3}\check{\Gamma}^s_{jr}\,g^{kr}\,g_{si}.\end{aligned}\tag{4.32}$$

Note that the skew-symmetric part of the Γ-symbols is determined by the *torsion tensor* $\mathbf{T}$ (see [5]) and by the structural constants c^k_{ij} (see their definition (4.15)):

$$\check{\Gamma}^k_{ij}=\frac{T^k_{ij}-c^k_{ij}}{2}.\tag{4.33}$$

Substituting (4.33) into (4.32), we derive the ultimate formula for Γ^k_{ij}:

$$\begin{aligned}\Gamma^k_{ij} &= \sum_{r=0}^{3}\frac{g^{kr}}{2}\left(L_{\Upsilon_i}(g_{jr})+L_{\Upsilon_j}(g_{ri})-L_{\Upsilon_r}(g_{ij})\right)-\\ &-\frac{c^k_{ij}}{2}+\sum_{r=0}^{3}\sum_{s=0}^{3}g^{kr}\,\frac{c^s_{ir}}{2}\,g_{sj}+\sum_{r=0}^{3}\sum_{s=0}^{3}g^{kr}\,\frac{c^s_{jr}}{2}\,g_{si}+\\ &+\frac{T^k_{ij}}{2}-\sum_{r=0}^{3}\sum_{s=0}^{3}g^{kr}\,\frac{T^s_{ir}}{2}\,g_{sj}-\sum_{r=0}^{3}\sum_{s=0}^{3}g^{kr}\,\frac{T^s_{jr}}{2}\,g_{si}.\end{aligned}\tag{4.34}$$

Definition 4.6. A real spinor connection $(\Gamma,\mathrm{A},\bar{\mathrm{A}})$ of the bundle of chiral spinors SM is called a *metric connection*, if it is concordant 1) with the spin-metric tensor $\mathbf{d}$, 2) with the Infeld-van der Waerden field $\mathbf{G}$, and 3) with the metric tensor $\mathbf{d}$.

The theorem 4.2 says that the conditions 1) and 2) are sufficient for a real spinor connection $(\Gamma,\mathrm{A},\bar{\mathrm{A}})$ to be a metric connection. The Einstein's theory of gravity, which is also called the General Relativity, is a theory without torsion, i. e. the torsion tensor $\mathbf{T}$ is taken to be zero in it: $\mathbf{T}=0$. Then (4.34) reduces to

$$\begin{aligned}\Gamma^k_{ij} &= \sum_{r=0}^{3}\frac{g^{kr}}{2}\left(L_{\Upsilon_i}(g_{jr})+L_{\Upsilon_j}(g_{ri})-L_{\Upsilon_r}(g_{ij})\right)-\\ &-\frac{c^k_{ij}}{2}+\sum_{r=0}^{3}\sum_{s=0}^{3}g^{kr}\,\frac{c^s_{ir}}{2}\,g_{sj}+\sum_{r=0}^{3}\sum_{s=0}^{3}g^{kr}\,\frac{c^s_{jr}}{2}\,g_{si}.\end{aligned}\tag{4.35}$$

According to the corollary 4.1, the Γ-components of a real metric spinor connection in *General Relativity* are the components of the Levi-Civita connection for the metric $\mathbf{g}$ and (4.35) is a frame version of the well-known formula

$$\Gamma^k_{ij}=\sum_{r=0}^{3}\frac{g^{kr}}{2}\left(\frac{\partial g_{jr}}{\partial x^i}+\frac{\partial g_{ri}}{\partial x^j}-\frac{\partial g_{ij}}{\partial x^r}\right).$$

The above calculations leading to the formula (4.35) are standard. They are similar to those performed in section 3 of [7].

Now let's study the concordance condition $\nabla\mathbf{d}=0$. In a frame relative coordinate form this condition is written as follows:

$$\nabla_r d_{ij}=L_{\Upsilon_r}(d_{ij})-\sum_{s=1}^{2}\mathrm{A}^s_{ri}\;d_{sj}-\sum_{s=1}^{2}\mathrm{A}^s_{rj}\;d_{is}=0.\tag{4.36}$$

By lowering the upper index s of A^s_{ri} we introduce the following quantities:

$$\mathrm{A}_{rij}=\sum_{s=1}^{2}\mathrm{A}^s_{ri}\;d_{sj}.\tag{4.37}$$

Then, due to (4.37), the equality (4.36) reduces to the following one:

$$\mathrm{A}_{rij}-\mathrm{A}_{rji}=L_{\Upsilon_r}(d_{ij}).\tag{4.38}$$

The formula (4.38) means that the skew-symmetric part of A_{rij} is determined by the Lie derivative $L_{\boldsymbol{\Upsilon}_r}(d_{ij})$ so that we can write

$$\mathrm{A}_{rij}=\hat{\mathrm{A}}_{rij}+\frac{L_{\boldsymbol{\Upsilon}_r}(d_{ij})}{2},\tag{4.39}$$

where $\hat{\mathrm{A}}_{rij}=\hat{\mathrm{A}}_{rji}$. Returning from (4.39) back to the quantities A^s_{ri}, we get

$$\mathrm{A}^s_{ri}=\sum_{j=1}^{2}\hat{\mathrm{A}}_{rij}\,d^{js}+\sum_{j=1}^{2}\frac{L_{\boldsymbol{\Upsilon}_r}(d_{ij})\;d^{js}}{2}.\tag{4.40}$$

Acting in a similar way, from $\nabla\bar{\mathbf{d}}=0$ we easily derive the formulas

$$\bar{\mathrm{A}}_{rij}=\hat{\bar{\mathrm{A}}}_{rij}+\frac{L_{\boldsymbol{\Upsilon}_r}(\bar{d}_{ij})}{2},\tag{4.41}$$

$$\bar{\mathrm{A}}^s_{ri}=\sum_{j=1}^{2}\hat{\bar{\mathrm{A}}}_{rij}\,\bar{d}^{js}+\sum_{j=1}^{2}\frac{L_{\boldsymbol{\Upsilon}_r}(\bar{d}_{ij})\;\bar{d}^{js}}{2},\tag{4.42}$$

where $\hat{\bar{\mathrm{A}}}_{rij}=\hat{\bar{\mathrm{A}}}_{rji}$. Thus, we have managed to reduce the concordance conditions $\nabla\mathbf{d}=0$ and $\nabla\bar{\mathbf{d}}=0$ to the symmetry conditions

$$\hat{\mathrm{A}}_{rij}=\hat{\mathrm{A}}_{rji},\qquad\qquad \hat{\bar{\mathrm{A}}}_{rij}=\hat{\bar{\mathrm{A}}}_{rji}\tag{4.43}$$

and to the formulas (4.40) and (4.42) for the A and $\bar{\mathrm{A}}$-components of a spinor connection $(\Gamma,\mathrm{A},\bar{\mathrm{A}})$.

The next step is to study the concordance conditions $\nabla\mathbf{G}=0$. Applying the formula (4.20) to $\nabla\mathbf{G}=0$, we derive the following equality:

$$L_{\boldsymbol{\Upsilon}_r}(G^{i\bar{i}}_q)+\sum_{s=1}^{2}G^{s\bar{i}}_q\;\mathrm{A}^i_{rs}+\sum_{\bar{s}=1}^{2}G^{i\bar{s}}_q\;\bar{\mathrm{A}}^{\bar{i}}_{r\bar{s}}-\sum_{p=0}^{3}G^{i\bar{i}}_p\,\Gamma^p_{rq}=0.\tag{4.44}$$

In order to transform (4.44) we multiply it by $G^{\,q}_{j\bar{j}}$ and sum it over the index q, meanwhile taking into account the second identity (3.12):

$$2\;\mathrm{A}^i_{rj}\;\delta^{\bar{i}}_{\bar{j}}+2\,\delta^i_j\,\bar{\mathrm{A}}^{\bar{i}}_{r\bar{j}}=\sum_{p=0}^{3}\sum_{q=0}^{3}G^{i\bar{i}}_p\,\Gamma^p_{rq}\,G^{\,q}_{j\bar{j}}-\sum_{q=0}^{3}L_{\boldsymbol{\Upsilon}_r}(G^{i\bar{i}}_q)\,G^{\,q}_{j\bar{j}}.\tag{4.45}$$

Then we apply the index lowering procedure to the indices i and $\bar{i}$ in (4.45). As a result, taking into account (4.37), we derive

$$\begin{aligned}2\;\mathrm{A}_{rji}\;\bar{d}_{\bar{j}\bar{i}}+2\,d_{ji}\,\bar{\mathrm{A}}_{r\bar{j}\bar{i}}=\sum_{s=1}^{2}\sum_{\bar{s}=1}^{2}\sum_{p=0}^{3}\sum_{q=0}^{3}d_{si}\,\bar{d}_{\bar{s}\bar{i}}\,G^{s\bar{s}}_p\,\Gamma^p_{rq}\,G^{\,q}_{j\bar{j}}\,-\\-\sum_{s=1}^{2}\sum_{\bar{s}=1}^{2}\sum_{q=0}^{3}d_{si}\,\bar{d}_{\bar{s}\bar{i}}\,L_{\boldsymbol{\Upsilon}_r}(G^{s\bar{s}}_q)\,G^{\,q}_{j\bar{j}}.\end{aligned}\tag{4.46}$$

Substituting (4.39) and (4.41) into (4.46), we derive

$$
\begin{aligned}
2\,\hat{\mathrm{A}}_{rji}\,\bar{d}_{\bar{j}\bar{i}}+2\,d_{ji}\,\hat{\bar{\mathrm{A}}}_{r\bar{j}\bar{i}}=\sum_{s=1}^{2}\sum_{\bar{s}=1}^{2}\sum_{p=0}^{3}\sum_{q=0}^{3}d_{si}\,\bar{d}_{\bar{s}\bar{i}}\,G^{s\bar{s}}_{p}\,\Gamma^{p}_{rq}\,G^{\,q}_{j\bar{j}}\,-\\
-\sum_{s=1}^{2}\sum_{\bar{s}=1}^{2}\sum_{q=0}^{3}d_{si}\,\bar{d}_{\bar{s}\bar{i}}\,L_{\boldsymbol{\Upsilon}_r}(G^{s\bar{s}}_{q})\,G^{\,q}_{j\bar{j}}-L_{\boldsymbol{\Upsilon}_r}(d_{ji})\,\bar{d}_{\bar{j}\bar{i}}-L_{\boldsymbol{\Upsilon}_r}(\bar{d}_{\bar{j}\bar{i}})\,d_{ji}.
\end{aligned}
\tag{4.47}
$$

The left hand side of the equality (4.47) is a sum of two terms. The first term is symmetric in i and j, see (4.43), while the other is skew-symmetric in these indices. Therefore, if we subdivide the right hand side of (4.47) into symmetric and skew-symmetric parts, we can write (4.47) as two separate equalities:

$$
\begin{aligned}
\hat{\mathrm{A}}_{rji}\,\bar{d}_{\bar{j}\bar{i}}=\sum_{s=1}^{2}\sum_{\bar{s}=1}^{2}\sum_{p=0}^{3}\sum_{q=0}^{3}\frac{d_{si}\,\bar{d}_{\bar{s}\bar{i}}\,G^{s\bar{s}}_{p}\,\Gamma^{p}_{rq}\,G^{\,q}_{j\bar{j}}+d_{sj}\,\bar{d}_{\bar{s}\bar{i}}\,G^{s\bar{s}}_{p}\,\Gamma^{p}_{rq}\,G^{\,q}_{i\bar{j}}}{4}\,-\\
-\sum_{s=1}^{2}\sum_{\bar{s}=1}^{2}\sum_{q=0}^{3}\frac{d_{si}\,\bar{d}_{\bar{s}\bar{i}}\,L_{\boldsymbol{\Upsilon}_r}(G^{s\bar{s}}_{q})\,G^{\,q}_{j\bar{j}}+d_{sj}\,\bar{d}_{\bar{s}\bar{i}}\,L_{\boldsymbol{\Upsilon}_r}(G^{s\bar{s}}_{q})\,G^{\,q}_{i\bar{j}}}{4},
\end{aligned}
\tag{4.48}
$$

$$
\begin{aligned}
d_{ji}\,\hat{\bar{\mathrm{A}}}_{r\bar{j}\bar{i}}=\sum_{s=1}^{2}\sum_{\bar{s}=1}^{2}\sum_{p=0}^{3}\sum_{q=0}^{3}\frac{d_{si}\,\bar{d}_{\bar{s}\bar{i}}\,G^{s\bar{s}}_{p}\,\Gamma^{p}_{rq}\,G^{\,q}_{j\bar{j}}-d_{sj}\,\bar{d}_{\bar{s}\bar{i}}\,G^{s\bar{s}}_{p}\,\Gamma^{p}_{rq}\,G^{\,q}_{i\bar{j}}}{4}\,-\\
-\sum_{s=1}^{2}\sum_{\bar{s}=1}^{2}\sum_{q=0}^{3}\frac{d_{si}\,\bar{d}_{\bar{s}\bar{i}}\,L_{\boldsymbol{\Upsilon}_r}(G^{s\bar{s}}_{q})\,G^{\,q}_{j\bar{j}}-d_{sj}\,\bar{d}_{\bar{s}\bar{i}}\,L_{\boldsymbol{\Upsilon}_r}(G^{s\bar{s}}_{q})\,G^{\,q}_{i\bar{j}}}{4}\,-\\
-\frac{L_{\boldsymbol{\Upsilon}_r}(d_{ji})\,\bar{d}_{\bar{j}\bar{i}}+L_{\boldsymbol{\Upsilon}_r}(\bar{d}_{\bar{j}\bar{i}})\,d_{ji}}{2}.
\end{aligned}
\tag{4.49}
$$

In two-dimensional case any equality skew-symmetric in two indices is equivalent to a scalar equality independent of these two indices. In the case of the equality (4.49) we can multiply it by d^{ij} and sum over the indices i and j. As a result we obtain the following equality equivalent to (4.49):

$$
\begin{aligned}
\hat{\bar{\mathrm{A}}}_{r\bar{j}\bar{i}}=-\frac{L_{\boldsymbol{\Upsilon}_r}(\bar{d}_{\bar{j}\bar{i}})}{2}+\sum_{s=1}^{2}\sum_{\bar{s}=1}^{2}\sum_{p=0}^{3}\sum_{q=0}^{3}\frac{\bar{d}_{\bar{s}\bar{i}}\,G^{s\bar{s}}_{p}\,\Gamma^{p}_{rq}\,G^{\,q}_{s\bar{j}}}{4}\,-\\
-\sum_{s=1}^{2}\sum_{\bar{s}=1}^{2}\sum_{q=0}^{3}\frac{\bar{d}_{\bar{s}\bar{i}}\,L_{\boldsymbol{\Upsilon}_r}(G^{s\bar{s}}_{q})\,G^{\,q}_{s\bar{j}}}{4}-\sum_{i=1}^{2}\sum_{j=1}^{2}\frac{L_{\boldsymbol{\Upsilon}_r}(d_{ji})\,d^{ij}\,\bar{d}_{\bar{j}\bar{i}}}{4}.
\end{aligned}
\tag{4.50}
$$

By definition the left hand side of (4.50) is symmetric in $\bar{i}$ and $\bar{j}$. Therefore it should be equal to the symmetric part of the right hand side

$$
\begin{aligned}
\hat{\bar{\mathrm{A}}}_{r\bar{j}\bar{i}}=\sum_{s=1}^{2}\sum_{\bar{s}=1}^{2}\sum_{p=0}^{3}\sum_{q=0}^{3}\frac{\bar{d}_{\bar{s}\bar{i}}\,G^{s\bar{s}}_{p}\,\Gamma^{p}_{rq}\,G^{\,q}_{s\bar{j}}+\bar{d}_{\bar{s}\bar{j}}\,G^{s\bar{s}}_{p}\,\Gamma^{p}_{rq}\,G^{\,q}_{s\bar{i}}}{8}\,-\\
-\sum_{s=1}^{2}\sum_{\bar{s}=1}^{2}\sum_{q=0}^{3}\frac{\bar{d}_{\bar{s}\bar{i}}\,L_{\boldsymbol{\Upsilon}_r}(G^{s\bar{s}}_{q})\,G^{\,q}_{s\bar{j}}+\bar{d}_{\bar{s}\bar{j}}\,L_{\boldsymbol{\Upsilon}_r}(G^{s\bar{s}}_{q})\,G^{\,q}_{s\bar{i}}}{8},
\end{aligned}
\tag{4.51}
$$

while the skew-symmetric part of the right hand side of (4.50) should be zero:

$$\begin{gathered}\sum_{s=1}^{2}\sum_{\bar s=1}^{2}\sum_{p=0}^{3}\sum_{q=0}^{3}\frac{\bar d_{\bar s\bar i}\,G^{s\bar s}_p\,\Gamma^p_{rq}\,G^{\,q}_{s\bar j}-\bar d_{\bar s\bar j}\,G^{s\bar s}_p\,\Gamma^p_{rq}\,G^{\,q}_{s\bar i}}{8}\,-\\ -\sum_{s=1}^{2}\sum_{\bar s=1}^{2}\sum_{q=0}^{3}\frac{\bar d_{\bar s\bar i}\,L_{\Upsilon_r}(G^{s\bar s}_q)\,G^{\,q}_{s\bar j}-\bar d_{\bar s\bar j}\,L_{\Upsilon_r}(G^{s\bar s}_q)\,G^{\,q}_{s\bar i}}{8}\,-\\ -\frac{L_{\Upsilon_r}(\bar d_{\bar j\bar i})}{2}-\sum_{i=1}^{2}\sum_{j=1}^{2}\frac{L_{\Upsilon_r}(d_{j\,i})\,d^{ij}\,\bar d_{\bar j\bar i}}{4}=0.\end{gathered}\tag{4.52}$$

Again, using the feature of the two-dimensional case, we can reduce (4.52) to an equality independent of $\bar i$ and $\bar j$. For this purpose let's multiply it by $\bar d^{\bar i\bar j}$ and sum over the indices $\bar i$ and $\bar j$. As a result we get

$$\begin{gathered}\sum_{s=1}^{2}\sum_{\bar s=1}^{2}\sum_{p=0}^{3}\sum_{q=0}^{3}\frac{G^{s\bar s}_p\,\Gamma^p_{rq}\,G^{\,q}_{s\bar s}}{4}-\sum_{s=1}^{2}\sum_{\bar s=1}^{2}\sum_{q=0}^{3}\frac{L_{\Upsilon_r}(G^{s\bar s}_q)\,G^{\,q}_{s\bar s}}{4}\,-\\ -\sum_{\bar i=1}^{2}\sum_{\bar j=1}^{2}\frac{L_{\Upsilon_r}(\bar d_{\bar j\bar i})\,\bar d^{\bar i\bar j}}{2}-\sum_{i=1}^{2}\sum_{j=1}^{2}\frac{L_{\Upsilon_r}(d_{j\,i})\,d^{ij}}{2}=0.\end{gathered}\tag{4.53}$$

Taking into account the first identity (3.12) and the formula (3.2), we can transform the equality (4.53) to a more symmetric form:

$$\begin{gathered}\sum_{p=0}^{3}\frac{\Gamma^p_{rp}}{2}-\sum_{s=1}^{2}\sum_{\bar s=1}^{2}\sum_{q=0}^{3}\frac{L_{\Upsilon_r}(G^{s\bar s}_q)\,d_{ns}\,\bar d_{\bar n\bar s}\,g^{qp}\,G^{n\bar n}_p}{4}\,-\\ -\sum_{\bar i=1}^{2}\sum_{\bar j=1}^{2}\frac{L_{\Upsilon_r}(\bar d_{\bar j\bar i})\,\bar d^{\bar i\bar j}}{2}-\sum_{i=1}^{2}\sum_{j=1}^{2}\frac{L_{\Upsilon_r}(d_{j\,i})\,d^{ij}}{2}=0.\end{gathered}\tag{4.54}$$

Note that the product $d_{ns}\,\bar d_{\bar n\bar s}\,g^{qp}$ in (4.54) is invariant under the simultaneous transposition of $s\longleftrightarrow n$, $\bar s\longleftrightarrow\bar n$, and $p\longleftrightarrow q$. Therefore, we can write

$$\begin{gathered}\sum_{p=0}^{3}\frac{\Gamma^p_{rp}}{2}-\sum_{\bar i=1}^{2}\sum_{\bar j=1}^{2}\frac{L_{\Upsilon_r}(\bar d_{\bar j\bar i})\,\bar d^{\bar i\bar j}}{2}-\sum_{i=1}^{2}\sum_{j=1}^{2}\frac{L_{\Upsilon_r}(d_{j\,i})\,d^{ij}}{2}\,-\\ -\sum_{s=1}^{2}\sum_{\bar s=1}^{2}\sum_{q=0}^{3}\sum_{n=1}^{2}\sum_{\bar n=1}^{2}\sum_{p=0}^{3}\frac{L_{\Upsilon_r}(G^{s\bar s}_q)\,d_{ns}\,\bar d_{\bar n\bar s}\,g^{qp}\,G^{n\bar n}_p+G^{s\bar s}_q\,d_{ns}\,\bar d_{\bar n\bar s}\,g^{qp}\,L_{\Upsilon_r}(G^{n\bar n}_p)}{8}=0.\end{gathered}$$

The Lie derivative L_{Υ_r} in the above equality acts as a first order linear differential operator. For this reason we can continue transforming the above equality:

$$\begin{gathered}\sum_{p=0}^{3}\frac{\Gamma^p_{rp}}{2}-\sum_{\bar i=1}^{2}\sum_{\bar j=1}^{2}\frac{L_{\Upsilon_r}(\bar d_{\bar j\bar i})\,\bar d^{\bar i\bar j}}{2}-\sum_{i=1}^{2}\sum_{j=1}^{2}\frac{L_{\Upsilon_r}(d_{j\,i})\,d^{ij}}{2}\,-\\ -\sum_{s=1}^{2}\sum_{\bar s=1}^{2}\sum_{q=0}^{3}\sum_{n=1}^{2}\sum_{\bar n=1}^{2}\sum_{p=0}^{3}\frac{L_{\Upsilon_r}(G^{s\bar s}_q\,d_{ns}\,\bar d_{\bar n\bar s}\,G^{n\bar n}_p)\,g^{qp}}{8}\,+\end{gathered}$$

$$+\sum_{s=1}^{2}\sum_{\bar{s}=1}^{2}\sum_{q=0}^{3}\sum_{n=1}^{2}\sum_{\bar{n}=1}^{2}\sum_{p=0}^{3}\frac{G^{s\bar{s}}_{q}\,L_{\Upsilon_r}(d_{ns})\,\bar{d}_{\bar{n}\bar{s}}\,g^{qp}\,G^{n\bar{n}}_{p}+G^{s\bar{s}}_{q}\,d_{ns}\,L_{\Upsilon_r}(\bar{d}_{\bar{n}\bar{s}})\,g^{qp}\,G^{n\bar{n}}_{p}}{8}=0.$$

Taking into account the second identity (3.12) and the formula (3.2), we find that the last term of the above equality cancels the second and the third terms in it. As a result, using (3.9), we can write it as follows:

$$\sum_{p=0}^{3}\frac{\Gamma^{p}_{rp}}{2}-\sum_{p=0}^{3}\sum_{q=0}^{3}\frac{L_{\Upsilon_r}(g_{qp})\,g^{qp}}{4}=0. \tag{4.55}$$

Using the formula (4.20), we can write (4.55) in a very simple form:

$$\sum_{p=0}^{3}\sum_{q=0}^{3}g^{qp}\;\nabla_{r}g_{qp}=0. \tag{4.56}$$

Thus, the formula (4.52) is reduced to (4.56), while (4.51) is equivalent to (4.50) provided (4.56) is fulfilled. From (4.50) and (4.41) we derive the following expression for $\bar{\mathrm{A}}$-components of the spinor connection $(\Gamma,\mathrm{A},\bar{\mathrm{A}})$:

$$\begin{gathered}\bar{\mathrm{A}}^{\bar{i}}_{r\bar{j}}=\sum_{s=1}^{2}\sum_{p=0}^{3}\sum_{q=0}^{3}\frac{G^{s\bar{i}}_{p}\,\Gamma^{p}_{rq}\,G^{\,q}_{s\bar{j}}}{4}-\\-\sum_{s=1}^{2}\sum_{q=0}^{3}\frac{L_{\Upsilon_r}(G^{s\bar{i}}_{q})\,G^{\,q}_{s\bar{j}}}{4}-\sum_{i=1}^{2}\sum_{j=1}^{2}\frac{L_{\Upsilon_r}(d_{ji})\,d^{ij}\,\delta^{\bar{i}}_{\bar{j}}}{4}.\end{gathered} \tag{4.57}$$

Both (4.56) and (4.57), when taken together, are equivalent to (4.49).

Having all done with (4.49), now we return back to the equality (4.48). The left hand side of this equality is skew-symmetric in $\bar{i}$ and $\bar{j}$. Like in the case of (4.47), subdividing the right hand side of (4.48) into two parts symmetric and skew-symmetric in $\bar{i}$ and $\bar{j}$, we write (4.48) as two separate equalities. Here is the first of these two equalities. It is skew-symmetric in $\bar{i}$ and $\bar{j}$:

$$\begin{gathered}\hat{\mathrm{A}}_{rji}\,\bar{d}_{\bar{j}\bar{i}}=\sum_{s=1}^{2}\sum_{\bar{s}=1}^{2}\sum_{p=0}^{3}\sum_{q=0}^{3}\frac{d_{si}\,\bar{d}_{\bar{s}\bar{i}}\,G^{s\bar{s}}_{p}\,\Gamma^{p}_{rq}\,G^{\,q}_{j\bar{j}}+d_{sj}\,\bar{d}_{\bar{s}\bar{i}}\,G^{s\bar{s}}_{p}\,\Gamma^{p}_{rq}\,G^{\,q}_{i\bar{j}}}{8}-\\-\sum_{s=1}^{2}\sum_{\bar{s}=1}^{2}\sum_{p=0}^{3}\sum_{q=0}^{3}\frac{d_{si}\,\bar{d}_{\bar{s}\bar{j}}\,G^{s\bar{s}}_{p}\,\Gamma^{p}_{rq}\,G^{\,q}_{j\bar{i}}+d_{sj}\,\bar{d}_{\bar{s}\bar{j}}\,G^{s\bar{s}}_{p}\,\Gamma^{p}_{rq}\,G^{\,q}_{i\bar{i}}}{8}-\\-\sum_{s=1}^{2}\sum_{\bar{s}=1}^{2}\sum_{q=0}^{3}\frac{d_{si}\,\bar{d}_{\bar{s}\bar{i}}\,L_{\Upsilon_r}(G^{s\bar{s}}_{q})\,G^{\,q}_{j\bar{j}}+d_{sj}\,\bar{d}_{\bar{s}\bar{i}}\,L_{\Upsilon_r}(G^{s\bar{s}}_{q})\,G^{\,q}_{i\bar{j}}}{8}+\\+\sum_{s=1}^{2}\sum_{\bar{s}=1}^{2}\sum_{q=0}^{3}\frac{d_{si}\,\bar{d}_{\bar{s}\bar{j}}\,L_{\Upsilon_r}(G^{s\bar{s}}_{q})\,G^{\,q}_{j\bar{i}}+d_{sj}\,\bar{d}_{\bar{s}\bar{j}}\,L_{\Upsilon_r}(G^{s\bar{s}}_{q})\,G^{\,q}_{i\bar{i}}}{8}.\end{gathered} \tag{4.58}$$

Due to the skew-symmetry (4.58) can be transformed to an equality independent of $\bar{i}$ and $\bar{j}$ at all. Multiplying it by $\bar{d}^{\bar{i}\bar{j}}$ and summing over $\bar{i}$ and $\bar{j}$, we get the following equality analogous to the equality (4.51):

$$\begin{aligned}\hat{\mathrm{A}}_{rji}=\sum_{s=1}^{2}\sum_{\bar{s}=1}^{2}\sum_{p=0}^{3}\sum_{q=0}^{3}\frac{d_{si}\,G^{s\bar{s}}_p\,\Gamma^p_{rq}\,G^{\,q}_{j\bar{s}}+d_{sj}\,G^{s\bar{s}}_p\,\Gamma^p_{rq}\,G^{\,q}_{i\bar{s}}}{8}\,-\\-\sum_{s=1}^{2}\sum_{\bar{s}=1}^{2}\sum_{q=0}^{3}\frac{d_{si}\,L_{\mathbf{\Upsilon}_r}(G^{s\bar{s}}_q)\,G^{\,q}_{j\bar{s}}+d_{sj}\,L_{\mathbf{\Upsilon}_r}(G^{s\bar{s}}_q)\,G^{\,q}_{i\bar{s}}}{8}.\end{aligned}\tag{4.59}$$

By analogy to (4.52) one can write the following equality:

$$\begin{aligned}\sum_{s=1}^{2}\sum_{\bar{s}=1}^{2}\sum_{p=0}^{3}\sum_{q=0}^{3}\frac{d_{si}\,G^{s\bar{s}}_p\,\Gamma^p_{rq}\,G^{\,q}_{j\bar{s}}-d_{sj}\,G^{s\bar{s}}_p\,\Gamma^p_{rq}\,G^{\,q}_{i\bar{s}}}{8}\,-\\-\sum_{s=1}^{2}\sum_{\bar{s}=1}^{2}\sum_{q=0}^{3}\frac{d_{si}\,L_{\mathbf{\Upsilon}_r}(G^{s\bar{s}}_q)\,G^{\,q}_{j\bar{s}}-d_{sj}\,L_{\mathbf{\Upsilon}_r}(G^{s\bar{s}}_q)\,G^{\,q}_{i\bar{s}}}{8}\,-\\-\frac{L_{\mathbf{\Upsilon}_r}(d_{ji})}{2}-\sum_{\bar{i}=1}^{2}\sum_{\bar{j}=1}^{2}\frac{L_{\mathbf{\Upsilon}_r}(\bar{d}_{\bar{j}\bar{i}})\,\bar{d}^{\bar{i}\bar{j}}\,d_{ji}}{4}=0.\end{aligned}\tag{4.60}$$

Acting in a similar way as in the case of (4.52), one can show that the equality (4.60) is equivalent to (4.56). Adding (4.60) to (4.59), we get

$$\begin{aligned}\hat{\mathrm{A}}_{rji}=-\frac{L_{\mathbf{\Upsilon}_r}(d_{ji})}{2}+\sum_{s=1}^{2}\sum_{\bar{s}=1}^{2}\sum_{p=0}^{3}\sum_{q=0}^{3}\frac{d_{si}\,G^{s\bar{s}}_p\,\Gamma^p_{rq}\,G^{\,q}_{j\bar{s}}}{4}\,-\\-\sum_{s=1}^{2}\sum_{\bar{s}=1}^{2}\sum_{q=0}^{3}\frac{d_{si}\,L_{\mathbf{\Upsilon}_r}(G^{s\bar{s}}_q)\,G^{\,q}_{j\bar{s}}}{4}-\sum_{\bar{i}=1}^{2}\sum_{\bar{j}=1}^{2}\frac{L_{\mathbf{\Upsilon}_r}(\bar{d}_{\bar{j}\bar{i}})\,\bar{d}^{\bar{i}\bar{j}}\,d_{ji}}{4}.\end{aligned}\tag{4.61}$$

The formula (4.61) is similar to (4.50). From (4.61) and (4.39) we derive

$$\begin{aligned}\mathrm{A}^i_{rj}=\sum_{\bar{s}=1}^{2}\sum_{p=0}^{3}\sum_{q=0}^{3}\frac{G^{i\bar{s}}_p\,\Gamma^p_{rq}\,G^{\,q}_{j\bar{s}}}{4}\,-\\-\sum_{\bar{s}=1}^{2}\sum_{q=0}^{3}\frac{L_{\mathbf{\Upsilon}_r}(G^{i\bar{s}}_q)\,G^{\,q}_{j\bar{s}}}{4}-\sum_{\bar{i}=1}^{2}\sum_{\bar{j}=1}^{2}\frac{L_{\mathbf{\Upsilon}_r}(\bar{d}_{\bar{j}\bar{i}})\,\bar{d}^{\bar{i}\bar{j}}\,\delta^i_j}{4}.\end{aligned}\tag{4.62}$$

The last step is to study the symmetric part of the equality (4.48). Symmetrizing (4.48) with respect to $\bar{i}$ and $\bar{j}$, we find that its symmetric part is written as

$$B_{ij\bar{i}\bar{j}}+B_{ji\bar{i}\bar{j}}=0,\tag{4.63}$$

where

$$\begin{aligned}B_{ij\bar{i}\bar{j}}=\sum_{s=1}^{2}\sum_{\bar{s}=1}^{2}\sum_{p=0}^{3}\sum_{q=0}^{3}\frac{d_{si}\,\bar{d}_{\bar{s}\bar{i}}\,G^{s\bar{s}}_p\,\Gamma^p_{rq}\,G^{\,q}_{j\bar{j}}+d_{sj}\,\bar{d}_{\bar{s}\bar{j}}\,G^{s\bar{s}}_p\,\Gamma^p_{rq}\,G^{\,q}_{i\bar{i}}}{8}\,-\\-\sum_{s=1}^{2}\sum_{\bar{s}=1}^{2}\sum_{q=0}^{3}\frac{d_{si}\,\bar{d}_{\bar{s}\bar{i}}\,L_{\mathbf{\Upsilon}_r}(G^{s\bar{s}}_q)\,G^{\,q}_{j\bar{j}}+d_{sj}\,\bar{d}_{\bar{s}\bar{j}}\,L_{\mathbf{\Upsilon}_r}(G^{s\bar{s}}_q)\,G^{\,q}_{i\bar{i}}}{8}.\end{aligned}\tag{4.64}$$

Using (4.25) and the formula (3.2), we transform (4.64) as follows:

$$
\begin{aligned}
B_{ij\bar{i}\bar{j}} &= \sum_{s,\ \bar{s},}^{2}\!\!\cdots\!\!\sum_{n,\ \bar{n}}^{2}\sum_{p,\ q,}^{3}\!\!\cdots\!\!\sum_{\alpha,\ \beta}^{3}\frac{(d_{si}\,\bar{d}_{\bar{s}\bar{i}}\,G^{s\bar{s}}_p\,g^{p\alpha})\,\Gamma_{r\beta\alpha}\,(d_{nj}\,\bar{d}_{\bar{n}\bar{j}}\,G^{n\bar{n}}_q\,g^{\beta q})}{8}\,+\\
&+\sum_{s,\ \bar{s},}^{2}\!\!\cdots\!\!\sum_{n,\ \bar{n}}^{2}\sum_{p,\ q,}^{3}\!\!\cdots\!\!\sum_{\alpha,\ \beta}^{3}\frac{(d_{sj}\,\bar{d}_{\bar{s}\bar{j}}\,G^{s\bar{s}}_p\,g^{p\alpha})\,\Gamma_{r\beta\alpha}\,(d_{ni}\,\bar{d}_{\bar{n}\bar{i}}\,G^{n\bar{n}}_q\,g^{\beta q})}{8}\,-\\
&-\sum_{s,\ \bar{s},}^{2}\!\!\cdots\!\!\sum_{n,\ \bar{n}}^{2}\sum_{p,\ q}^{3}\frac{(d_{si}\,\bar{d}_{\bar{s}\bar{i}}\,L_{\boldsymbol{\Upsilon}_r}(G^{s\bar{s}}_q))\,g^{qp}\,(d_{nj}\,\bar{d}_{\bar{n}\bar{j}}\,G^{n\bar{n}}_p)}{8}\,-\\
&-\sum_{s,\ \bar{s},}^{2}\!\!\cdots\!\!\sum_{n,\ \bar{n}}^{2}\sum_{p,\ q}^{3}\frac{(d_{si}\,\bar{d}_{\bar{s}\bar{i}}\,G^{s\bar{s}}_q\,g^{qp}\,(d_{nj}\,\bar{d}_{\bar{n}\bar{j}}\,L_{\boldsymbol{\Upsilon}_r}(G^{n\bar{n}}_p))}{8}.
\end{aligned}
\tag{4.65}
$$

Remember that the Lie derivative $L_{\boldsymbol{\Upsilon}_r}$ acts as a first order linear differential operator. Therefore, the formula (4.65) can be written as

$$
\begin{aligned}
B_{ij\bar{i}\bar{j}} &= -\sum_{s,\ \bar{s},}^{2}\!\!\cdots\!\!\sum_{n,\ \bar{n}}^{2}\sum_{p,\ q}^{3}\frac{d_{si}\,\bar{d}_{\bar{s}\bar{i}}\,d_{nj}\,\bar{d}_{\bar{n}\bar{j}}\,L_{\boldsymbol{\Upsilon}_r}(G^{s\bar{s}}_q\,g^{qp}\,G^{n\bar{n}}_p)}{8}\,+\\
&+\sum_{s,\ \bar{s},}^{2}\!\!\cdots\!\!\sum_{n,\ \bar{n}}^{2}\sum_{p,\ q}^{3}\frac{(d_{si}\,\bar{d}_{\bar{s}\bar{i}}\,G^{s\bar{s}}_q)\,L_{\boldsymbol{\Upsilon}_r}(g^{qp})\,(d_{nj}\,\bar{d}_{\bar{n}\bar{j}}\,G^{n\bar{n}}_p)}{8}\,+\\
&+\sum_{s,\ \bar{s},}^{2}\!\!\cdots\!\!\sum_{n,\ \bar{n}}^{2}\sum_{p,\ q,}^{3}\!\!\cdots\!\!\sum_{\alpha,\ \beta}^{3}\frac{(d_{si}\,\bar{d}_{\bar{s}\bar{i}}\,G^{s\bar{s}}_p\,g^{p\alpha})\,\Gamma_{r\beta\alpha}\,(d_{nj}\,\bar{d}_{\bar{n}\bar{j}}\,G^{n\bar{n}}_q\,g^{\beta q})}{8}\,+\\
&+\sum_{s,\ \bar{s},}^{2}\!\!\cdots\!\!\sum_{n,\ \bar{n}}^{2}\sum_{p,\ q,}^{3}\!\!\cdots\!\!\sum_{\alpha,\ \beta}^{3}\frac{(d_{sj}\,\bar{d}_{\bar{s}\bar{j}}\,G^{s\bar{s}}_p\,g^{p\alpha})\,\Gamma_{r\beta\alpha}\,(d_{ni}\,\bar{d}_{\bar{n}\bar{i}}\,G^{n\bar{n}}_q\,g^{\beta q})}{8}.
\end{aligned}
\tag{4.66}
$$

Remember that g^{qp} form the inverse matrix for g_{qp}. Therefore, we have

$$
L_{\boldsymbol{\Upsilon}_r}(g^{qp}) = -\sum_{\alpha=0}^{3}\sum_{\beta=0}^{3} g^{p\alpha}\,L_{\boldsymbol{\Upsilon}_r}(g_{\alpha\beta})\,g^{\beta q}
\tag{4.67}
$$

Substituting (4.67) into (4.66) and applying (3.10) and (4.20) to it, we derive

$$
\begin{aligned}
B_{ij\bar{i}\bar{j}} &= -\sum_{s,\ \bar{s},}^{2}\!\!\cdots\!\!\sum_{n,\ \bar{n}}^{2}\sum_{p,\ q}^{3}\frac{d_{si}\,\bar{d}_{\bar{s}\bar{i}}\,d_{nj}\,\bar{d}_{\bar{n}\bar{j}}\,L_{\boldsymbol{\Upsilon}_r}(d^{sn}\,\bar{d}^{\bar{s}\bar{n}})}{4}\,-\\
&-\sum_{s,\ \bar{s},}^{2}\!\!\cdots\!\!\sum_{n,\ \bar{n}}^{2}\sum_{p,\ q,}^{3}\!\!\cdots\!\!\sum_{\alpha,\ \beta}^{3}\frac{(d_{si}\,\bar{d}_{\bar{s}\bar{i}}\,G^{s\bar{s}}_p\,g^{p\alpha})\,\nabla_r g_{\alpha\beta}\,(d_{nj}\,\bar{d}_{\bar{n}\bar{j}}\,G^{n\bar{n}}_q\,g^{\beta q})}{8}.
\end{aligned}
\tag{4.68}
$$

The first term in (4.68) is skew-symmetric in i and j. It vanishes when we substitute (4.68) into (4.63). As a result the equality (4.63) takes the form

$$
\sum_{\alpha=0}^{3}\sum_{\beta=0}^{3} G^{\alpha}_{i\bar{i}}\ \nabla_r g_{\alpha\beta}\ G^{\beta}_{j\bar{j}} + \sum_{\alpha=0}^{3}\sum_{\beta=0}^{3} G^{\alpha}_{j\bar{i}}\ \nabla_r g_{\alpha\beta}\ G^{\beta}_{i\bar{j}} = 0.
\tag{4.69}
$$

Thus the concordance condition $\nabla\mathbf{G} = 0$ is equivalent to the formulas (4.57), and (4.62) provided $\nabla\mathbf{d} = 0$, $\nabla\bar{\mathbf{d}} = 0$, and the equalities (4.56) and (4.69) are fulfilled. This result of the above calculations can be stated as the following theorem.

Theorem 4.3. *The concordance conditions $\nabla\mathbf{d} = 0$, $\nabla\bar{\mathbf{d}} = 0$, and $\nabla\mathbf{G} = 0$ for a spinor connection $(\Gamma, \mathrm{A}, \bar{\mathrm{A}})$ are equivalent to the formulas* (4.34)*,* (4.57)*, and* (4.62) *for its components in an arbitrary frame pair $(U,\,\mathbf{\Upsilon}_0,\,\mathbf{\Upsilon}_1,\,\mathbf{\Upsilon}_2,\,\mathbf{\Upsilon}_3)$ and $(U,\,\mathbf{\Psi}_1,\,\mathbf{\Psi}_2)$ of the bundles TM and SM respectively.*

Proof. Note that $\nabla\mathbf{d} = 0$, $\nabla\bar{\mathbf{d}} = 0$, and $\nabla\mathbf{G} = 0$ imply $\nabla\mathbf{g} = 0$. The latter concordance condition is equivalent to the formula (4.34). Moreover, due to this condition the equalities (4.56) and (4.69) are fulfilled. Then from $\nabla\mathbf{d} = 0$, $\nabla\bar{\mathbf{d}} = 0$, and $\nabla\mathbf{G} = 0$ the formulas (4.57) and (4.62) are derived.

Conversely, the formula (4.34) leads to $\nabla\mathbf{g} = 0$ and to the equalities (4.56) and (4.69). The equality (4.69) is equivalent to (4.63). The equality (4.56) is equivalent to (4.60). Being combined with (4.62), the equality (4.60) leads to (4.58) and (4.42). Both (4.63) and (4.58) yield (4.48).

The equality (4.56) is equivalent to (4.52). Being combined with (4.57), the equality (4.52) leads to (4.49) and (4.40). The equalities (4.40) and (4.42) are equivalent to $\nabla\mathbf{d} = 0$ and $\nabla\bar{\mathbf{d}} = 0$. An finally, from (4.48) and (4.49) by applying (4.40) and (4.42) we derive $\nabla\mathbf{G} = 0$. The theorem is proved. $\square$

Corollary 4.2. *The components of a real metric connection $(\Gamma, \mathrm{A}, \bar{\mathrm{A}})$ with zero torsion $\mathbf{T} = 0$ for the bundle of chiral spinors SM are given by the explicit formulas* (4.35)*,* (4.57)*, and* (4.62)*.*

5. Dirac spinors.

Let M be a space-time manifold and let SM be a spinor bundle over M introduced by the definition 1.1. By $S^{\dagger}M$ (see (2.2)) we denote the Hermitian conjugate bundle for SM. Taking both SM and $S^{\dagger}M$, we construct their direct sum

$$DM = SM \oplus S^{\dagger}M. \tag{5.1}$$

The direct sum (5.1) is called the *Dirac bundle* associated with the spinor bundle SM. This is a four-dimensional complex bundle over M. The bundles SM and $S^{\dagger}M$, when treated as the constituents of DM, are called *chiral bundles*.

Due to the expansion the Dirac bundle DM acquires from SM its three basic spin-tensorial fields: the *spin-metric tensor* $\mathbf{d}$, the *chirality operator* $\mathbf{H}$, and the *Hermitian spin-metric tensor* $\mathbf{D}$, which is also called the *Dirac form*. The definitions of these three fields can be found in section 3 of [6].

Definition 5.1. A frame $(U,\,\mathbf{\Psi}_1,\,\mathbf{\Psi}_2,\,\mathbf{\Psi}_3,\,\mathbf{\Psi}_4)$ of the Dirac bundle DM is called an *orthonormal frame* if the spin-metric tensor $\mathbf{d}$ is represented by the following skew-symmetric matrix in this frame:

$$d_{ij} = d(\mathbf{\Psi}_i, \mathbf{\Psi}_j) = \begin{Vmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{Vmatrix}. \tag{5.2}$$

Definition 5.2. A frame $(U,\,\mathbf{\Psi}_1,\,\mathbf{\Psi}_2,\,\mathbf{\Psi}_3,\,\mathbf{\Psi}_4)$ of the Dirac bundle DM is called an *anti-orthonormal frame* if the spin-metric tensor $\mathbf{d}$ is represented by the matrix opposite to the matrix (5.2) in this frame:

$$d_{ij} = d(\mathbf{\Psi}_i, \mathbf{\Psi}_j) = \begin{Vmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{Vmatrix}. \tag{5.3}$$

Definition 5.3. A frame $(U,\,\mathbf{\Psi}_1,\,\mathbf{\Psi}_2,\,\mathbf{\Psi}_3,\,\mathbf{\Psi}_4)$ of the Dirac bundle DM is called a *chiral frame* if the chirality operator $\mathbf{H}$ given by the following matrix in this frame:

$$H^i_j = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{Vmatrix}. \tag{5.4}$$

Definition 5.4. A frame $(U,\,\mathbf{\Psi}_1,\,\mathbf{\Psi}_2,\,\mathbf{\Psi}_3,\,\mathbf{\Psi}_4)$ of the Dirac bundle DM is called an *anti-chiral frame* if the chirality operator $\mathbf{H}$ given by the diagonal matrix opposite to the matrix (5.4) in this frame:

$$H^i_j = \begin{Vmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Vmatrix}. \tag{5.5}$$

Definition 5.5. A frame $(U,\,\mathbf{\Psi}_1,\,\mathbf{\Psi}_2,\,\mathbf{\Psi}_3,\,\mathbf{\Psi}_4)$ of the Dirac bundle DM is called a *self-adjoint frame* if the Hermitian spin-metric tensor $\mathbf{D}$ (the Dirac form) is represented by the following matrix in this frame:

$$D_{i\bar{j}} = D(\mathbf{\Psi}_{\bar{j}}, \mathbf{\Psi}_i) = \begin{Vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{Vmatrix}. \tag{5.6}$$

Definition 5.6. A frame $(U,\,\mathbf{\Psi}_1,\,\mathbf{\Psi}_2,\,\mathbf{\Psi}_3,\,\mathbf{\Psi}_4)$ of the Dirac bundle DM is called an *anti-self-adjoint frame* if the Hermitian spin-metric tensor $\mathbf{D}$ (the Dirac form) is represented by the matrix opposite to the matrix (5.6) in this frame:

$$D_{i\bar{j}} = D(\mathbf{\Psi}_{\bar{j}}, \mathbf{\Psi}_i) = \begin{Vmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{Vmatrix}. \tag{5.7}$$

In [6] the P and T reflection operations were studied and the following four types of frames in the Dirac bundle DM were considered:

(1) canonically orthonormal chiral frames;
(2) P-reverse anti-chiral frames;
(3) T-reverse anti-chiral frames;
(4) PT-reverse chiral frames.

Canonically orthonormal chiral frames are simultaneously orthonormal, chiral, and self-adjoint frames. This is the basic type of frames most closely related to the expansion (5.1). Each canonically orthonormal chiral frame of the Dirac bundle $(U,\,\mathbf{\Psi}_1,\,\mathbf{\Psi}_2,\,\mathbf{\Psi}_3,\,\mathbf{\Psi}_4)$ is produced from some orthonormal frame $(U,\,\mathbf{\Psi}_1,\,\mathbf{\Psi}_2)$ of the chiral bundle SM as follows:

$$\mathbf{\Psi}_1=\mathbf{\Psi}_1^{\text{chiral}},\qquad \mathbf{\Psi}_2=\mathbf{\Psi}_2^{\text{chiral}},\qquad \mathbf{\Psi}_3=\overline{\boldsymbol{\vartheta}}{}^1_{\text{chiral}},\qquad \mathbf{\Psi}_4=\overline{\boldsymbol{\vartheta}}{}^2_{\text{chiral}}. \tag{5.8}$$

P**-reverse anti-chiral frames** are self-adjoint, but anti-orthonormal and anti-chiral. Any P-reverse anti-chiral frame $(U,\,\tilde{\mathbf{\Psi}}_1,\,\tilde{\mathbf{\Psi}}_2,\,\tilde{\mathbf{\Psi}}_3,\,\tilde{\mathbf{\Psi}}_4)$, is produced from some canonically orthonormal chiral frame $(U,\,\mathbf{\Psi}_1,\,\mathbf{\Psi}_2,\,\mathbf{\Psi}_3,\,\mathbf{\Psi}_4)$ by P-inversion:

$$\tilde{\mathbf{\Psi}}_1=\mathbf{\Psi}_3,\qquad \tilde{\mathbf{\Psi}}_2=\mathbf{\Psi}_4,\qquad \tilde{\mathbf{\Psi}}_3=\mathbf{\Psi}_1,\qquad \tilde{\mathbf{\Psi}}_4=\mathbf{\Psi}_2. \tag{5.9}$$

T**-reverse anti-chiral frames** are orthonormal, anti-chiral, and anti-self-adjoint. Any T-reverse anti-chiral frame $(U,\,\tilde{\mathbf{\Psi}}_1,\,\tilde{\mathbf{\Psi}}_2,\,\tilde{\mathbf{\Psi}}_3,\,\tilde{\mathbf{\Psi}}_4)$, is produced from some canonically orthonormal chiral frame $(U,\,\mathbf{\Psi}_1,\,\mathbf{\Psi}_2,\,\mathbf{\Psi}_3,\,\mathbf{\Psi}_4)$ by T-inversion:

$$\tilde{\mathbf{\Psi}}_1=i\,\mathbf{\Psi}_3,\qquad \tilde{\mathbf{\Psi}}_2=i\,\mathbf{\Psi}_4,\qquad \tilde{\mathbf{\Psi}}_3=-i\,\mathbf{\Psi}_1,\qquad \tilde{\mathbf{\Psi}}_4=-i\,\mathbf{\Psi}_2. \tag{5.10}$$

PT**-reverse chiral frames** are anti-orthonormal, chiral, and anti-self-adjoint. Any PT-reverse chiral frame $(U,\,\tilde{\mathbf{\Psi}}_1,\,\tilde{\mathbf{\Psi}}_2,\,\tilde{\mathbf{\Psi}}_3,\,\tilde{\mathbf{\Psi}}_4)$, is produced from some canonically orthonormal chiral frame $(U,\,\mathbf{\Psi}_1,\,\mathbf{\Psi}_2,\,\mathbf{\Psi}_3,\,\mathbf{\Psi}_4)$ by PT-inversion:

$$\tilde{\mathbf{\Psi}}_1=i\,\mathbf{\Psi}_1,\qquad \tilde{\mathbf{\Psi}}_2=i\,\mathbf{\Psi}_2,\qquad \tilde{\mathbf{\Psi}}_3=-i\,\mathbf{\Psi}_3,\qquad \tilde{\mathbf{\Psi}}_4=-i\,\mathbf{\Psi}_4. \tag{5.11}$$

All of the above facts are easily derived from (5.2), (5.3), (5.4), (5.5), (5.6), and (5.7). The P, T, and PT-inversions introduced in (5.9), (5.10), and (5.11) are not actual operations over spinors, they are frame transformations only. The formula (5.8) defines a frame construction operation.

The frames of all of the above four types are canonically associated with some frames in TM. The frame association is given by the diagram

$$\begin{array}{|c|}\hline \text{Canonically orthonormal}\\ \text{chiral frames}\\ \hline\end{array}\ \longrightarrow\ \begin{array}{|c|}\hline \text{Positively polarized}\\ \text{right orthonormal frames}\\ \hline\end{array}$$
$$\begin{array}{|c|}\hline P\text{-reverse}\\ \text{anti-chiral frames}\\ \hline\end{array}\ \longrightarrow\ \begin{array}{|c|}\hline \text{Positively polarized}\\ \text{left orthonormal frames}\\ \hline\end{array}$$
$$\begin{array}{|c|}\hline T\text{-reverse}\\ \text{anti-chiral frames}\\ \hline\end{array}\ \longrightarrow\ \begin{array}{|c|}\hline \text{Negatively polarized}\\ \text{right orthonormal frames}\\ \hline\end{array} \tag{5.12}$$
$$\begin{array}{|c|}\hline PT\text{-reverse}\\ \text{chiral frames}\\ \hline\end{array}\ \longrightarrow\ \begin{array}{|c|}\hline \text{Negatively polarized}\\ \text{left orthonormal frames}\\ \hline\end{array}$$

Like SM, the Dirac bundle DM is a complex vector bundle over the smooth real space-time manifold M. For this reason there is a semilinear involution of complex conjugation τ acting upon spin-tensorial fields associated with DM. This involution τ is canonically associated with DM, it is introduces in a way similar to τ for SM (see details in [6]). Applying τ to the spin-metric tensor $\mathbf{d}$, we get

$$\bar{\mathbf{d}} = \tau(\mathbf{d}). \tag{5.13}$$

This is a spin-tensorial field of the type $(0,0|0,2|0,0)$, while $\mathbf{d}$ itself is a field of the type $(0,2|0,0|0,0)$. The following formulas are analogous to (2.18) and (2.19):

$$\tau(\mathbf{X}) = \sum^{4}_{\substack{i_1,\,\ldots\,,\,i_\nu\\ j_1,\,\ldots\,,\,j_\gamma\\ \bar{i}_1,\,\ldots\,,\,\bar{i}_\alpha\\ \bar{j}_1,\,\ldots\,,\,\bar{j}_\beta}}\!\!\ldots\sum^{4}\sum^{3}_{\substack{h_1,\,\ldots\,,\,h_m\\ k_1,\,\ldots\,,\,k_n}}\!\!\ldots\sum^{3} \tau X^{i_1\ldots\,i_\nu\,\bar{i}_1\ldots\,\bar{i}_\alpha\,h_1\ldots\,h_m}_{j_1\ldots\,j_\gamma\,\bar{j}_1\ldots\,\bar{j}_\beta\,k_1\ldots\,k_n}\;\boldsymbol{\Psi}^{j_1\ldots\,j_\gamma\,\bar{j}_1\ldots\,\bar{j}_\beta\,k_1\ldots\,k_n}_{i_1\ldots\,i_\nu\,\bar{i}_1\ldots\,\bar{i}_\alpha\,h_1\ldots\,h_m}, \tag{5.14}$$

where

$$\tau X^{i_1\ldots\,i_\nu\,\bar{i}_1\ldots\,\bar{i}_\alpha\,h_1\ldots\,h_m}_{j_1\ldots\,j_\gamma\,\bar{j}_1\ldots\,\bar{j}_\beta\,k_1\ldots\,k_n} = \overline{X^{\bar{i}_1\ldots\,\bar{i}_\alpha\,i_1\ldots\,i_\nu\,h_1\ldots\,h_m}_{\bar{j}_1\ldots\,\bar{j}_\beta\,j_1\ldots\,j_\gamma\,k_1\ldots\,k_n}}. \tag{5.15}$$

Applying (5.14) and (5.15) to (5.13), we derive the components of $\bar{\mathbf{d}}$:

$$\bar{d}_{\bar{i}\bar{j}} = \overline{d_{\bar{i}\bar{j}}}. \tag{5.16}$$

The local fields $\boldsymbol{\Psi}^{j_1\ldots\,j_\gamma\,\bar{j}_1\ldots\,\bar{j}_\beta\,k_1\ldots\,k_n}_{i_1\ldots\,i_\nu\,\bar{i}_1\ldots\,\bar{i}_\alpha\,h_1\ldots\,h_m}$ in the expansion (5.16) are introduced in a way similar to that of (2.13). As for the formulas (5.14), (5.15), and (5.16), they hold for arbitrary frame pairs $(U,\,\boldsymbol{\Psi}_1,\,\boldsymbol{\Psi}_2,\,\boldsymbol{\Psi}_3,\,\boldsymbol{\Psi}_4)$ and $(U,\,\boldsymbol{\Upsilon}_0,\,\boldsymbol{\Upsilon}_1,\,\boldsymbol{\Upsilon}_2,\,\boldsymbol{\Upsilon}_3)$, not only for those listed in the diagram (5.12).

Let's apply the formula (5.15) to the formula (5.6) or to the formula (5.7). As a result we find that the Dirac field $\mathbf{D}$ is a real spin-tensorial field:

$$\tau(\mathbf{D}) = \mathbf{D}. \tag{5.17}$$

From (5.17) for the Dirac form $D(\mathbf{X},\mathbf{Y}) = C(\mathbf{D}\otimes\tau(\mathbf{X})\otimes\mathbf{Y})$ we derive

$$D(\mathbf{X},\mathbf{Y}) = \sum^{4}_{i=1}\sum^{4}_{j=1} D_{i\bar{j}}\,\overline{X^{\bar{j}}}\,Y^i = \overline{D(\mathbf{Y},\mathbf{X})}. \tag{5.18}$$

The identity (5.18) shows that the Dirac form $D(\mathbf{X},\mathbf{Y})$ is a Hermitian form.

6. Dirac's γ-field, γ-symbols, and γ-matrices.

The Infeld-van der Waerden symbols are not expanded to the Dirac bundle. Instead of them, here we have the Dirac's γ-field $\boldsymbol{\gamma}$. This is a spin-tensorial field of the type $(1,1|0,0|0,1)$. Its components are called the *Dirac's γ-symbols*. In a frame pair of any type listed on the diagram (5.12), γ-symbols are given explicitly:

$$\begin{aligned}
&\gamma^1_{1\,0} = 0, &&\gamma^1_{2\,0} = 0, &&\gamma^1_{3\,0} = 1, &&\gamma^1_{4\,0} = 0,\\
&\gamma^2_{1\,0} = 0, &&\gamma^2_{2\,0} = 0, &&\gamma^2_{3\,0} = 0, &&\gamma^2_{4\,0} = 1,\\
&\gamma^3_{1\,0} = 1, &&\gamma^3_{2\,0} = 0, &&\gamma^3_{3\,0} = 0, &&\gamma^3_{4\,0} = 0,\\
&\gamma^4_{1\,0} = 0, &&\gamma^4_{2\,0} = 1, &&\gamma^4_{3\,0} = 0, &&\gamma^4_{4\,0} = 0,
\end{aligned} \tag{6.1}$$

$$
\begin{aligned}
&\gamma^1_{1\,1}=0, &&\gamma^1_{2\,1}=0, &&\gamma^1_{3\,1}=0, &&\gamma^1_{4\,1}=1,\\
&\gamma^2_{1\,1}=0, &&\gamma^2_{2\,1}=0, &&\gamma^2_{3\,1}=1, &&\gamma^2_{4\,1}=0,\\
&\gamma^3_{1\,1}=0, &&\gamma^3_{2\,1}=-1, &&\gamma^3_{3\,1}=0, &&\gamma^3_{4\,1}=0,\\
&\gamma^4_{1\,1}=-1, &&\gamma^4_{2\,1}=0, &&\gamma^4_{3\,1}=0, &&\gamma^4_{4\,1}=0,
\end{aligned}
\tag{6.2}
$$

$$
\begin{aligned}
&\gamma^1_{1\,2}=0, &&\gamma^1_{2\,2}=0, &&\gamma^1_{3\,2}=0, &&\gamma^1_{4\,2}=-i,\\
&\gamma^2_{1\,2}=0, &&\gamma^2_{2\,2}=0, &&\gamma^2_{3\,2}=i, &&\gamma^2_{4\,2}=0,\\
&\gamma^3_{1\,2}=0, &&\gamma^3_{2\,2}=i, &&\gamma^3_{3\,2}=0, &&\gamma^3_{4\,2}=0,\\
&\gamma^4_{1\,2}=-i, &&\gamma^4_{2\,2}=0, &&\gamma^4_{3\,2}=0, &&\gamma^4_{4\,2}=0,
\end{aligned}
\tag{6.3}
$$

$$
\begin{aligned}
&\gamma^1_{1\,3}=0, &&\gamma^1_{2\,3}=0, &&\gamma^1_{3\,3}=1, &&\gamma^1_{4\,3}=0,\\
&\gamma^2_{1\,3}=0, &&\gamma^2_{2\,3}=0, &&\gamma^2_{3\,3}=0, &&\gamma^2_{4\,3}=-1,\\
&\gamma^3_{1\,3}=-1, &&\gamma^3_{2\,3}=0, &&\gamma^3_{3\,3}=0, &&\gamma^3_{4\,3}=0,\\
&\gamma^4_{1\,3}=0, &&\gamma^4_{2\,3}=1, &&\gamma^4_{3\,3}=0, &&\gamma^4_{4\,3}=0.
\end{aligned}
\tag{6.4}
$$

The second lower index of the γ-symbols (6.1), (6.2), (6.3), and (6.4) is a spacial index. By fixing this index, we can arrange γ-symbols into four square matrices

$$
\gamma_k=\begin{Vmatrix}\gamma^1_{1k} & \gamma^1_{2k} & \gamma^1_{3k} & \gamma^1_{4k}\\ \gamma^2_{1k} & \gamma^2_{2k} & \gamma^2_{3k} & \gamma^2_{4k}\\ \gamma^3_{1k} & \gamma^3_{2k} & \gamma^3_{3k} & \gamma^3_{4k}\\ \gamma^4_{1k} & \gamma^4_{2k} & \gamma^4_{3k} & \gamma^4_{4k}\end{Vmatrix},\qquad k=0,\,1,\,2,\,3.
\tag{6.5}
$$

The matrices (6.5) are called *Dirac matrices*. One can write them explicitly:

$$
\begin{aligned}
&\gamma_0=\begin{Vmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{Vmatrix}, &&\gamma_1=\begin{Vmatrix}0&0&0&1\\0&0&1&0\\0&-1&0&0\\-1&0&0&0\end{Vmatrix},\\
&\gamma_2=\begin{Vmatrix}0&0&0&-i\\0&0&i&0\\0&i&0&0\\-i&0&0&0\end{Vmatrix}, &&\gamma_3=\begin{Vmatrix}0&0&1&0\\0&0&0&-1\\-1&0&0&0\\0&1&0&0\end{Vmatrix}.
\end{aligned}
\tag{6.6}
$$

The Dirac matrices (6.6) are very popular in physics. However, dealing with them, one should remember that each separate matrix has no spin-tensorial interpretation.

The most popular property of the Dirac matrices (6.6) is written in terms of their anticommutators $\{\gamma_i,\,\gamma_j\}=\gamma_i\cdot\gamma_j-\gamma_j\cdot\gamma_i$:

$$
\{\gamma_i,\,\gamma_j\}=2\,g_{ij}\,\mathbf{1}.
\tag{6.7}
$$

Here $\mathbf{1}$ is the unit matrix. Due to this property they define the 4-dimensional representation of the Clifford algebra $\mathcal{C}l(1,3,\mathbb{R})$ (see [8]). In terms of the γ-symbols the formula (6.7) is written as follows:

$$\sum_{b=1}^{4}\gamma^{a}_{bi}\,\gamma^{b}_{cj}+\sum_{b=1}^{4}\gamma^{a}_{bj}\,\gamma^{b}_{ci}=2\,g_{ij}\,\delta^{a}_{c}. \tag{6.8}$$

Apart from (6.8), there are also some analogs of the properties of Infeld-van der Waerden symbols (3.8), (3.9), (3.10), and (3.11). Here is the most simple of them:

$$\sum_{a=1}^{4}\sum_{b=1}^{4}\sum_{e=1}^{4}\sum_{h=1}^{4}\gamma^{a}_{bi}\,d_{ae}\,d^{bh}\,\gamma^{e}_{hj}=4\,g_{ij}. \tag{6.9}$$

The identity (6.9) is an analog of (3.9). It is derived from the following more simple identity with the use of the anticommutator relationship (6.8):

$$\sum_{a=1}^{4}\sum_{b=1}^{4}\gamma^{a}_{bi}\,d_{ae}\,d^{bh}=\gamma^{h}_{ei}. \tag{6.10}$$

By d^{bh} in (6.9) and (6.10) we denote the components of the dual spin-metric tensor. By tradition we denote it by the same symbol $\mathbf{d}$. Its components d^{bh} form the matrix inverse to d_{ae}.

The inverse Dirac's γ-field $\boldsymbol{\gamma}$ is a spin-tensorial field of the type $(1,1|0,0|1,0)$. Its components are obtained from γ^{i}_{jm} by raising the lower index m:

$$\gamma^{im}_{j}=\sum_{k=0}^{3}\gamma^{i}_{jk}\;g^{km}. \tag{6.11}$$

The formula (6.11) is an analog of the formula (3.2). The quantities γ^{im}_{j} obtained through this formula are called the *inverse γ-symbols*. From (6.9) and (6.10), taking into account (6.11), now we derive

$$\sum_{e=1}^{4}\sum_{h=1}^{4}\gamma^{h}_{ej}\;\gamma^{ei}_{h}=4\,\delta^{i}_{j}. \tag{6.12}$$

The formula (6.12) is an analog of the first formula (3.12). By raising the index i in (6.10) we obtain the following equality for the inverse γ-symbols:

$$\sum_{a=1}^{4}\sum_{b=1}^{4}\gamma^{ai}_{b}\,d_{ae}\,d^{bh}=\gamma^{hi}_{e}. \tag{6.13}$$

Then rising both indices i and j in (6.9) we derive the identity

$$\sum_{a=1}^{4}\sum_{b=1}^{4}\sum_{e=1}^{4}\sum_{h=1}^{4}\gamma^{ai}_{b}\,d_{ae}\,d^{bh}\,\gamma^{ej}_{h}=4\,g^{ij}. \tag{6.14}$$

The identity (6.14) is analogous to (3.11). As for the identities (6.10) and (6.13), they have no analogs in chiral spinors.

The relation of Dirac's γ-field and the chirality operator $\mathbf{H}$ is determined by the structure of γ-matrices (6.6). By means of direct calculations we prove that

$$\{\gamma_m,\,\mathbf{H}\}=0. \tag{6.15}$$

Like in (6.7), in (6.15) we have the anticommutator $\{\gamma_m,\,\mathbf{H}\}=\gamma_m\cdot\mathbf{H}+\mathbf{H}\cdot\gamma_m$. In a coordinate form the identity (6.15) is written as

$$\sum_{b=1}^{4}\gamma^a_{bm}\,H^b_c+\sum_{b=1}^{4}H^a_b\,\gamma^b_{cm}=0. \tag{6.16}$$

As for $\mathbf{d}$ and $\mathbf{H}$, their relation is described by the identity similar to (6.16):

$$\sum_{b=1}^{4}d_{ab}\,H^b_c=\sum_{b=1}^{4}H^b_a\,d_{bc}. \tag{6.17}$$

The identity (6.17) means that the chirality operator $\mathbf{H}$ is a symmetric operator with respect to the bilinear form of the spin-metric tensor $\mathbf{d}$, i. e.

$$d(\mathbf{H}(\mathbf{X}),\mathbf{Y})=d(\mathbf{X},\mathbf{H}(\mathbf{Y})) \tag{6.18}$$

for any two spinors $\mathbf{X}$ and $\mathbf{Y}$. In the case if the Dirac form (the form of the Hermitian spin-metric tensor $\mathbf{D}$) we have the identity similar to (6.18):

$$D(\mathbf{H}(\mathbf{X}),\mathbf{Y})=-D(\mathbf{X},\mathbf{H}(\mathbf{Y})). \tag{6.19}$$

The identity (6.19) means that $\mathbf{H}$ is an anti-Hermitian operator with respect to the Hermitian form D. In a coordinate form (6.19) is written as

$$\sum_{\bar{a}=1}^{4}D_{i\bar{a}}\,\overline{H^{\bar{a}}_{\bar{i}}}=-\sum_{a=1}^{4}H^a_i\,D_{a\bar{i}}. \tag{6.20}$$

Returning back to the Dirac's γ-symbols, we can write

$$\begin{gathered}\sum_{m=0}^{3}\sum_{n=0}^{3}\gamma^a_{bm}\,\gamma^e_{hn}\;g^{mn}=\delta^a_h\;\delta^e_b-H^a_h\;H^e_b+\\+\,d^{ae}\,d_{bh}-\sum_{r=1}^{4}\sum_{s=1}^{4}H^a_r\,d^{re}\,d_{bs}\,H^s_h,\end{gathered} \tag{6.21}$$

$$\begin{gathered}\sum_{m=0}^{3}\sum_{n=0}^{3}\gamma^{am}_b\,\gamma^{en}_h\;g_{mn}=\delta^a_h\;\delta^e_b-H^a_h\;H^e_b+\\+\,d^{ae}\,d_{bh}-\sum_{r=1}^{4}\sum_{s=1}^{4}H^a_r\,d^{re}\,d_{bs}\,H^s_h.\end{gathered} \tag{6.22}$$

The identities (6.21) and (6.22) are analogs of (3.8) and (3.10). Taking into account

(6.11), we can transform (6.21) and (6.22) to the following identity:

$$\begin{aligned}\sum_{m=0}^{3}\gamma_b^{am}\,\gamma_{hm}^{e}&=\delta_h^a\,\delta_b^e-H_h^a\,H_b^e+\\&+d^{ae}\,d_{bh}-\sum_{r=1}^{4}\sum_{s=1}^{4}H_r^a\,d^{re}\,d_{bs}\,H_h^s.\end{aligned}\tag{6.23}$$

The identity (6.23) is an analog of the second identity (3.12).

Let's take the components of the Dirac form (the Hermitian spin-metric tensor **D**) and raise their indices. As a result we obtain

$$D^{i\bar{i}}=\sum_{a=1}^{4}\sum_{\bar{a}=1}^{4}d^{ia}\,D_{a\bar{a}}\;\bar{d}^{\bar{a}\bar{i}}.\tag{6.24}$$

The matrix $D^{i\bar{i}}$ is inverse to $D_{i\bar{i}}$ in the sense of the following equalities:

$$\sum_{\bar{a}=1}^{4}D_{j\bar{a}}\,D^{i\bar{a}}=\delta_j^i,\qquad\qquad\sum_{\bar{a}=1}^{4}D_{a\bar{j}}\,D^{a\bar{i}}=\delta_{\bar{j}}^{\bar{i}}.\tag{6.25}$$

Due to (6.25) the spin-tensorial field **D** of the type $(1,0|1,0|0,0)$ determined by the matrix (6.24) is called the *inverse Hermitian spin-metric tensor*. Using both $D^{i\bar{i}}$ and $D_{i\bar{i}}$, we define the following quantities:

$$\gamma_m^{i\bar{i}}=\sum_{a=1}^{4}\gamma_{am}^{i}\,D^{a\bar{i}},\qquad\qquad\gamma_{i\bar{i}}^{m}=\sum_{a=1}^{4}\gamma_i^{am}\,D_{a\bar{i}}.\tag{6.26}$$

The quantities (6.26) are called *direct and inverse Hermitian γ-symbols*. They define two spin-tensorial fields of the types $(1,0|1,0|0,1)$ and $(0,1|0,1|1,0)$ respectively. We denote these fields by the same symbol $\boldsymbol{\gamma}$, as well as the initial fields from which they are produced. The fields (6.26) are real fields:

$$\tau(\boldsymbol{\gamma})=\boldsymbol{\gamma}.\tag{6.27}$$

Indeed, by means of direct calculations we can prove that

$$\gamma_m^{i\bar{i}}=\overline{\gamma_m^{\bar{i}i}},\qquad\qquad\gamma_{i\bar{i}}^{m}=\overline{\gamma_{\bar{i}i}^{m}}.\tag{6.28}$$

The equalities (6.28) are coordinate representations of the equality (6.27). In terms of the initial γ-symbols they can be written as

$$\sum_{\bar{a}=1}^{4}D_{i\bar{a}}\,\overline{\gamma_{\bar{i}m}^{\bar{a}}}=\sum_{a=1}^{4}\gamma_{im}^{a}\,D_{a\bar{i}}.\tag{6.29}$$

The equality (6.29) is similar to (6.20). Moreover, we have

$$\sum_{a=1}^{4}d_{ia}\,\gamma_{jm}^{a}=-\sum_{a=1}^{4}\gamma_{im}^{a}\,d_{aj}.\tag{6.30}$$

The equality (6.30) is derived from (6.10), it is similar to (6.20) and (6.29).

7. Spinor connections for Dirac spinors.

Let $(U,\,\boldsymbol{\Upsilon}_0,\,\boldsymbol{\Upsilon}_1,\,\boldsymbol{\Upsilon}_2,\,\boldsymbol{\Upsilon}_3)$ and $(U,\,\boldsymbol{\Psi}_1,\,\boldsymbol{\Psi}_2,\,\boldsymbol{\Psi}_3,\,\boldsymbol{\Psi}_4)$ be two frames with a common domain U of the bundles TM and DM respectively. Let $(\tilde{U},\,\tilde{\boldsymbol{\Upsilon}}_0,\,\tilde{\boldsymbol{\Upsilon}}_1,\,\tilde{\boldsymbol{\Upsilon}}_2,\,\tilde{\boldsymbol{\Upsilon}}_3)$ and $(\tilde{U},\,\tilde{\boldsymbol{\Psi}}_1,\,\tilde{\boldsymbol{\Psi}}_2,\,\tilde{\boldsymbol{\Psi}}_3,\,\tilde{\boldsymbol{\Psi}}_4)$ be other two such frames. Assume that $U\cap\tilde{U}\neq\varnothing$. Then at each point $p\in U\cap\tilde{U}\neq\varnothing$ one can write the transition formulas (1.4). Instead of (1.6) here we write the following transition formulas:

$$\tilde{\boldsymbol{\Psi}}_i=\sum_{j=1}^{4}\mathfrak{S}^j_i\,\boldsymbol{\Psi}_j,\qquad\qquad \boldsymbol{\Psi}_i=\sum_{j=1}^{4}\mathfrak{T}^j_i\,\tilde{\boldsymbol{\Psi}}_j.\tag{7.1}$$

Using the transition matrices from (1.4) and (7.1), one can define the θ-parameters. They are introduced by the formulas which are almost the same as the formulas (4.6), (4.7), (4.8), and (4.9) in section 4:

$$\tilde{\theta}^k_{ij}=\sum_{a=0}^{3}T^k_a\,L_{\tilde{\boldsymbol{\Upsilon}}_i}(S^a_j)=-\sum_{a=0}^{3}L_{\tilde{\boldsymbol{\Upsilon}}_i}(T^k_a)\,S^a_j,\tag{7.2}$$

$$\tilde{\vartheta}^k_{ij}=\sum_{a=1}^{4}\mathfrak{T}^k_a\,L_{\tilde{\boldsymbol{\Upsilon}}_i}(\mathfrak{S}^a_j)=-\sum_{a=1}^{4}L_{\tilde{\boldsymbol{\Upsilon}}_i}(\mathfrak{T}^k_a)\,\mathfrak{S}^a_j,\tag{7.3}$$

$$\theta^k_{ij}=\sum_{a=0}^{3}S^k_a\,L_{\boldsymbol{\Upsilon}_i}(T^a_j)=-\sum_{a=0}^{3}L_{\boldsymbol{\Upsilon}_i}(S^k_a)\,T^a_j,\tag{7.4}$$

$$\vartheta^k_{ij}=\sum_{a=1}^{4}\mathfrak{S}^k_a\,L_{\boldsymbol{\Upsilon}_i}(\mathfrak{T}^a_j)=-\sum_{a=1}^{4}L_{\boldsymbol{\Upsilon}_i}(\mathfrak{S}^k_a)\,\mathfrak{T}^a_j.\tag{7.5}$$

The only difference is that the indices i and k in (7.3) and (7.5) run over the range from 1 to 4. The formulas (7.2) and (7.4) coincide with (4.6) and (4.8) exactly. For this reason θ-parameters $\tilde{\theta}^k_{ij}$ and θ^k_{ij} here coincide with those in section 4 and the relationships (4.10), (4.11), and (4.14) for them are valid.

Definition 7.1. A *spinor connection* in the Dirac bundle DM is a geometric object such that in each frame pair $(U,\,\boldsymbol{\Upsilon}_0,\,\boldsymbol{\Upsilon}_1,\,\boldsymbol{\Upsilon}_2,\,\boldsymbol{\Upsilon}_3)$ and $(U,\,\boldsymbol{\Psi}_1,\,\boldsymbol{\Psi}_2,\,\boldsymbol{\Psi}_3,\,\boldsymbol{\Psi}_4)$ of the bundles TM and DM it is given by three arrays of smooth complex-valued functions

$$\begin{aligned}
&\Gamma^k_{ij}=\Gamma^k_{ij}(p),\quad i,j,k=0,\,\ldots\,,3,\\
&\mathrm{A}^k_{ij}=\mathrm{A}^k_{ij}(p),\quad i=0,\,\ldots\,,3,\quad j,k=1,\,\ldots\,,4,\\
&\bar{\mathrm{A}}^k_{ij}=\bar{\mathrm{A}}^k_{ij}(p),\quad i=0,\,\ldots\,,3,\quad j,k=1,\,\ldots\,,4,
\end{aligned}$$

where $p\in U$, such that when passing from the frame pair $(U,\,\boldsymbol{\Upsilon}_0,\,\boldsymbol{\Upsilon}_1,\,\boldsymbol{\Upsilon}_2,\,\boldsymbol{\Upsilon}_3)$ and $(U,\,\boldsymbol{\Psi}_1,\,\boldsymbol{\Psi}_2,\,\boldsymbol{\Psi}_3,\,\boldsymbol{\Psi}_4)$ to some other frame pair $(\tilde{U},\,\tilde{\boldsymbol{\Upsilon}}_0,\,\tilde{\boldsymbol{\Upsilon}}_1,\,\tilde{\boldsymbol{\Upsilon}}_2,\,\tilde{\boldsymbol{\Upsilon}}_3)$ and $(\tilde{U},\,\tilde{\boldsymbol{\Psi}}_1,\,\tilde{\boldsymbol{\Psi}}_2,\tilde{\boldsymbol{\Psi}}_3,\,\tilde{\boldsymbol{\Psi}}_4)$ with $U\cap\tilde{U}\neq\varnothing$ these functions are transformed as follows:

$$\Gamma^k_{ij}=\sum_{b=0}^{3}\sum_{a=0}^{3}\sum_{c=0}^{3}S^k_a\,T^b_j\,T^c_i\;\tilde{\Gamma}^a_{c\,b}+\theta^k_{ij},\tag{7.6}$$

$$\mathrm{A}^k_{ij}=\sum_{b=1}^{4}\sum_{a=1}^{4}\sum_{c=0}^{3}\mathfrak{S}^k_a\,\mathfrak{T}^b_j\,T^c_i\;\tilde{\mathrm{A}}^a_{c\,b}+\vartheta^k_{ij},\tag{7.7}$$

$$\bar{\mathrm{A}}^k_{ij}=\sum_{b=1}^{4}\sum_{a=1}^{4}\sum_{c=0}^{3}\overline{\mathfrak{S}^k_a}\;\overline{\mathfrak{T}^b_j}\,T^c_i\;\tilde{\bar{\mathrm{A}}}^a_{c\,b}+\overline{\vartheta^k_{ij}}.\tag{7.8}$$

The components of the transition matrices S, T, $\mathfrak{S}$, and $\mathfrak{T}$ in (7.6), (7.7), and (7.8) are taken from (1.4) and (7.1), while the quantities θ^k_{ij} and ϑ^k_{ij} are defined in (7.4) and (7.5). The covariant differential ∇ associated with the spinor connection $(\Gamma,\mathrm{A},\bar{\mathrm{A}})$ introduced in the definition 7.1 is a differential operator

$$\nabla\colon\ D^\alpha_\beta\bar{D}^\nu_\gamma T^m_n M\to D^\alpha_\beta\bar{D}^\nu_\gamma T^m_{n+1}M.\tag{7.9}$$

In a coordinate form the operator (7.9) is represented by a covariant derivative:

$$\begin{aligned}
&\nabla_{k_{n+1}}X^{i_1\ldots\,i_\alpha\,\bar{i}_1\ldots\,\bar{i}_\nu\,h_1\ldots\,h_m}_{j_1\ldots\,j_\beta\,\bar{j}_1\ldots\,\bar{j}_\gamma\,k_1\ldots\,k_n}=L_{\boldsymbol{\Upsilon}_{k_{n+1}}}\bigl(X^{i_1\ldots\,i_\alpha\,\bar{i}_1\ldots\,\bar{i}_\nu\,h_1\ldots\,h_m}_{j_1\ldots\,j_\beta\,\bar{j}_1\ldots\,\bar{j}_\gamma\,k_1\ldots\,k_n}\bigr)-\\
&+\sum_{\mu=1}^{\alpha}\sum_{v_\mu=1}^{4}\mathrm{A}^{i_\mu}_{k_{n+1}\,v_\mu}\;X^{i_1\ldots\,v_\mu\,\ldots\,i_\varepsilon\,\bar{i}_1\ldots\,\bar{i}_\nu\,h_1\ldots\,h_m}_{j_1\ldots\,\ldots\,\ldots\,j_\beta\,\bar{j}_1\ldots\,\bar{j}_\gamma\,k_1\ldots\,k_n}-\\
&\qquad-\sum_{\mu=1}^{\beta}\sum_{w_\mu=1}^{4}\mathrm{A}^{w_\mu}_{k_{n+1}\,j_\mu}\;X^{i_1\ldots\,\ldots\,\ldots\,i_\alpha\,\bar{i}_1\ldots\,\bar{i}_\nu h_1\ldots\,h_m}_{j_1\ldots\,w_\mu\,\ldots\,j_\beta\,\bar{j}_1\ldots\,\bar{j}_\gamma k_1\ldots\,k_n}+\\
&+\sum_{\mu=1}^{\nu}\sum_{v_\mu=1}^{4}\bar{\mathrm{A}}^{\bar{i}_\mu}_{k_{n+1}\,v_\mu}\;X^{i_1\ldots\,i_\alpha\,\bar{i}_1\ldots\,v_\mu\,\ldots\,\bar{i}_\nu\,h_1\ldots\,h_m}_{j_1\ldots\,j_\beta\,\bar{j}_1\ldots\,\ldots\,\ldots\,\bar{j}_\gamma\,k_1\ldots\,k_n}-\\
&\qquad-\sum_{\mu=1}^{\gamma}\sum_{w_\mu=1}^{4}\bar{\mathrm{A}}^{w_\mu}_{k_{n+1}\,\bar{j}_\mu}\;X^{i_1\ldots\,i_\alpha\,\bar{i}_1\ldots\,\ldots\,\ldots\,\bar{i}_\nu\,h_1\ldots\,h_m}_{j_1\ldots\,j_\beta\,\bar{j}_1\ldots\,w_\mu\,\ldots\,\bar{j}_\gamma\,k_1\ldots\,k_n}+\\
&+\sum_{\mu=1}^{m}\sum_{v_\mu=0}^{3}\Gamma^{h_\mu}_{k_{n+1}\,v_\mu}\;X^{i_1\ldots\,i_\alpha\,\bar{i}_1\ldots\,\bar{i}_\nu\,h_1\ldots\,v_\mu\,\ldots\,h_m}_{j_1\ldots\,j_\beta\,\bar{j}_1\ldots\,\bar{j}_\gamma\,k_1\ldots\,\ldots\,\ldots\,k_n}-\\
&\qquad-\sum_{\mu=1}^{n}\sum_{w_\mu=0}^{3}\Gamma^{w_\mu}_{k_{n+1}\,k_\mu}\;X^{i_1\ldots\,i_\alpha\,\bar{i}_1\ldots\,\bar{i}_\nu\,h_1\ldots\,\ldots\,\ldots\,h_m}_{j_1\ldots\,j_\beta\,\bar{j}_1\ldots\,\bar{j}_\gamma\,k_1\ldots\,w_\mu\,\ldots\,k_n}.
\end{aligned}\tag{7.10}$$

The formula (7.9) is analogous to (4.19), while (7.10) is an analog of (4.20).

Definition 7.2. A spinor connection $(\Gamma,\mathrm{A},\bar{\mathrm{A}})$ of the bundle of Dirac spinors DM is called *concordant with the complex conjugation* or a *real connection* if the corresponding covariant differential (7.9) commute with the involution τ, i. e. if $\nabla(\tau(\mathbf{X}))=\tau(\nabla\mathbf{X})$ for any spin-tensorial field $\mathbf{X}$.

Definition 7.3. A spinor connection $(\Gamma,\mathrm{A},\bar{\mathrm{A}})$ of the bundle of Dirac spinors DM is called *concordant with the Dirac's γ-field* if $\nabla\boldsymbol{\gamma}=0$.

Definition 7.4. A spinor connection $(\Gamma,\mathrm{A},\bar{\mathrm{A}})$ of the bundle of Dirac spinors DM is called *concordant with the spin-metric tensor* if $\nabla\mathbf{d}=0$.

Definition 7.5. A spinor connection $(\Gamma, \mathrm{A}, \bar{\mathrm{A}})$ of the bundle of Dirac spinors DM is called *concordant with the metric tensor* if $\nabla\mathbf{g} = 0$.

Definition 7.6. A spinor connection $(\Gamma, \mathrm{A}, \bar{\mathrm{A}})$ of the bundle of Dirac spinors DM is called *concordant with the chirality operator* if $\nabla\mathbf{H} = 0$.

Definition 7.7. A spinor connection $(\Gamma, \mathrm{A}, \bar{\mathrm{A}})$ of the bundle of Dirac spinors DM is called *concordant with the Hermitian spin-metric tensor* if $\nabla\mathbf{D} = 0$.

Theorem 7.1. *A spinor connection $(\Gamma, \mathrm{A}, \bar{\mathrm{A}})$ of the bundle of Dirac spinors DM is concordant with the complex conjugation τ if and only if*

$$\Gamma^k_{ij} = \overline{\Gamma^k_{ij}}, \qquad\qquad \bar{\mathrm{A}}^k_{ij} = \overline{\mathrm{A}^k_{ij}}. \tag{7.11}$$

The theorem 7.1 is an analog of the theorem 4.1. Its proof is obvious due to the formulas (5.14), (5.15), and (7.10).

The Dirac γ-symbols are more numerous than the Infeld-van der Waerden symbols. For this reason they contain more information and they are more self-sufficient. Instead of the theorem 4.2 here we have.

Theorem 7.2. *Any spinor connection $(\Gamma, \mathrm{A}, \bar{\mathrm{A}})$ of the bundle of Dirac spinors DM concordant with the Dirac γ-field $\boldsymbol{\gamma}$ is concordant with the metric tensor $\mathbf{g}$ as well, i. e. $\nabla\boldsymbol{\gamma} = 0$ implies $\nabla\mathbf{g} = 0$.*

Proof. In order to prove this theorem it is sufficient to apply the identity (6.8). By setting $c = a$ and summing over the index a from (6.8) we derive

$$\sum_{a=1}^{4}\sum_{b=1}^{4}\gamma^a_{bi}\,\gamma^b_{aj} + \sum_{a=1}^{4}\sum_{b=1}^{4}\gamma^a_{bj}\,\gamma^b_{ai} = 2\sum_{a=1}^{4}\sum_{b=1}^{4}\gamma^a_{bi}\,\gamma^b_{aj} = 8\,g_{ij}. \tag{7.12}$$

Applying the covariant derivative ∇_k to both sides of the equality (7.12) and taking into account that $\nabla_k\gamma^a_{bi} = 0$ and $\nabla_k\gamma^b_{aj}$, we derive $\nabla_k g_{ij} = 0$. $\square$

Theorem 7.3. *A spinor connection $(\Gamma, \mathrm{A}, \bar{\mathrm{A}})$ of the bundle of Dirac spinors DM concordant with the Dirac γ-field and with the spin-metric tensor $\mathbf{d}$ is concordant with the chirality operator $\mathbf{H}$ too, i. e. $\nabla\boldsymbol{\gamma} = 0$ and $\nabla\mathbf{d} = 0$ imply $\nabla\mathbf{H} = 0$.*

Proof. In order to prove this theorem we apply the identity (6.23). Due to the previous theorem 7.2 from $\nabla_k\gamma^i_{jm} = 0$ it follows that $\nabla_k g_{ij} = 0$ and $\nabla_k g^{ij} = 0$. Then from $\nabla_k g^{ij} = 0$ and $\nabla_k\gamma^i_{jm} = 0$ due to the formula (6.11) we get $\nabla_k\gamma^{im}_j = 0$. Now applying ∇_k to both sides of (6.23) and using $\nabla\mathbf{d} = 0$, we derive

$$\sum_{r=1}^{4}\sum_{s=1}^{4}\bigl(\nabla_k H^a_r\,d^{re}\,d_{bs}\,H^s_h + H^a_r\,d^{re}\,d_{bs}\,\nabla_k H^s_h\bigr) + \nabla_k H^a_h\;H^e_b + H^a_h\;\nabla_k H^e_b = 0.$$

Let's multiply this equality by H^b_q and sum it over the index b. As a result we get

$$\begin{aligned}\sum_{r=1}^{4}\nabla_k H^a_r\,d^{re}\,d_{qh} + \sum_{b=1}^{4}\sum_{r=1}^{4}\sum_{s=1}^{4}H^a_r\,d^{re}\,d_{qb}\,H^b_s\,\nabla_k H^s_h\,+\\ +\,\nabla_k H^a_h\,\delta^e_q + \sum_{b=1}^{4}H^a_h\,H^b_q\,\nabla_k H^e_b = 0.\end{aligned} \tag{7.13}$$

In deriving (7.13) we used the formula (6.17) and the identity $\mathbf{H}^2 = \mathbf{id}$. In a coordinate form the identity $\mathbf{H}^2 = \mathbf{id}$ is written as follows:

$$\sum_{b=1}^{4} H^e_b\,H^b_h = \delta^e_h. \tag{7.14}$$

The formula (7.14) is easily derived from the matrix representation of the chirality operator (5.4) or (5.5). From (7.14) we easily derive

$$\sum_{e=1}^{4}\sum_{b=1}^{4} H^e_b\,H^b_e = 4. \tag{7.15}$$

Applying the covariant derivative ∇_k to (7.15), we find that

$$\sum_{e=1}^{4}\sum_{b=1}^{4}\bigl(\nabla_k H^e_b\,H^b_e + H^e_b\,\nabla_k H^b_e\bigr) = 2\sum_{e=1}^{4}\sum_{b=1}^{4} H^b_e\,\nabla_k H^e_b = 0. \tag{7.16}$$

Now let's set $q = e$ in the equality (7.13) and recall that the matrix d^{re} is inverse to the matrix d_{eb} when summing over the index e. As a result from (7.13) we get

$$6\,\nabla_k H^a_h + \sum_{e=1}^{4}\sum_{b=1}^{4} H^a_h\,H^b_e\,\nabla_k H^e_b = 0. \tag{7.17}$$

Applying (7.16) to (7.17), we see that the second term in (7.17) is zero. Hence, we have $\nabla_k H^a_h = 0$. The theorem 7.3 is proved. $\square$

Theorem 7.4. *A real spinor connection* $(\Gamma, \mathrm{A}, \bar{\mathrm{A}})$ *of the bundle of Dirac spinors* DM *concordant with the Dirac* γ*-field and with the spin-metric tensor* $\mathbf{d}$ *is concordant with the Dirac form* $\mathbf{D}$ *as well, i. e.* $\nabla\boldsymbol{\gamma} = 0$ *and* $\nabla\mathbf{d} = 0$ *imply* $\nabla\mathbf{D} = 0$.

Proof. The proof of this theorem is based on the following identity for γ-symbols:

$$\begin{gathered}\sum_{m=0}^{3}\sum_{n=0}^{3}\gamma^a_{bm}\,g_{mn}\,\overline{\gamma^e_{hn}} = -\sum_{s=1}^{4}\sum_{\bar r=1}^{4} D_{s\bar h}\,d^{sa}\,D_{b\bar r}\,\bar d^{\bar r\bar e} +\\ +\sum_{r=1}^{4}\sum_{s=1}^{4}\sum_{\bar r=1}^{4}\sum_{q=1}^{4} H^q_s\,D_{q\bar h}\,d^{sa}\,H^r_b\,D_{r\bar r}\,\bar d^{\bar r\bar e} -\\ -\sum_{s=1}^{4}\sum_{\bar r=1}^{4} d^{sa}\,D_{s\bar r}\,\bar d^{\bar r\bar e}\,D_{b\bar h} + \sum_{r=1}^{4}\sum_{s=1}^{4}\sum_{\bar r=1}^{4}\sum_{q=1}^{4} d^{sa}\,H^r_s\,D_{r\bar r}\,\bar d^{\bar r\bar e}\,H^q_b\,D_{q\bar h}.\end{gathered} \tag{7.18}$$

The identity (7.18) is proved by direct calculations. Let's apply the covariant derivative ∇_k to both sides of the identity (7.18). When doing it we should remember that $\nabla\boldsymbol{\gamma} = 0$ and $\nabla\mathbf{d} = 0$ imply $\nabla\mathbf{g} = 0$ and $\nabla\mathbf{H} = 0$ due to the theorems 7.2 and 7.3. Moreover, $\nabla\boldsymbol{\gamma} = 0$ implies $\nabla\tau(\boldsymbol{\gamma}) = 0$ since $(\Gamma, \mathrm{A}, \bar{\mathrm{A}})$ is assumed to be a real spinor connection. As a result, taking into account all the above arguments, from (7.18) we derive the following identity for the components of the Dirac form $\mathbf{D}$:

$$
\begin{aligned}
&-\sum_{s=1}^{4}\sum_{\bar{r}=1}^{4}\nabla_k D_{s\bar{h}}\,d^{sa}\,D_{b\bar{r}}\,\bar{d}^{\bar{r}\bar{e}}-\sum_{s=1}^{4}\sum_{\bar{r}=1}^{4}D_{s\bar{h}}\,d^{sa}\,\nabla_k D_{b\bar{r}}\,\bar{d}^{\bar{r}\bar{e}}+\\
&+\sum_{r=1}^{4}\sum_{s=1}^{4}\sum_{\bar{r}=1}^{4}\sum_{q=1}^{4}d^{sa}\bigl(H^q_s\,\nabla_k D_{q\bar{h}}\,H^r_b\,D_{r\bar{r}}+H^q_s\,D_{q\bar{h}}\,H^r_b\,\nabla_k D_{r\bar{r}}\bigr)\,\bar{d}^{\bar{r}\bar{e}}-\\
&-\sum_{s=1}^{4}\sum_{\bar{r}=1}^{4}d^{sa}\,\nabla_k D_{s\bar{r}}\,\bar{d}^{\bar{r}\bar{e}}\,D_{b\bar{h}}-\sum_{s=1}^{4}\sum_{\bar{r}=1}^{4}d^{sa}\,D_{s\bar{r}}\,\bar{d}^{\bar{r}\bar{e}}\,\nabla_k D_{b\bar{h}}+\\
&+\sum_{r=1}^{4}\sum_{s=1}^{4}\sum_{\bar{r}=1}^{4}\sum_{q=1}^{4}d^{sa}\bigl(H^r_s\,\nabla_k D_{r\bar{r}}\,H^q_b\,D_{q\bar{h}}+H^r_s\,D_{r\bar{r}}\,H^q_b\,\nabla_k D_{q\bar{h}}\bigr)\,\bar{d}^{\bar{r}\bar{e}}=0.
\end{aligned}
$$

Note that each term in the above identity contains the sum over the index s and the factor d^{sa}, where a is a free index. Similarly, each term contains the sum over the index $\bar{r}$ and the factor $\bar{d}^{\bar{r}\bar{e}}$, where $\bar{e}$ is a free index. The quantities d^{sa} and $\bar{d}^{\bar{r}\bar{e}}$ form two non-degenerate matrices. Therefore, we can cancel them in the above equality and simultaneously omit the sums over s and $\bar{r}$:

$$
\begin{aligned}
&-\nabla_k D_{s\bar{h}}\,D_{b\bar{r}}-D_{s\bar{h}}\,\nabla_k D_{b\bar{r}}+\sum_{r=1}^{4}\sum_{q=1}^{4}H^q_s\,\nabla_k D_{q\bar{h}}\,H^r_b\,D_{r\bar{r}}+\\
&+\sum_{r=1}^{4}\sum_{q=1}^{4}H^q_s\,D_{q\bar{h}}\,H^r_b\,\nabla_k D_{r\bar{r}}-\nabla_k D_{s\bar{r}}\,D_{b\bar{h}}-D_{s\bar{r}}\,\nabla_k D_{b\bar{h}}+\\
&+\sum_{r=1}^{4}\sum_{q=1}^{4}H^r_s\,\nabla_k D_{r\bar{r}}\,H^q_b\,D_{q\bar{h}}+\sum_{r=1}^{4}\sum_{q=1}^{4}H^r_s\,D_{r\bar{r}}\,H^q_b\,\nabla_k D_{q\bar{h}}=0.
\end{aligned}
\tag{7.19}
$$

Now let's multiply (7.19) by $D^{b\bar{h}}$ and sum over the indices b and $\bar{h}$. The quantities $D^{b\bar{h}}$ are defined through the formula (6.24). Taking into account (6.25), we derive

$$
\begin{aligned}
&-5\,\nabla_k D_{s\bar{r}}+\sum_{r=1}^{4}\sum_{q=1}^{4}\sum_{b=1}^{4}\sum_{\bar{h}=1}^{4}H^q_s\,\nabla_k D_{q\bar{h}}\,D^{b\bar{h}}\,H^r_b\,D_{r\bar{r}}-\\
&-\sum_{b=1}^{4}\sum_{\bar{h}=1}^{4}D_{s\bar{r}}\,\nabla_k D_{b\bar{h}}\,D^{b\bar{h}}+\sum_{r=1}^{4}\sum_{b=1}^{4}H^r_s\,\nabla_k D_{r\bar{r}}\,H^b_b+\\
&+\sum_{r=1}^{4}\sum_{q=1}^{4}\sum_{b=1}^{4}\sum_{\bar{h}=1}^{4}H^r_s\,D_{r\bar{r}}\,H^q_b\,\nabla_k D_{q\bar{h}}\,D^{b\bar{h}}=0.
\end{aligned}
\tag{7.20}
$$

In deriving (7.20), apart from (6.25), we used the equality (7.14). Remember, that $\mathbf{H}$ is a traceless operator (see its matrix (5.4)). This means that

$$
\operatorname{tr}\mathbf{H}=\sum_{b=1}^{4}H^b_b=0.
\tag{7.21}
$$

Due to (7.21) the fourth term in (7.20) is zero. The third term in the left hand side of (7.20) is zero too. This fact is proved with the use of the formula (6.24):

$$\begin{aligned}
0=\sum_{b=1}^{4}\sum_{\bar h=1}^{4}\nabla_k(D_{b\bar h}\,D^{b\bar h})=\sum_{b=1}^{4}\sum_{\bar h=1}^{4}\sum_{a=1}^{4}\sum_{\bar a=1}^{4}\nabla_k(D_{b\bar h}\,d^{ba}\,D_{a\bar a}\,\bar d^{\bar a\bar h})=\\
=\sum_{b=1}^{4}\sum_{\bar h=1}^{4}\sum_{a=1}^{4}\sum_{\bar a=1}^{4}\nabla_kD_{b\bar h}\,d^{ba}\,D_{a\bar a}\,\bar d^{\bar a\bar h}+\sum_{b=1}^{4}\sum_{\bar h=1}^{4}\sum_{a=1}^{4}\sum_{\bar a=1}^{4}D_{b\bar h}\,d^{ba}\,\nabla_kD_{a\bar a}\,\bar d^{\bar a\bar h}=\\
=\sum_{b=1}^{4}\sum_{\bar h=1}^{4}\sum_{a=1}^{4}\sum_{\bar a=1}^{4}\nabla_kD_{b\bar h}\,d^{ba}\,D_{a\bar a}\,\bar d^{\bar a\bar h}+\sum_{b=1}^{4}\sum_{\bar h=1}^{4}\sum_{a=1}^{4}\sum_{\bar a=1}^{4}\nabla_kD_{a\bar a}\,d^{ab}\,D_{b\bar h}\,\bar d^{\bar h\bar a}=\\
=2\sum_{b=1}^{4}\sum_{\bar h=1}^{4}\sum_{a=1}^{4}\sum_{\bar a=1}^{4}\nabla_kD_{b\bar h}\,d^{ba}\,D_{a\bar a}\,\bar d^{\bar a\bar h}=2\sum_{b=1}^{4}\sum_{\bar h=1}^{4}\nabla_kD_{b\bar h}\,D^{b\bar h}.
\end{aligned}$$

In the above calculations we used (6.25) and the skew-symmetry of $\mathbf{d}$ and $\bar{\mathbf{d}}$. The last term in the left hand side of (7.20) is also zero. This fact is derived from (7.21):

$$\begin{aligned}
0=\sum_{b=1}^{4}\nabla_kH^b_b=\sum_{q=1}^{4}\sum_{b=1}^{4}\sum_{\bar h=1}^{4}\nabla_k(H^q_b\,D_{q\bar h}\,D^{b\bar h})=\sum_{q=1}^{4}\sum_{b=1}^{4}\sum_{\bar h=1}^{4}\sum_{a=1}^{4}\sum_{\bar a=1}^{4}\bigl(H^q_b\,\times\\
\times\,\nabla_kD_{q\bar h}\,d^{ba}\,D_{a\bar a}\,\bar d^{\bar a\bar h}+H^q_b\,D_{q\bar h}\,d^{ba}\,\nabla_kD_{a\bar a}\,\bar d^{\bar a\bar h}\bigr)=\sum_{q=1}^{4}\sum_{b=1}^{4}\sum_{\bar h=1}^{4}\sum_{a=1}^{4}\sum_{\bar a=1}^{4}H^q_b\,\times\\
\times\,\nabla_kD_{q\bar h}\,d^{ba}\,D_{a\bar a}\,\bar d^{\bar a\bar h}+\sum_{q=1}^{4}\sum_{b=1}^{4}\sum_{\bar h=1}^{4}\sum_{a=1}^{4}\sum_{\bar a=1}^{4}H^a_b\,\nabla_kD_{a\bar a}\,d^{bq}\,D_{q\bar h}\,\bar d^{\bar h\bar a}=\\
=2\sum_{q=1}^{4}\sum_{b=1}^{4}\sum_{\bar h=1}^{4}\sum_{a=1}^{4}\sum_{\bar a=1}^{4}H^q_b\,\nabla_kD_{q\bar h}\,d^{ba}\,D_{a\bar a}\,\bar d^{\bar a\bar h}=2\sum_{q=1}^{4}\sum_{b=1}^{4}\sum_{\bar h=1}^{4}H^q_b\,\nabla_kD_{q\bar h}\,D^{b\bar h}.
\end{aligned}$$

In these calculations we used (6.25), the skew-symmetry $\mathbf{d}$ and $\bar{\mathbf{d}}$, and the equality

$$\sum_{b=1}^{4}H^q_b\,d^{ba}=\sum_{b=1}^{4}d^{qb}\,H^a_b. \tag{7.22}$$

The equality (7.22) is easily derived from (6.17). And finally, we need to transform the second term in the left hand side of (7.20):

$$\begin{aligned}
\sum_{r=1}^{4}\sum_{q=1}^{4}\sum_{b=1}^{4}\sum_{\bar h=1}^{4}H^q_s\,\nabla_kD_{q\bar h}\,D^{b\bar h}\,H^r_b\,D_{r\bar r}=-\sum_{\bar s=1}^{4}\sum_{q=1}^{4}\sum_{b=1}^{4}\sum_{\bar h=1}^{4}H^q_s\,\nabla_kD_{q\bar h}\,\times\\
\times\,D^{b\bar h}\,D_{b\bar s}\,\overline{H^{\bar s}_{\bar r}}=-\sum_{q=1}^{4}\sum_{\bar h=1}^{4}H^q_s\,\nabla_kD_{q\bar h}\,\overline{H^{\bar h}_{\bar r}}=-\sum_{q=1}^{4}\sum_{\bar h=1}^{4}H^q_s\,\nabla_k(D_{q\bar h}\,\overline{H^{\bar h}_{\bar r}})=\\
=\sum_{q=1}^{4}\sum_{r=1}^{4}H^q_s\,\nabla_k(H^r_q\,D_{r\bar r})=\sum_{q=1}^{4}\sum_{r=1}^{4}H^q_s\,H^r_q\,\nabla_kD_{r\bar r}=\nabla_kD_{s\bar r}.
\end{aligned}\tag{7.23}$$

In deriving (7.23) we used the equalities (6.20), (6.25), and (7.14). Now, substituting (7.23) back into (7.20) and recalling that the third, the fourth, and the fifth terms in the left hand side of this equality do vanish, we find that (7.20) is reduced to $\nabla_k D_{s\bar{r}} = 0$. Thus, $\nabla\mathbf{D} = 0$. The theorem 7.4 is proved. $\square$

Definition 7.8. A real spinor connection $(\Gamma, \mathrm{A}, \bar{\mathrm{A}})$ of the bundle of Dirac spinors DM concordant with the spin-metric tensor $\mathbf{d}$ and with Dirac γ-field $\boldsymbol{\gamma}$ is called a *metric connection*.

According to the above theorems 7.2, 7.3, 7.4, a metric connection $(\Gamma, \mathrm{A}, \bar{\mathrm{A}})$ is concordant with the metric tensor $\mathbf{g}$, with the chirality operator $\mathbf{H}$, with the Dirac form $\mathbf{D}$, and with many other fields produced from these basic fields.

8. Explicit formulas.

Let $(\Gamma, \mathrm{A}, \bar{\mathrm{A}})$ be a metric connection of the bundle of Dirac spinors DM. Since $\nabla\mathbf{g} = 0$ for this connection, its Γ-components are uniquely determined by the torsion tensor $\mathbf{T}$. They are given by the explicit formula (4.34). If $\mathbf{T} = 0$, this formula reduces to (4.35). Our goal in this section is to study A-components of a metric connection. Its $\bar{\mathrm{A}}$-components then are expressed through A-components according to the formula (7.11).

In order to derive an explicit formula for the A-components of a metric connection we introduce some auxiliary objects. Using the chirality operator $\mathbf{H}$, we define two projector operators $\overset{\bullet}{\mathbf{H}}$ and $\overset{\circ}{\mathbf{H}}$ by means of the following formulas:

$$\overset{\bullet}{\mathbf{H}} = \frac{\mathbf{id} + \mathbf{H}}{2}, \qquad\qquad \overset{\circ}{\mathbf{H}} = \frac{\mathbf{id} - \mathbf{H}}{2}. \tag{8.1}$$

Their components are given by the formulas

$$\overset{\bullet}{H}{}^i_j = \frac{\delta^i_j + H^i_j}{2}, \qquad\qquad \overset{\circ}{H}{}^i_j = \frac{\delta^i_j - H^i_j}{2}, \tag{8.2}$$

Due to (8.1) and (8.2) we have the expansions

$$\mathbf{id} = \overset{\bullet}{\mathbf{H}} + \overset{\circ}{\mathbf{H}}, \qquad\qquad \delta^i_j = \overset{\bullet}{H}{}^i_j + \overset{\circ}{H}{}^i_j. \tag{8.3}$$

Now, relying on the expansion (8.3), we introduce the following quantities:

$$\overset{\bullet\bullet}{\gamma}{}^i_{jm} = \sum_{r=1}^{4}\sum_{s=1}^{4} \overset{\bullet}{H}{}^i_r\,\overset{\bullet}{H}{}^s_j\,\gamma^r_{sm}, \qquad\qquad \overset{\circ\circ}{\gamma}{}^i_{jm} = \sum_{r=1}^{4}\sum_{s=1}^{4} \overset{\circ}{H}{}^i_r\,\overset{\circ}{H}{}^s_j\,\gamma^r_{sm}, \tag{8.4}$$

$$\overset{\circ\bullet}{\gamma}{}^i_{jm} = \sum_{r=1}^{4}\sum_{s=1}^{4} \overset{\circ}{H}{}^i_r\,\overset{\bullet}{H}{}^s_j\,\gamma^r_{sm}, \qquad\qquad \overset{\bullet\circ}{\gamma}{}^i_{jm} = \sum_{r=1}^{4}\sum_{s=1}^{4} \overset{\bullet}{H}{}^i_r\,\overset{\circ}{H}{}^s_j\,\gamma^r_{sm}. \tag{8.5}$$

From (8.4), (8.5), and from the expansion (8.3) we derive the reconstruction formula

$$\gamma^i_{jm} = \overset{\bullet\bullet}{\gamma}{}^i_{jm} + \overset{\bullet\circ}{\gamma}{}^i_{jm} + \overset{\circ\bullet}{\gamma}{}^i_{jm} + \overset{\circ\circ}{\gamma}{}^i_{jm}. \tag{8.6}$$

By means of direct calculations we find that the quantities (8.4) are identically zero: $\overset{\bullet\bullet}{\gamma}{}^{i}_{jm} = 0$ and $\overset{\circ\circ}{\gamma}{}^{i}_{jm} = 0$. Therefore, the expansion (8.6) reduces to the following one:

$$\gamma^{i}_{jm} = \overset{\bullet\circ}{\gamma}{}^{i}_{jm} + \overset{\circ\bullet}{\gamma}{}^{i}_{jm}.$$

Now let's proceed with the A-components of a metric connection $(\Gamma, \mathrm{A}, \bar{\mathrm{A}})$. By analogy to the formulas (8.4) and (8.5) we introduce the following quantities:

$$\overset{\bullet\bullet}{\mathrm{A}}{}^{i}_{kj} = \sum_{r=1}^{4}\sum_{s=1}^{4} \overset{\bullet}{H}{}^{i}_{r}\,\overset{\bullet}{H}{}^{s}_{j}\ \mathrm{A}^{r}_{ks}, \qquad \overset{\circ\circ}{\mathrm{A}}{}^{i}_{kj} = \sum_{r=1}^{4}\sum_{s=1}^{4} \overset{\circ}{H}{}^{i}_{r}\,\overset{\circ}{H}{}^{s}_{j}\ \mathrm{A}^{r}_{ks}, \tag{8.7}$$

$$\overset{\circ\bullet}{\mathrm{A}}{}^{i}_{kj} = \sum_{r=1}^{4}\sum_{s=1}^{4} \overset{\circ}{H}{}^{i}_{r}\,\overset{\bullet}{H}{}^{s}_{j}\ \mathrm{A}^{r}_{ks}, \qquad \overset{\bullet\circ}{\mathrm{A}}{}^{i}_{kj} = \sum_{r=1}^{4}\sum_{s=1}^{4} \overset{\bullet}{H}{}^{i}_{r}\,\overset{\circ}{H}{}^{s}_{j}\ \mathrm{A}^{r}_{ks}\,. \tag{8.8}$$

In this case the quantities (8.7) are not necessarily zero. Therefore, we have all of the four terms (8.7) and (8.8) in the expansion

$$\mathrm{A}^{i}_{kj} = \overset{\bullet\bullet}{\mathrm{A}}{}^{i}_{kj} + \overset{\bullet\circ}{\mathrm{A}}{}^{i}_{kj} + \overset{\circ\bullet}{\mathrm{A}}{}^{i}_{kj} + \overset{\circ\circ}{\mathrm{A}}{}^{i}_{kj}. \tag{8.9}$$

A metric connection $(\Gamma, \mathrm{A}, \bar{\mathrm{A}})$ is concordant with $\boldsymbol{\gamma}$ and $\mathbf{H}$. Hence, from $\nabla\boldsymbol{\gamma} = 0$ and $\nabla\mathbf{H} = 0$ we derive the following equalities:

$$\nabla_{k}\overset{\bullet}{H}{}^{i}_{j} = 0, \qquad \nabla_{k}\overset{\circ}{H}{}^{i}_{j} = 0, \qquad \nabla_{k}\,\overset{\bullet\circ}{\gamma}{}^{i}_{jm} = 0, \qquad \nabla_{k}\,\overset{\circ\bullet}{\gamma}{}^{i}_{jm} = 0. \tag{8.10}$$

Applying the formula (7.10) to the first equality (8.10), we get

$$L_{\boldsymbol{\Upsilon}_k}(\overset{\bullet}{H}{}^{i}_{s}) + \sum_{r=1}^{4}\mathrm{A}^{i}_{kr}\ \overset{\bullet}{H}{}^{r}_{s} - \sum_{r=1}^{4}\mathrm{A}^{r}_{ks}\ \overset{\bullet}{H}{}^{i}_{r} = 0. \tag{8.11}$$

Let's multiply (8.11) by $\overset{\circ}{H}{}^{s}_{j}$ and sum it over the index s:

$$\sum_{s=1}^{4}\overset{\circ}{H}{}^{s}_{j}\,L_{\boldsymbol{\Upsilon}_k}(\overset{\bullet}{H}{}^{i}_{s}) + \sum_{s=1}^{4}\sum_{r=1}^{4}\mathrm{A}^{i}_{kr}\ \overset{\bullet}{H}{}^{r}_{s}\,\overset{\circ}{H}{}^{s}_{j} - \sum_{s=1}^{4}\sum_{r=1}^{4}\overset{\bullet}{H}{}^{i}_{r}\,\overset{\circ}{H}{}^{s}_{j}\ \mathrm{A}^{r}_{ks} = 0. \tag{8.12}$$

The projectors (8.1) are complementary to each other. Therefore their product is zero: $\overset{\bullet}{\mathbf{H}}\cdot\overset{\circ}{\mathbf{H}} = 0$. Applying this equality to (8.12) we derive

$$\overset{\bullet\circ}{\mathrm{A}}{}^{i}_{kj} = \sum_{s=1}^{4}\overset{\circ}{H}{}^{s}_{j}\,L_{\boldsymbol{\Upsilon}_k}(\overset{\bullet}{H}{}^{i}_{s}). \tag{8.13}$$

In a similar way from the second equality (8.10), we obtain

$$\overset{\circ\bullet}{\mathrm{A}}{}^{i}_{kj} = \sum_{s=1}^{4}\overset{\bullet}{H}{}^{s}_{j}\,L_{\boldsymbol{\Upsilon}_k}(\overset{\circ}{H}{}^{i}_{s}). \tag{8.14}$$

Now let's apply the formula (7.10) to the third and to the fourth equalities (8.10). As a result we get the following two formulas:

$$L_{\boldsymbol{\Upsilon}_k}(\overset{\bullet\circ}{\gamma}{}^{a}_{bm})+\sum_{r=1}^{4}\mathrm{A}^{a}_{kr}\,\overset{\bullet\circ}{\gamma}{}^{r}_{bm}-\sum_{r=1}^{4}\mathrm{A}^{r}_{kb}\,\overset{\bullet\circ}{\gamma}{}^{a}_{rm}=\sum_{r=0}^{3}\Gamma^{r}_{km}\,\overset{\bullet\circ}{\gamma}{}^{a}_{br},\tag{8.15}$$

$$L_{\boldsymbol{\Upsilon}_k}(\overset{\circ\bullet}{\gamma}{}^{a}_{bm})+\sum_{r=1}^{4}\mathrm{A}^{a}_{kr}\,\overset{\circ\bullet}{\gamma}{}^{r}_{bm}-\sum_{r=1}^{4}\mathrm{A}^{r}_{kb}\,\overset{\circ\bullet}{\gamma}{}^{a}_{rm}=\sum_{r=0}^{3}\Gamma^{r}_{km}\,\overset{\circ\bullet}{\gamma}{}^{a}_{br}.\tag{8.16}$$

In order to transform (8.15) and (8.16) we need some identities for the γ-symbols (8.5). From the identity (6.23) we derive two formulas

$$\sum_{m=0}^{3}\sum_{n=0}^{3}\overset{\bullet\circ}{\gamma}{}^{a}_{bm}\,g^{mn}\,\overset{\bullet\circ}{\gamma}{}^{e}_{hn}=2\,\overset{\bullet}{d}{}^{ae}\,\overset{\circ}{d}_{bh},\tag{8.17}$$

$$\sum_{m=0}^{3}\sum_{n=0}^{3}\overset{\circ\bullet}{\gamma}{}^{a}_{bm}\,g^{mn}\,\overset{\circ\bullet}{\gamma}{}^{e}_{hn}=2\,\overset{\circ}{d}{}^{ae}\,\overset{\bullet}{d}_{bh}.\tag{8.18}$$

In the formulas (8.17) and (8.18) we used the following notations:

$$\begin{aligned}
&\overset{\bullet}{d}{}^{ae}=\sum_{r=1}^{4}\sum_{s=1}^{4}\overset{\bullet}{H}{}^{a}_{r}\,d^{rs}\,\overset{\bullet}{H}{}^{e}_{s}, &&\overset{\circ}{d}{}^{ae}=\sum_{r=1}^{4}\sum_{s=1}^{4}\overset{\circ}{H}{}^{a}_{r}\,d^{rs}\,\overset{\circ}{H}{}^{e}_{s},\\
&\overset{\bullet}{d}_{bh}=\sum_{r=1}^{4}\sum_{s=1}^{4}\overset{\bullet}{H}{}^{r}_{b}\,d_{rs}\,\overset{\bullet}{H}{}^{s}_{h}, &&\overset{\circ}{d}_{bh}=\sum_{r=1}^{4}\sum_{s=1}^{4}\overset{\circ}{H}{}^{r}_{b}\,d_{rs}\,\overset{\circ}{H}{}^{s}_{h}.
\end{aligned}\tag{8.19}$$

The matrices (8.19) are skew-symmetric and degenerate. However, in some cases they can be used for raising and lowering spinor indices:

$$\begin{aligned}
&\overset{\bullet\bullet}{\mathrm{A}}_{kij}=\sum_{s=1}^{4}\overset{\bullet\bullet}{\mathrm{A}}{}^{s}_{ki}\,\overset{\bullet}{d}_{sj}, &&\overset{\circ\circ}{\mathrm{A}}_{kij}=\sum_{s=1}^{4}\overset{\circ\circ}{\mathrm{A}}{}^{s}_{ki}\,\overset{\circ}{d}_{sj},\\
&\overset{\bullet\bullet}{\mathrm{A}}{}^{i}_{kj}=\sum_{s=1}^{4}\overset{\bullet\bullet}{\mathrm{A}}_{kjs}\,\overset{\bullet}{d}{}^{si}, &&\overset{\circ\circ}{\mathrm{A}}{}^{i}_{kj}=\sum_{s=1}^{4}\overset{\circ\circ}{\mathrm{A}}_{kjs}\,\overset{\circ}{d}{}^{si}.
\end{aligned}\tag{8.20}$$

Applying the formula (8.17) to (8.15), we derive the following equality:

$$\begin{gathered}
\frac{1}{2}\sum_{m=0}^{3}\sum_{n=0}^{3}L_{\boldsymbol{\Upsilon}_k}(\overset{\bullet\circ}{\gamma}{}^{a}_{bm})\,g^{mn}\,\overset{\bullet\circ}{\gamma}{}^{e}_{hn}+\sum_{r=1}^{4}\mathrm{A}^{a}_{kr}\,\overset{\bullet}{d}{}^{re}\,\overset{\circ}{d}_{bh}-\\
-\sum_{r=1}^{4}\mathrm{A}^{r}_{kb}\,\overset{\bullet}{d}{}^{ae}\,\overset{\circ}{d}_{rh}=\frac{1}{2}\sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{r=0}^{3}\Gamma^{r}_{km}\,\overset{\bullet\circ}{\gamma}{}^{a}_{br}\,g^{mn}\,\overset{\bullet\circ}{\gamma}{}^{e}_{hn},
\end{gathered}\tag{8.21}$$

Let's multiply (8.21) by $\overset{\bullet}{H}{}^{q}_{a}\,\overset{\circ}{H}{}^{b}_{j}$ and sum over the indices a and b. Then we get

$$\begin{gathered}
\frac{1}{2}\sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{a=1}^{4}\sum_{b=1}^{4}L_{\boldsymbol{\Upsilon}_k}(\overset{\bullet\circ}{\gamma}{}^{a}_{bm})\,g^{mn}\,\overset{\bullet\circ}{\gamma}{}^{e}_{hn}\,\overset{\bullet}{H}{}^{q}_{a}\,\overset{\circ}{H}{}^{b}_{j}+\sum_{r=1}^{4}\overset{\bullet\bullet}{\mathrm{A}}{}^{q}_{kr}\,\overset{\bullet}{d}{}^{re}\,\overset{\circ}{d}_{jh}-\\
-\overset{\circ\circ}{\mathrm{A}}_{kjh}\,\overset{\bullet}{d}{}^{qe}=\frac{1}{2}\sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{r=0}^{3}\Gamma^{r}_{km}\,\overset{\bullet\circ}{\gamma}{}^{q}_{jr}\,g^{mn}\,\overset{\bullet\circ}{\gamma}{}^{e}_{hn}.
\end{gathered}\tag{8.22}$$

Now let's multiply (8.22) by $\overset{\bullet}{d}_{eq}$ and sum over the indices e and q. This yields

$$\begin{aligned}2\,\overset{\circ\circ}{\mathrm{A}}_{kjh} &= \frac{1}{2}\sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{a=1}^{4}\sum_{b=1}^{4}\sum_{e=1}^{4} L_{\boldsymbol{\Upsilon}_k}(\overset{\bullet\circ}{\gamma}{}^{a}_{bm})\,g^{mn}\,\overset{\bullet\circ}{\gamma}{}^{e}_{hn}\,\overset{\circ}{H}{}^{b}_{j}\,\overset{\bullet}{d}_{ea}\,+\\ &+\sum_{r=1}^{4}\overset{\bullet\bullet}{\mathrm{A}}{}^{r}_{kr}\,\overset{\circ}{d}_{jh}-\frac{1}{2}\sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{r=0}^{3}\sum_{e=1}^{4}\sum_{q=1}^{4}\Gamma^{r}_{km}\,\overset{\bullet\circ}{\gamma}{}^{q}_{jr}\,g^{mn}\,\overset{\bullet\circ}{\gamma}{}^{e}_{hn}\,\overset{\bullet}{d}_{eq}.\end{aligned}\tag{8.23}$$

Then let's raise the lower index h in (8.23) by means of the matrix $\overset{\circ}{d}{}^{hi}$, i.e. applying the second formula (8.20) to $\overset{\circ\circ}{\mathrm{A}}_{kjh}$ in the left hand side. As a result we get

$$\begin{aligned}\overset{\circ\circ}{\mathrm{A}}{}^{i}_{kj} &= \frac{1}{4}\sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{a=1}^{4}\sum_{b=1}^{4} L_{\boldsymbol{\Upsilon}_k}(\overset{\bullet\circ}{\gamma}{}^{a}_{bm})\,g^{mn}\,\overset{\circ\bullet}{\gamma}{}^{i}_{an}\,\overset{\circ}{H}{}^{b}_{j}\,+\\ &+\frac{1}{2}\sum_{r=1}^{4}\overset{\bullet\bullet}{\mathrm{A}}{}^{r}_{kr}\,\overset{\circ}{H}{}^{i}_{j}-\frac{1}{4}\sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{r=0}^{3}\sum_{q=1}^{4}\Gamma^{r}_{km}\,\overset{\bullet\circ}{\gamma}{}^{q}_{jr}\,g^{mn}\,\overset{\circ\bullet}{\gamma}{}^{i}_{qn}.\end{aligned}\tag{8.24}$$

Acting in a similar way, i.e. applying the formula (8.18) to (8.16) and then performing some calculations analogous to the above ones, we derive the formula

$$\begin{aligned}\overset{\bullet\bullet}{\mathrm{A}}{}^{i}_{kj} &= \frac{1}{4}\sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{a=1}^{4}\sum_{b=1}^{4} L_{\boldsymbol{\Upsilon}_k}(\overset{\circ\bullet}{\gamma}{}^{a}_{bm})\,g^{mn}\,\overset{\bullet\circ}{\gamma}{}^{i}_{an}\,\overset{\bullet}{H}{}^{b}_{j}\,+\\ &+\frac{1}{2}\sum_{r=1}^{4}\overset{\circ\circ}{\mathrm{A}}{}^{r}_{kr}\,\overset{\bullet}{H}{}^{i}_{j}-\frac{1}{4}\sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{r=0}^{3}\sum_{q=1}^{4}\Gamma^{r}_{km}\,\overset{\circ\bullet}{\gamma}{}^{q}_{jr}\,g^{mn}\,\overset{\bullet\circ}{\gamma}{}^{i}_{qn}.\end{aligned}\tag{8.25}$$

Remember that for a metric connection $\nabla\mathbf{d}=0$ and $\nabla\mathbf{H}=0$. Then from (8.2) and (8.19) we get the following equalities for $\overset{\bullet}{d}_{ab}$ and $\overset{\circ}{d}_{ab}$:

$$\nabla_k\overset{\bullet}{d}_{ab}=0, \qquad\qquad \nabla_k\overset{\circ}{d}_{ab}=0.\tag{8.26}$$

In an expanded form these equalities (8.26) are written as

$$L_{\boldsymbol{\Upsilon}_k}(\overset{\bullet}{d}_{ab})-\sum_{r=1}^{4}\mathrm{A}^{r}_{ka}\,\overset{\bullet}{d}_{rb}-\sum_{r=1}^{4}\mathrm{A}^{r}_{kb}\,\overset{\bullet}{d}_{ar}=0,\tag{8.27}$$

$$L_{\boldsymbol{\Upsilon}_k}(\overset{\circ}{d}_{ab})-\sum_{r=1}^{4}\mathrm{A}^{r}_{ka}\,\overset{\circ}{d}_{rb}-\sum_{r=1}^{4}\mathrm{A}^{r}_{kb}\,\overset{\circ}{d}_{ar}=0.\tag{8.28}$$

Let's multiply (8.27) by $\overset{\bullet}{d}{}^{ba}$ and multiply (8.28) by $\overset{\circ}{d}{}^{ba}$. Then let's sum both equalities over the indices a and b. As a result we get

$$\sum_{r=1}^{4}\overset{\bullet\bullet}{\mathrm{A}}{}^{r}_{kr}=\frac{1}{2}\sum_{a=1}^{4}\sum_{b=1}^{4}L_{\boldsymbol{\Upsilon}_k}(\overset{\bullet}{d}_{ab})\,\overset{\bullet}{d}{}^{ba}, \qquad \sum_{r=1}^{4}\overset{\circ\circ}{\mathrm{A}}{}^{r}_{kr}=\frac{1}{2}\sum_{a=1}^{4}\sum_{b=1}^{4}L_{\boldsymbol{\Upsilon}_k}(\overset{\circ}{d}_{ab})\,\overset{\circ}{d}{}^{ba}.$$

Substituting these formulas back into (8.24) and (8.25), we derive

$$
\begin{gathered}
\overset{\circ\circ}{\mathrm{A}}{}^{i}_{kj}=\frac{1}{4}\sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{a=1}^{4}\sum_{b=1}^{4}L_{\boldsymbol{\Upsilon}_k}(\overset{\bullet\circ}{\gamma}{}^{a}_{bm})\,g^{mn}\,\overset{\circ\bullet}{\gamma}{}^{i}_{an}\,\overset{\circ}{H}{}^{b}_{j}\,+\\
+\frac{1}{4}\sum_{a=1}^{4}\sum_{b=1}^{4}L_{\boldsymbol{\Upsilon}_k}(\overset{\bullet}{d}_{ab})\,\overset{\bullet}{d}{}^{ba}\,\overset{\circ}{H}{}^{i}_{j}-\frac{1}{4}\sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{r=0}^{3}\sum_{q=1}^{4}\Gamma^{r}_{km}\,\overset{\bullet\circ}{\gamma}{}^{q}_{jr}\,g^{mn}\,\overset{\circ\bullet}{\gamma}{}^{i}_{qn},
\end{gathered}
\tag{8.29}
$$

$$
\begin{gathered}
\overset{\bullet\bullet}{\mathrm{A}}{}^{i}_{kj}=\frac{1}{4}\sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{a=1}^{4}\sum_{b=1}^{4}L_{\boldsymbol{\Upsilon}_k}(\overset{\circ\bullet}{\gamma}{}^{a}_{bm})\,g^{mn}\,\overset{\bullet\circ}{\gamma}{}^{i}_{an}\,\overset{\bullet}{H}{}^{b}_{j}\,+\\
+\frac{1}{4}\sum_{a=1}^{4}\sum_{b=1}^{4}L_{\boldsymbol{\Upsilon}_k}(\overset{\circ}{d}_{ab})\,\overset{\circ}{d}{}^{ba}\,\overset{\bullet}{H}{}^{i}_{j}-\frac{1}{4}\sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{r=0}^{3}\sum_{q=1}^{4}\Gamma^{r}_{km}\,\overset{\circ\bullet}{\gamma}{}^{q}_{jr}\,g^{mn}\,\overset{\bullet\circ}{\gamma}{}^{i}_{qn}.
\end{gathered}
\tag{8.30}
$$

The formula (8.29) is an analog of the formula (4.57), while (8.30) is an analog of (4.62). Moreover, these formulas are reduced to the corresponding formulas (4.57) and (4.62) if we choose a spinor frame $(U,\,\mathbf{\Psi}_1,\,\mathbf{\Psi}_2,\,\mathbf{\Psi}_3,\,\mathbf{\Psi}_4)$ concordant with the expansion (5.1). This fact leads to the following theorem.

Theorem 8.1. *Any metric connection* $(\Gamma,\mathrm{A},\bar{\mathrm{A}})$ *of the bundle of Dirac spinors* DM *is a unique extension of some metric connection of the chiral bundle* SM*.*

According to the formula (8.9), the final step in deriving an explicit formula for the A-components of a metric connection $(\Gamma,\mathrm{A},\bar{\mathrm{A}})$ is to add the above four formulas (8.13), (8.14), (8.29), and (8.30). This yields

$$
\begin{gathered}
\mathrm{A}^{i}_{kj}=\sum_{b=1}^{4}\overset{\circ}{H}{}^{b}_{j}\,L_{\boldsymbol{\Upsilon}_k}(\overset{\bullet}{H}{}^{i}_{b})+\sum_{b=1}^{4}\overset{\bullet}{H}{}^{b}_{j}\,L_{\boldsymbol{\Upsilon}_k}(\overset{\circ}{H}{}^{i}_{b})\,+\\
+\sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{a=1}^{4}\sum_{b=1}^{4}\frac{L_{\boldsymbol{\Upsilon}_k}(\overset{\bullet\circ}{\gamma}{}^{a}_{bm})\,g^{mn}\,\overset{\circ\bullet}{\gamma}{}^{i}_{an}\,\overset{\circ}{H}{}^{b}_{j}+L_{\boldsymbol{\Upsilon}_k}(\overset{\circ\bullet}{\gamma}{}^{a}_{bm})\,g^{mn}\,\overset{\bullet\circ}{\gamma}{}^{i}_{an}\,\overset{\bullet}{H}{}^{b}_{j}}{4}\,-\\
-\sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{r=0}^{3}\sum_{q=1}^{4}\frac{\Gamma^{r}_{km}\,\overset{\bullet\circ}{\gamma}{}^{q}_{jr}\,g^{mn}\,\overset{\circ\bullet}{\gamma}{}^{i}_{qn}+\Gamma^{r}_{km}\,\overset{\circ\bullet}{\gamma}{}^{q}_{jr}\,g^{mn}\,\overset{\bullet\circ}{\gamma}{}^{i}_{qn}}{4}\,+\\
+\frac{1}{4}\sum_{a=1}^{4}\sum_{b=1}^{4}L_{\boldsymbol{\Upsilon}_k}(\overset{\bullet}{d}_{ab})\,\overset{\bullet}{d}{}^{ba}\,\overset{\circ}{H}{}^{i}_{j}+\frac{1}{4}\sum_{a=1}^{4}\sum_{b=1}^{4}L_{\boldsymbol{\Upsilon}_k}(\overset{\circ}{d}_{ab})\,\overset{\circ}{d}{}^{ba}\,\overset{\bullet}{H}{}^{i}_{j}.
\end{gathered}
\tag{8.31}
$$

This formula (8.31) can be simplified a little bit. For this purpose we need the following identities derived from the formula (6.23):

$$
\sum_{m=0}^{3}\sum_{n=0}^{3}\overset{\bullet\circ}{\gamma}{}^{a}_{bm}\,g^{mn}\,\overset{\circ\bullet}{\gamma}{}^{e}_{hn}=2\,\overset{\bullet}{H}{}^{a}_{h}\,\overset{\circ}{H}{}^{e}_{b},
\tag{8.32}
$$

$$
\sum_{m=0}^{3}\sum_{n=0}^{3}\overset{\circ\bullet}{\gamma}{}^{a}_{bm}\,g^{mn}\,\overset{\bullet\circ}{\gamma}{}^{e}_{hn}=2\,\overset{\circ}{H}{}^{a}_{h}\,\overset{\bullet}{H}{}^{e}_{b}.
\tag{8.33}
$$

The identities (8.32) and (8.33) are analogous to (8.17) and (8.18). Before applying them to (8.31) we need to set $e=i$ and $h=a$ in them and then sum over the index

a. As a result we transform them to the following two identities:

$$\sum_{a=1}^{4}\sum_{m=0}^{3}\sum_{n=0}^{3}\overset{\bullet\circ}{\gamma}{}^{a}_{bm}\,g^{mn}\,\overset{\circ\bullet}{\gamma}{}^{i}_{an}=4\,\overset{\circ}{H}{}^{i}_{b},\qquad\sum_{a=1}^{4}\sum_{m=0}^{3}\sum_{n=0}^{3}\overset{\circ\bullet}{\gamma}{}^{a}_{bm}\,g^{mn}\,\overset{\bullet\circ}{\gamma}{}^{i}_{an}=4\,\overset{\bullet}{H}{}^{i}_{b}.$$

Now, applying these identities to the second line in (8.31), we derive

$$\begin{aligned}\mathrm{A}^{i}_{kj}=&\sum_{a=1}^{4}\sum_{b=1}^{4}\frac{L_{\boldsymbol{\Upsilon}_k}(\overset{\bullet}{d}_{ab})\,\overset{\bullet}{d}{}^{ba}\,\overset{\circ}{H}{}^{i}_{j}+L_{\boldsymbol{\Upsilon}_k}(\overset{\circ}{d}_{ab})\,\overset{\circ}{d}{}^{ba}\,\overset{\bullet}{H}{}^{i}_{j}}{4}+\\&+\sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{a=1}^{4}\frac{L_{\boldsymbol{\Upsilon}_k}(\overset{\bullet\circ}{\gamma}{}^{a}_{jm})\,g^{mn}\,\overset{\circ\bullet}{\gamma}{}^{i}_{an}+L_{\boldsymbol{\Upsilon}_k}(\overset{\circ\bullet}{\gamma}{}^{a}_{jm})\,g^{mn}\,\overset{\bullet\circ}{\gamma}{}^{i}_{an}}{4}-\\&-\sum_{m=0}^{3}\sum_{n=0}^{3}\sum_{r=0}^{3}\sum_{a=1}^{4}\frac{\Gamma^{r}_{km}\,\overset{\bullet\circ}{\gamma}{}^{a}_{jr}\,g^{mn}\,\overset{\circ\bullet}{\gamma}{}^{i}_{an}+\Gamma^{r}_{km}\,\overset{\circ\bullet}{\gamma}{}^{a}_{jr}\,g^{mn}\,\overset{\bullet\circ}{\gamma}{}^{i}_{an}}{4}.\end{aligned}\tag{8.34}$$

A metric connection is a real connection. According to (7.11), the $\bar{\mathrm{A}}$-components of a metric connection are obtained from A^{i}_{kj} by complex conjugation:

$$\bar{\mathrm{A}}^{i}_{kj}=\overline{\mathrm{A}^{i}_{kj}}.\tag{8.35}$$

Theorem 8.2. *For any skew-symmetric real spin-tensorial field* $\mathbf{T}$ *of the type* $(0,0|0,0|1,2)$ *there is a unique metric connection* $(\Gamma,\mathrm{A},\bar{\mathrm{A}})$ *in* DM *with the torsion tensor* $\mathbf{T}$*. Its components in an arbitrary frame pair* (U, $\boldsymbol{\Upsilon}_0$, $\boldsymbol{\Upsilon}_1$, $\boldsymbol{\Upsilon}_2$, $\boldsymbol{\Upsilon}_3$) *and* (U, $\boldsymbol{\Psi}_1$, $\boldsymbol{\Psi}_2$, $\boldsymbol{\Psi}_3$, $\boldsymbol{\Psi}_4$) *are given by the formulas* (4.34)*,* (8.34)*, and* (8.35)*.*

Corollary 8.1. *The components of a real metric connection* $(\Gamma,\mathrm{A},\bar{\mathrm{A}})$ *with zero torsion* $\mathbf{T}=0$ *for the bundle of Dirac spinors* SM *are given by the explicit formulas* (4.35)*,* (8.34)*, and* (8.35)*.*

The theorem 8.2 and the corollary 8.1 are analogous to the theorem 4.3 and its corollary 4.2 in the case of chiral spinors. As for the relation of metric connections of Dirac and chiral spinors, it is described by the theorem 8.1

References

1. Dubrovin B. A., Novikov S. P., Fomenko A. T., *Modern geometry.* Vol. I. *Methods and applications*, Nauka publishers, Moscow, 1986.
2. Penrose R., Rindler W., *Spinors and space-time.* Vol. I. *Two-spinor calculus and relativistic fields*, Cambridge University Press, 1984.
3. Sharipov R. A., *Spinor functions of spinors and the concept of extended spinor fields*, e-print math.DG/0511350 in Electronic Archive http://arXiv.org.
4. Sharipov R. A., *Classical electrodynamics and theory of relativity*, Bashkir State University, Ufa, 1997; see also physics/0311011 in Electronic Archive http://arXiv.org and r-sharipov/r4-b5.htm in GeoCities.
5. Sharipov R. A., *Commutation relationships and curvature spin-tensors for extended spinor connections*, e-print math.DG/0512396 in Electronic Archive http://arXiv.org.
6. Sharipov R. A., *A note on Dirac spinors in a non-flat space-time of general relativity*, e-print math.DG/0601262 in Electronic Archive http://arXiv.org.
7. Comer J., Sharipov R. A., *On the geometry of a dislocated medium*, e-print math-ph/0502007 in Electronic Archive http://arXiv.org.
8. Kirillov A. A., *Elements of the theory of representations*, Nauka publishers, Moscow, 1978.

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