TENSOR FUNCTIONS OF TENSORS AND THE CONCEPT OF EXTENDED TENSOR FIELDS.

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Abstract. Tensor fields depending on other tensor fields are considered. The concept of extended tensor fields is introduced and the theory of differentiation for such fields is developed.

1. Tensors and tensor fields on manifolds.

Let M be some n-dimensional smooth real manifold. Then each point $p \in M$ has some neighborhood U bijectively mapped onto an open set V in the n-dimensional space $\mathbb{R}^n$. This means that any point $p \in U$ is associated with some unique vector in V with the coordinates $x^1(p), \ldots, x^n(p)$. The set $V \subset \mathbb{R}^n$ is called a *local map* or a *local chart* of the manifold M, while the numbers $x^1(p), \ldots, x^n(p)$ are called the *coordinates* of the point p in the local chart V. In a not too formal terminology the set $U \subset M$ is also called a *local map* or a *local chart*.

The whole manifold is covered by local charts. If two local charts U and $\tilde{U}$ do overlap, i. e. if $U \cap \tilde{U} \neq \varnothing$, then the so-called *transition functions* arise:

$$\begin{cases} \tilde{x}^1 = \tilde{x}^1(x^1, \ldots, x^n), \\ \ldots\ldots\ldots\ldots\ldots \\ \tilde{x}^n = \tilde{x}^n(x^1, \ldots, x^n), \end{cases} \qquad \begin{cases} x^1 = x^1(\tilde{x}^1, \ldots, \tilde{x}^n), \\ \ldots\ldots\ldots\ldots\ldots \\ x^n = x^n(\tilde{x}^1, \ldots, \tilde{x}^n). \end{cases} \tag{1.1}$$

They relate the local coordinates of a point $p \in U \cap \tilde{U}$ in two charts. In the case of smooth manifolds the transition functions (1.1) for all pairs of overlapping maps are smooth functions. The partial derivatives of (1.1) form the transition matrices:

$$S^i_j = \frac{\partial x^i}{\partial \tilde{x}^j}, \qquad T^i_j = \frac{\partial \tilde{x}^i}{\partial x^j}. \tag{1.2}$$

They are inverse to each other: $S = T^{-1}$. By tradition S is called the *direct transition matrix*, while T is called the *inverse transition matrix*.

All what was said just above is a standard definition of a smooth real manifold. We give it here in order to make this paper understandable not only to professional mathematicians, but to physicists and to students majoring in physics and engineering. With the same purpose in mind, below we shall combine the coordinate and coordinate-free approaches.

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Continuing our introductory section, let's consider the following differential operators associated with two local coordinate systems in M:

$$\mathbf{E}_i = \frac{\partial}{\partial x^i}, \qquad\qquad \tilde{\mathbf{E}}_i = \frac{\partial}{\partial \tilde{x}^i}. \tag{1.3}$$

From (1.1) and (1.2) it is easy to derive that the differential operators $\mathbf{E}_i$ and $\tilde{\mathbf{E}}_j$ are related to each other through the following equalities:

$$\tilde{\mathbf{E}}_j = \sum_{i=1}^{n} S^i_j\ \mathbf{E}_i, \qquad\qquad \mathbf{E}_i = \sum_{j=1}^{n} T^j_i\ \tilde{\mathbf{E}}_j. \tag{1.4}$$

The three-dimensional Euclidean space $\mathbb{E}$, which we observe in our everyday life, is an example of a smooth manifold. In this case $x^1, \ldots, x^n$ and $\tilde{x}^1, \ldots, \tilde{x}^n$ are interpreted as Cartesian and/or curvilinear coordinates in $\mathbb{E}$. The differential operators (1.3) can be associated with the frame vectors of moving frames of two curvilinear coordinate systems (see [1] and [2]) because (1.4) coincide with the corresponding relationships for the frame vectors.

A smooth surface in the space $\mathbb{E}$ is another example of a smooth manifold. In this case the differential operators (1.3) can be associated with the tangent vectors forming a basis in the tangent plane to that surface (see [2]). This is the reason why in the case of an arbitrary smooth manifold M the operators (1.3) are called *tangent vectors*. If some point $p \in U$ is fixed, then the *tangent space* $T_p(M)$ is defined as the span of the vectors (1.3) at that point:

$$T_p(M) = \big\langle \mathbf{E}_1, \ldots, \mathbf{E}_n \big\rangle = \big\langle \tilde{\mathbf{E}}_1, \ldots, \tilde{\mathbf{E}}_n \big\rangle. \tag{1.5}$$

The tangent spaces $T_p(M)$ and $T_q(M)$ of different points $p \neq q$ are understood as two different n-dimensional vector spaces[1].

Definition 1.1. A vector field $\mathbf{X}$ is a vector-valued function that maps each point $p \in M$ to some vector $\mathbf{X}(p) \in T_p(M)$.

This is an invariant (coordinate-free) definition of a vector field. Due to (1.5) one can expand the vector $\mathbf{X}(p)$ in two different bases:

$$\mathbf{X}(p) = \sum_{i=1}^{n} X^i\ \mathbf{E}_i, \qquad\qquad \mathbf{X}(p) = \sum_{j=1}^{n} \tilde{X}^j\ \tilde{\mathbf{E}}_i. \tag{1.6}$$

Here $X^i = X^i(p) = X^i(x^1, \ldots, x^n)$ and $\tilde{X}^i = \tilde{X}^i(p) = \tilde{X}^i(\tilde{x}^1, \ldots, \tilde{x}^n)$ are the components of the vector $\mathbf{X}(p)$ in two different local charts. Due to (1.4) and (1.6) they are related to each other according to the formulas:

$$X^i = \sum_{j=1}^{n} S^i_j\ \tilde{X}^j, \qquad\qquad \tilde{X}^j = \sum_{i=1}^{n} T^j_i\ X^i. \tag{1.7}$$

The formulas (1.7) form the base for coordinate definition of a vector field (see [1] and [2]). Here this definition is formulated as follows.

[1] Informally, one can imagine a manifold M as a cat with the hairs $T_p(M)$ growing from each point p on its skin.

Definition 1.2. A vector field $\mathbf{X}$ is a geometric object in each local chart represented by its components $X^i = X^i(x^1, \ldots, x^n)$ and such that under a change of a local chart its components are transformed according to the formulas (1.7).

Let $T^*_p(M)$ be the dual space for the tangent space $T_p(M)$ (see [3] for the definition of a dual space). Then consider the following tensor product[1]:

$$T^r_s(p, M) = \overbrace{T_p(M) \otimes \ldots \otimes T_p(M)}^{r \text{ times}} \otimes \underbrace{T^*_p(M) \otimes \ldots \otimes T^*_p(M)}_{s \text{ times}}. \tag{1.8}$$

The tensor product (1.8) is also a vector space associated with the point p. So, each point of a smooth manifold carries a great many mathematical constructs[1] including but not limited to those considered in this paper. Some of these constructs correspond to real physical fields, others are waiting their time to be associated with something in the nature.

Definition 1.3. A tensor field $\mathbf{X}$ of the type (r, s) is a tensor-valued function that maps each point $p \in M$ to some tensor $\mathbf{X}(p) \in T^r_s(p, M)$.

In order to represent a tensor field in a local map we need to have some basis in the space (1.8). The differentials $dx^1, \ldots, dx^n$ form a basis in the dual space $T^*_p(M)$. This basis is dual to the basis of tangent vectors $\mathbf{E}_1, \ldots, \mathbf{E}_n$ in $T_p(M)$:

$$\langle dx^i \,|\, \mathbf{E}_j \rangle = \delta^i_j = \begin{cases} 1 & \text{for } \ i = j, \\ 0 & \text{for } \ i \neq j. \end{cases} \tag{1.9}$$

By angular brackets in (1.9) we denote the scalar product of a vector and a covector (see definition in Chapter III of [3]). Now let's denote

$$\mathbf{E}^{j_1 \ldots\, j_s}_{i_1 \ldots\, i_r} = \mathbf{E}_{i_1} \otimes \ldots \otimes \mathbf{E}_{i_r} \otimes dx^{j_1} \otimes \ldots \otimes dx^{j_s}. \tag{1.10}$$

The local[2] tensor fields (1.10) form a basis in the space (1.8) at all points $p \in U$. Therefore, any tensor field $\mathbf{X}$ of the type (r, s) admits the expansion

$$\mathbf{X} = \sum_{i_1=1}^{n} \ldots \sum_{i_r=1}^{n} \sum_{j_1=1}^{n} \ldots \sum_{j_s=1}^{n} X^{i_1 \ldots\, i_r}_{j_1 \ldots\, j_s} \, \mathbf{E}^{j_1 \ldots\, j_s}_{i_1 \ldots\, i_r}. \tag{1.11}$$

The coefficients $X^{i_1 \ldots\, i_r}_{j_1 \ldots\, j_s} = X^{i_1 \ldots\, i_r}_{j_1 \ldots\, j_s}(x^1, \ldots, x^n)$ in the expansion (1.11) are called the *components* of the tensor field $\mathbf{X}$ in the local chart U. Under a change of a local chart they are transformed as follows:

$$X^{i_1 \ldots\, i_r}_{j_1 \ldots\, j_s} = \sum^{\mathrm{n}}_{\substack{h_1, \ldots, h_r \\ k_1, \ldots, k_s}} \!\!\ldots\!\! \sum^{\mathrm{n}} S^{i_1}_{h_1} \ldots S^{i_r}_{h_r} \; T^{k_1}_{j_1} \ldots T^{k_s}_{j_s} \; \tilde{X}^{h_1 \ldots\, h_r}_{k_1 \ldots\, k_s}, \tag{1.12}$$

[1] One can imagine $T^*_p(M)$ and $T^r_s(p, M)$ as other hairs on the skin of that our cat growing from the same point p as $T_p(M)$. However, I don't know if some real animal can have a bunch of hairs on the same root.

[2] The tensor fields (1.10) are defined only within the local chart U.

The transformation rule can be inverted. The inverse transformation is written as

$$\tilde{X}^{i_1\ldots\,i_r}_{j_1\ldots\,j_s} = \sum^{\mathrm{n}}_{\substack{h_1,\,\ldots\,,\,h_r\\ k_1,\,\ldots\,,\,k_s}}\!\!\ldots\!\sum^{\mathrm{n}} T^{i_1}_{h_1}\ldots\,T^{i_r}_{h_r}\;S^{k_1}_{j_1}\ldots\,S^{k_s}_{j_s}\;X^{h_1\ldots\,h_r}_{k_1\ldots\,k_s}. \tag{1.13}$$

An alternative definition of a tensor field is based on the transformation rules (1.12) and (1.13) for its components.

Definition 1.4. A tensor field $\mathbf{X}$ of the type (r,s) is a geometric object in each local chart represented by its components $X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s} = X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}(x^1,\,\ldots\,,x^n)$ and such that under a change of a local chart its components obey the transformation rules (1.12) and (1.13).

2. Tangent bundle, cotangent bundle, and other tensor bundles.

Let p be some point of a smooth real manifold M and let $\mathbf{v}\in T_p(M)$ be some tangent vector at the point p. The set of all pairs $q=(p,\mathbf{v})$ forms another smooth real manifold. It is called the *tangent bundle*[1] of M and denoted by TM. The map $\pi\colon TM\to M$ that takes a point $q=(p,\mathbf{v})$ of TM to the point $p\in M$ is called the *canonical projection* of the tangent bundle TM onto the base manifold M. If a local chart U on M is given, a point $q=(p,\mathbf{v})$ of TM is represented by $2n$ variables

$$x^1,\,\ldots\,,x^n,\,v^1,\,\ldots\,,v^n, \tag{2.1}$$

where $x^1,\,\ldots\,,x^n$ are the local coordinates of the point $p=\pi(q)$ and $v^1,\,\ldots\,,v^n$ are the components of the tangent vector $\mathbf{v}$:

$$\mathbf{v}=v^1\;\mathbf{E}_1+\ldots+v^n\;\mathbf{E}_n. \tag{2.2}$$

Hence, we have $\dim(TM)=2\,\dim(M)$. Tangent bundles naturally arise in considering Newtonian dynamical systems with holonomic constraints (see theses [4], [5], and the series of papers [6–23]). In mechanics a base manifold M is called a *configuration space*, TM is called a *phase space*, and (2.2) is interpreted as the *velocity vector* of a point moving within M.

Let $\mathbf{p}\in T^*_p(M)$ be some covector at the point $p\in M$. The set of all pairs $q=(p,\mathbf{p})$ forms a smooth real manifold which is called the *cotangent bundle* of M. It is denoted T^*M. In the case of cotangent bundle T^*M we also have the *canonical projection* $\pi\colon T^*M\to M$ that takes a point $q=(p,\mathbf{p})$ to the point p of the base manifold M. In a local chart a point $q=(p,\mathbf{p})$ is represented by $2n$ variables

$$x^1,\,\ldots\,,x^n,\,p_1,\,\ldots\,,p_n, \tag{2.3}$$

where $x^1,\,\ldots\,,x^n$ are the local coordinates of the point $p=\pi(q)$ and $p_1,\,\ldots\,,p_n$ are the components of the covector $\mathbf{p}\in T^*_p(M)$:

$$\mathbf{p}=p_1\;dx^1+\ldots+p_n\;dx^n. \tag{2.4}$$

[1] Being more strict, TM is called the *total space* of a tangent bundle, while a *tangent bundle* itself is a whole construct including a total space, a base, and a projection map $\pi\colon TM\to M$. However, in this paper we use more loose terminology.

For the dimension of a cotangent bundle we have $\dim(T^*M) = 2\dim(M)$. Cotangent bundles naturally arise when one passes from Lagrangian dynamical systems to the corresponding Hamiltonian dynamical systems. In this case T^*M is interpreted as the **p**-representation of a phase space TM, while TM is understood as the **v**-representation of T^*M (see papers [24–30]). In Hamiltonian dynamics the covector (2.4) is called the *momentum covector*.

Tensor bundles are defined by analogy to TM and T^*M. Let p be some point of the base manifold M and let $\mathbf{T}$ be some tensor of the type (r,s) at this point. The set of all pairs $q = (p, \mathbf{T})$ forms a smooth real manifold T^r_sM of the dimension $\dim(T^r_sM) = n + n^{r+s}$. It is called the *tensor bundle* of the type (r,s) over the base M. The map $\pi\colon T^r_sM \to M$ that takes a point $q = (p, \mathbf{T})$ of T^r_sM to the point $p \in M$ is called the *canonical projection* of the tensor bundle T^r_sM onto the base manifold M. Like in (2.1) and (2.3), we can specify the set of variables associated with a point $q = (p, \mathbf{T})$ of the tensor bundle T^r_sM and with a local chart $U \subset M$:

$$x^1,\,\ldots\,,x^n,\,T^{1\,\ldots\,1}_{1\,\ldots\,1},\,\ldots\,,T^{n\,\ldots\,n}_{n\,\ldots\,n}. \tag{2.5}$$

The number of variables in (2.5) determines the dimension $\dim(T^r_sM) = n + n^{r+s}$ of the tensor bundle T^r_sM. Note that if we consider a tensor field $\mathbf{T}$, its components $T^{i_1\ldots\,i_r}_{j_1\ldots\,j_s} = T^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}(x^1,\,\ldots\,,x^n)$ are functions of $x^1,\,\ldots\,,x^n$, while $T^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}$ in (2.5) are independent variables. For another chart $\tilde U \subset M$ we have the other set of variables

$$\tilde x^1,\,\ldots\,,\tilde x^n,\,\tilde T^{1\,\ldots\,1}_{1\,\ldots\,1},\,\ldots\,,\tilde T^{n\,\ldots\,n}_{n\,\ldots\,n}. \tag{2.6}$$

If the charts U and $\tilde U$ are overlapping, then we can write the transition functions

$$\begin{cases}\tilde x^1 = \tilde x^1(x^1,\,\ldots\,,x^n),\\ \ldots\ldots\ldots\ldots\ldots\\ \tilde x^n = \tilde x^n(x^1,\,\ldots\,,x^n),\\ \tilde T^{i_1\ldots\,i_r}_{j_1\ldots\,j_s} = \sum\limits^{\mathrm{n}}_{\substack{h_1,\,\ldots\,,h_r\\ k_1,\,\ldots\,,k_s}}\!\!\ldots\!\sum\limits^{\mathrm{n}} T^{i_1}_{h_1}\ldots\,T^{i_r}_{h_r}\;S^{k_1}_{j_1}\ldots\,S^{k_s}_{j_s}\;T^{h_1\ldots\,h_r}_{k_1\ldots\,k_s},\end{cases} \tag{2.7}$$

where the transition matrices T and S are the same as in (1.13) — they are determined by (1.2). Here are the inverse transition functions relating (2.5) and (2.6):

$$\begin{cases}x^1 = x^1(\tilde x^1,\,\ldots\,,\tilde x^n),\\ \ldots\ldots\ldots\ldots\ldots\\ x^n = x^n(\tilde x^1,\,\ldots\,,\tilde x^n),\\ T^{i_1\ldots\,i_r}_{j_1\ldots\,j_s} = \sum\limits^{\mathrm{n}}_{\substack{h_1,\,\ldots\,,h_r\\ k_1,\,\ldots\,,k_s}}\!\!\ldots\!\sum\limits^{\mathrm{n}} S^{i_1}_{h_1}\ldots\,S^{i_r}_{h_r}\;T^{k_1}_{j_1}\ldots\,T^{k_s}_{j_s}\;\tilde T^{h_1\ldots\,h_r}_{k_1\ldots\,k_s}.\end{cases} \tag{2.8}$$

The transformations (2.7) and (2.8) play the same role for the tensor bundle T^r_sM as the transformations (1.1) for the base manifold M. Note that they are linear with respect to $T^{h_1\ldots\,h_r}_{k_1\ldots\,k_s}$ and $\tilde T^{h_1\ldots\,h_r}_{k_1\ldots\,k_s}$, but they are nonlinear with respect to the base variables $x^1,\,\ldots\,,x^n$ and $\tilde x^1,\,\ldots\,,\tilde x^n$.

3. Composite tensor bundles.

Suppose that we have several tensor bundles over the same base manifold M. Let's denote them $T^{r_1}_{s_1}M, \ldots, T^{r_Q}_{s_Q}M$ and consider their direct sum over M:

$$T^{r_1\ldots\,r_Q}_{s_1\ldots\,s_Q}M = T^{r_1}_{s_1}M \oplus \ldots \oplus T^{r_Q}_{s_Q}M \tag{3.1}$$

(see the definition of a direct sum in [31]). We shall call (3.1) the *composite tensor bundle*. A point q of the composite tensor bundle (3.1) is a list

$$q = (p,\, \mathbf{T}[1],\, \ldots\,,\, \mathbf{T}[Q]), \tag{3.2}$$

where p is a point of the base M and $\mathbf{T}[1], \ldots, \mathbf{T}[Q]$ are some tensors of the types $(r_1, s_1), \ldots, (r_Q, s_Q)$ at the point p. The *canonical projection*

$$\pi\colon\, T^{r_1\ldots\,r_Q}_{s_1\ldots\,s_Q}M \to M$$

is defined as a map that takes a point q of the form (3.2) to the point $p \in M$. If a local chart U in the base manifold M is given, a point $q \in T^{r_1\ldots\,r_Q}_{s_1\ldots\,s_Q}M$ such that $\pi(q) \in U$ is represented by the following set of variables:

$$x^1,\, \ldots\,,\, x^n,\, T^{1\,\ldots\,1}_{1\,\ldots\,1}[1],\, \ldots\,,\, T^{n\,\ldots\,n}_{n\,\ldots\,n}[1],\, \ldots\,,\, T^{1\,\ldots\,1}_{1\,\ldots\,1}[Q],\, \ldots\,,\, T^{n\,\ldots\,n}_{n\,\ldots\,n}[Q]. \tag{3.3}$$

Taking another chart $\tilde{U}$, we get the other set of variables

$$\tilde{x}^1,\, \ldots\,,\, \tilde{x}^n,\, \tilde{T}^{1\,\ldots\,1}_{1\,\ldots\,1}[1],\, \ldots\,,\, \tilde{T}^{n\,\ldots\,n}_{n\,\ldots\,n}[1],\, \ldots\,,\, \tilde{T}^{1\,\ldots\,1}_{1\,\ldots\,1}[Q],\, \ldots\,,\, \tilde{T}^{n\,\ldots\,n}_{n\,\ldots\,n}[Q]. \tag{3.4}$$

If these charts are overlapping, then we have a system of transition functions

$$\begin{cases} \tilde{x}^1 = \tilde{x}^1(x^1,\, \ldots\,, x^n),\\ \dots\dots\dots\dots\dots\\ \tilde{x}^n = \tilde{x}^n(x^1,\, \ldots\,, x^n),\\ \tilde{T}^{i_1\ldots\, i_r}_{j_1\ldots\, j_s}[P] = \sum\limits^{\mathrm{n}}_{\substack{h_1,\,\ldots\,,\,h_r\\ k_1,\,\ldots\,,\,k_s}}\!\!\ldots\!\sum\limits^{\mathrm{n}} T^{i_1}_{h_1}\,\ldots\,T^{i_r}_{h_r}\;S^{k_1}_{j_1}\,\ldots\,S^{k_s}_{j_s}\;T^{h_1\ldots\, h_r}_{k_1\ldots\, k_s}[P], \end{cases} \tag{3.5}$$

where $r = r_P$, $s = s_P$, and the integer number P runs from 1 to Q. Similarly, we can write the system of the inverse transition functions for (3.5):

$$\begin{cases} x^1 = x^1(\tilde{x}^1,\, \ldots\,, \tilde{x}^n),\\ \dots\dots\dots\dots\dots\\ x^n = x^n(\tilde{x}^1,\, \ldots\,, \tilde{x}^n),\\ T^{i_1\ldots\, i_r}_{j_1\ldots\, j_s}[P] = \sum\limits^{\mathrm{n}}_{\substack{h_1,\,\ldots\,,\,h_r\\ k_1,\,\ldots\,,\,k_s}}\!\!\ldots\!\sum\limits^{\mathrm{n}} S^{i_1}_{h_1}\,\ldots\,S^{i_r}_{h_r}\;T^{k_1}_{j_1}\,\ldots\,T^{k_s}_{j_s}\;\tilde{T}^{h_1\ldots\, h_r}_{k_1\ldots\, k_s}[P]. \end{cases} \tag{3.6}$$

Here again $r = r_P$, $s = s_P$, and P runs from 1 to Q. The formulas (3.5) and (3.6) generalize (2.7) and (2.8) for the case of composite tensor bundles. They relate two sets of variables (3.3) and (3.4).

The composite tensor bundle $N = T^{r_1\ldots\,r_Q}_{s_1\ldots\,s_Q}M$ is a manifold (like M itself) and (3.3) are the local coordinates of its point q in some local chart. Therefore, one can consider the tangent space $T_q(N)$. By analogy to (1.5) it is defined as the span of all partial derivatives with respect to the variables (3.3). Let's denote them

$$\mathbf{U}_i = \frac{\partial}{\partial x^i}, \qquad\qquad \mathbf{V}^{j_1\ldots\,j_s}_{i_1\ldots\,i_r}[P] = \frac{\partial}{\partial T^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}[P]}, \tag{3.7}$$

where $r = r_P$, $s = s_P$ and P runs from 1 to Q. Note that $\mathbf{U}_i$ in (3.7) are different from $\mathbf{E}_i$ in (1.3). Passing from (3.3) to another set of local coordinates (3.4), one should define the other set of partial derivatives spanning the tangent space $T_q(N)$:

$$\tilde{\mathbf{U}}_i = \frac{\partial}{\partial \tilde{x}^i}, \qquad\qquad \tilde{\mathbf{V}}^{j_1\ldots\,j_s}_{i_1\ldots\,i_r}[P] = \frac{\partial}{\partial \tilde{T}^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}[P]}. \tag{3.8}$$

From (3.4) and (3.5) one easily derives the transformations formulas

$$\begin{cases} \tilde{\mathbf{V}}^{j_1\ldots\,j_s}_{i_1\ldots\,i_r}[P] = \sum\limits^{\mathrm{n}}_{\substack{h_1,\,\ldots\,,\,h_r\\ k_1,\,\ldots\,,\,k_s}}\!\!\ldots\!\!\sum\limits^{\mathrm{n}} S^{h_1}_{i_1}\,\ldots\,S^{h_r}_{i_r}\;T^{j_1}_{k_1}\,\ldots\,T^{j_s}_{k_s}\;\mathbf{V}^{k_1\ldots\,k_s}_{h_1\ldots\,h_r}[P], \\[3ex] \tilde{\mathbf{U}}_j = \sum\limits^{n}_{i=1} S^i_j\,\mathbf{U}_i + \sum\limits^{Q}_{P=1}\sum\limits^{\mathrm{n}}_{\substack{i_1,\,\ldots\,,\,i_r\\ j_1,\,\ldots\,,\,j_s}}\!\!\ldots\!\!\sum\limits^{\mathrm{n}}\sum\limits^{\mathrm{n}}_{\substack{h_1,\,\ldots\,,\,h_r\\ k_1,\,\ldots\,,\,k_s}}\!\!\ldots\!\!\sum\limits^{\mathrm{n}} S^{i_1}_{h_1}\,\ldots\,S^{i_r}_{h_r}\;T^{k_1}_{j_1}\,\ldots\,T^{k_s}_{j_s}\,\times \\[3ex] \qquad\times\left(\sum\limits^{r}_{m=1}\sum\limits^{n}_{v_m=1}\left(\sum\limits^{n}_{h=1} T^{h_m}_h\,\frac{\partial S^h_{v_m}}{\partial \tilde{x}^j}\right)\tilde{T}^{h_1\ldots\,v_m\ldots\,h_r}_{k_1\ldots\,\ldots\,\ldots\,k_s}[P]\,+\right. \\[3ex] \left.\qquad+\sum\limits^{s}_{m=1}\sum\limits^{n}_{w_m=1}\left(\sum\limits^{n}_{h=1}\frac{\partial T^{w_m}_h}{\partial \tilde{x}^j}\,S^h_{k_m}\right)\tilde{T}^{h_1\ldots\,\ldots\,\ldots\,h_r}_{k_1\ldots\,w_m\ldots\,k_s}[P]\right)\mathbf{V}^{j_1\ldots\,j_s}_{i_1\ldots\,i_r}[P]. \end{cases} \tag{3.9}$$

The transformation formulas (3.9) express the tangent vectors (3.8) through the tangent vectors (3.7) in $T_q(N)$. In order to simplify these formulas we introduce the following θ-parameters defined through the transition matrices (1.2):

$$\tilde{\theta}^k_{ij} = \sum^{n}_{h=1} T^k_h\,\frac{\partial S^h_j}{\partial \tilde{x}^i} = \sum^{n}_{h=1} T^k_h\,\frac{\partial x^h}{\partial \tilde{x}^i\,\partial \tilde{x}^j}. \tag{3.10}$$

Looking at the right hand side of (3.10), we see that $\tilde{\theta}^k_{ij}$ are symmetric:

$$\tilde{\theta}^k_{ij} = \tilde{\theta}^k_{j\,i}.$$

Moreover, note that S and T are inverse to each other. From this fact we derive

$$0 = \frac{\partial \delta^k_j}{\partial \tilde{x}^i} = \frac{\partial}{\partial \tilde{x}^i}\left(\sum^{n}_{h=1} T^k_h\,S^h_j\right) = \sum^{n}_{h=1} T^k_h\frac{\partial S^h_j}{\partial \tilde{x}^j} + \sum^{n}_{h=1}\frac{\partial T^k_h}{\partial \tilde{x}^i}\,S^h_j. \tag{3.11}$$

Now, comparing (3.10) and (3.11), we find that $\tilde{\theta}^k_{ij}$ can also be defined as

$$\tilde{\theta}^k_{ij} = -\sum_{h=1}^{n} \frac{\partial T^k_h}{\partial \tilde{x}^i}\, S^h_j. \tag{3.12}$$

Applying (3.10) and (3.12) to the formula (3.9), we can simplify it as follows:

$$\begin{cases} \tilde{\mathbf{V}}^{j_1\dots\,j_s}_{i_1\dots\,i_r}[P] = \sum\limits^{\mathrm{n}}_{\substack{h_1,\,\dots\,,\,h_r\\ k_1,\,\dots\,,\,k_s}}\!\!\dots\!\!\sum\limits^{\mathrm{n}} S^{h_1}_{i_1}\,\dots\,S^{h_r}_{i_r}\;T^{j_1}_{k_1}\,\dots\,T^{j_s}_{k_s}\;\mathbf{V}^{k_1\dots\,k_s}_{h_1\dots\,h_r}[P],\\[2ex] \tilde{\mathbf{U}}_j = \sum\limits_{i=1}^{n} S^i_j\,\mathbf{U}_i + \sum\limits_{P=1}^{Q}\sum\limits^{\mathrm{n}}_{\substack{i_1,\,\dots\,,\,i_r\\ j_1,\,\dots\,,\,j_s}}\!\!\dots\!\!\sum\limits^{\mathrm{n}}\sum\limits^{\mathrm{n}}_{\substack{h_1,\,\dots\,,\,h_r\\ k_1,\,\dots\,,\,k_s}}\!\!\dots\!\!\sum\limits^{\mathrm{n}}\Bigg(\sum\limits_{m=1}^{r}\sum\limits_{v_m=1}^{n}\tilde{\theta}^{h_m}_{j\,v_m}\,\times\\[2ex] \quad\times\,\tilde{T}^{h_1\dots\,v_m\dots\,h_r}_{k_1\dots\,\dots\,\dots\,k_s}[P] - \sum\limits_{m=1}^{s}\sum\limits_{w_m=1}^{n}\tilde{\theta}^{w_m}_{j\,k_m}\,\tilde{T}^{h_1\dots\,\dots\,\dots\,h_r}_{k_1\dots\,w_m\dots\,k_s}[P]\Bigg)\times\\[2ex] \qquad\times\,S^{i_1}_{h_1}\,\dots\,S^{i_r}_{h_r}\;T^{k_1}_{j_1}\,\dots\,T^{k_s}_{j_s}\;\mathbf{V}^{j_1\dots\,j_s}_{i_1\dots\,i_r}[P]. \end{cases} \tag{3.13}$$

The inverse transformation formulas expressing the tangent vectors (3.7) trough (3.8) are written by analogy to (3.13). They look like

$$\begin{cases} \mathbf{V}^{j_1\dots\,j_s}_{i_1\dots\,i_r}[P] = \sum\limits^{\mathrm{n}}_{\substack{h_1,\,\dots\,,\,h_r\\ k_1,\,\dots\,,\,k_s}}\!\!\dots\!\!\sum\limits^{\mathrm{n}} T^{h_1}_{i_1}\,\dots\,T^{h_r}_{i_r}\;S^{j_1}_{k_1}\,\dots\,S^{j_s}_{k_s}\;\tilde{\mathbf{V}}^{k_1\dots\,k_s}_{h_1\dots\,h_r}[P],\\[2ex] \mathbf{U}_j = \sum\limits_{i=1}^{n} T^i_j\,\tilde{\mathbf{U}}_i + \sum\limits_{P=1}^{Q}\sum\limits^{\mathrm{n}}_{\substack{i_1,\,\dots\,,\,i_r\\ j_1,\,\dots\,,\,j_s}}\!\!\dots\!\!\sum\limits^{\mathrm{n}}\sum\limits^{\mathrm{n}}_{\substack{h_1,\,\dots\,,\,h_r\\ k_1,\,\dots\,,\,k_s}}\!\!\dots\!\!\sum\limits^{\mathrm{n}}\Bigg(\sum\limits_{m=1}^{r}\sum\limits_{v_m=1}^{n}\theta^{h_m}_{j\,v_m}\,\times\\[2ex] \quad\times\,T^{h_1\dots\,v_m\dots\,h_r}_{k_1\dots\,\dots\,\dots\,k_s}[P] - \sum\limits_{m=1}^{s}\sum\limits_{w_m=1}^{n}\theta^{w_m}_{j\,k_m}\,T^{h_1\dots\,\dots\,\dots\,h_r}_{k_1\dots\,w_m\dots\,k_s}[P]\Bigg)\times\\[2ex] \qquad\times\,T^{i_1}_{h_1}\,\dots\,T^{i_r}_{h_r}\;S^{k_1}_{j_1}\,\dots\,S^{k_s}_{j_s}\;\tilde{\mathbf{V}}^{j_1\dots\,j_s}_{i_1\dots\,i_r}[P]. \end{cases} \tag{3.14}$$

The θ-parameters in (3.14) are defined by analogy to (3.10) and (3.12):

$$\theta^k_{ij} = \sum_{h=1}^{n} S^k_h\,\frac{\partial T^h_j}{\partial x^i} = \sum_{h=1}^{n} S^k_h\,\frac{\partial \tilde{x}^h}{\partial x^i\,\partial x^j} = -\sum_{h=1}^{n}\frac{\partial S^k_h}{\partial x^i}\,T^h_j. \tag{3.15}$$

Moreover, we easily derive the following two formulas:

$$\theta^k_{ij} = -\sum_{h=1}^{n}\sum_{p=1}^{n}\sum_{q=1}^{n}\tilde{\theta}^h_{p\,q} S^k_h\,T^p_i\,T^q_j, \tag{3.16}$$

$$\tilde{\theta}^k_{ij} = -\sum_{h=1}^{n}\sum_{p=1}^{n}\sum_{q=1}^{n}\theta^h_{p\,q} T^k_h\,S^p_i\,S^q_j. \tag{3.17}$$

The formulas (3.16) and (3.17) relate the θ-parameters given by the formula (3.14) with those given by the formulas (3.10) and (3.12).

4. Extended tensor fields.

Definition 4.1. Let $N = T^{r_1 \dots r_Q}_{s_1 \dots s_Q} M$ be a composite tensor bundle over a smooth real manifold M. An extended tensor field $\mathbf{X}$ of the type (α, β) is a tensor-valued function in $N = T^{r_1 \dots r_Q}_{s_1 \dots s_Q} M$ such that it takes each point $q \in N$ to some tensor $\mathbf{X}(q) \in T^{\alpha}_{\beta}(p, M)$, where $p = \pi(q)$ is the projection of q.

Informally speaking, an extended tensor field $\mathbf{X}$ is a tensorial function with one point argument $p \in M$ and Q tensorial arguments $\mathbf{T}[1], \dots, \mathbf{T}[Q]$. In a local chart $U \subset M$ it is represented by its components $X^{i_1 \dots i_\alpha}_{j_1 \dots j_\beta}$ each of which is a function of the variables (3.3). When passing from U to an overlapping chart $\tilde{U}$ the components of an extended tensor field are transformed according to the standard formula (1.13), while their arguments are transformed according to the formula (3.5). Tensor bundles, which we considered above, are used to realize these arguments geometrically in a coordinate-free form.

The term «extended tensor field» was first introduced in [11] in order to describe the force field of a Newtonian dynamical system. Indeed, writing the Newton's second law for a point mass $m\,\mathbf{a} = \mathbf{F}(\mathbf{r}, \mathbf{v})$, where $\mathbf{v} = \dot{\mathbf{r}}$ is its velocity and $\mathbf{a} = \ddot{\mathbf{r}}$ is its acceleration, we encounter a vector-function $\mathbf{F}$ with two arguments $\mathbf{r}$ and $\mathbf{v}$. The first argument represents a point of the 3-dimensional space $\mathbb{E}$, while the second argument is a vector attached to that point. So, both they form a point of the tangent bundle $T\mathbb{E}$. The standard 3-dimensional Euclidean space $\mathbb{E}$ is a very simple thing, its tangent bundle $T\mathbb{E}$ can be treated as a 6-dimensional space parametrized by pairs of 3-dimensional vectors. However, even in this trivial case, choosing curvilinear coordinates in $\mathbb{E}$, we find that the vectors $\mathbf{r}$ and $\mathbf{v}$ are slightly different in their nature.

Another place, where we find extended tensor fields with the arguments in a tangent bundle TM, is the geometry of Finsler (see [32]). Here the metric tensor $\mathbf{g}$ depends on the velocity vector $\mathbf{v}$. Some generalizations of the Finslerian geometry motivated by the Lagrangian dynamics were suggested in [27] and [28]. A different approach to understanding extended tensor fields with the arguments in a tangent bundle TM was used in [33].

Extended tensor fields with the arguments in a cotangent bundle T^*M are natural in Hamiltonian mechanics. Indeed, if a Hamiltonian dynamical system is produced from a Lagrangian dynamical system, its Hamilton function H depends on a point of its configuration space M and on a momentum covector $\mathbf{p}$ at this point.

More complicated extended tensor fields can be found in physics of continuous media and in field theories. For example, in [34] we find that the specific free energy function $f(T, \hat{\mathbf{G}})$ depends on the temperature T and the elastic part of the deformation tensor. Such a function should certainly be treated as an extended scalar field with the arguments in $N = T^{00}_{02}\mathbb{E}$ since the temperature T is a scalar field and $\hat{\mathbf{G}}$ is a tensor field of the type $(0, 2)$.

5. The algebra of extended tensor fields.

Suppose that some composite tensor bundle $T^{r_1 \dots r_Q}_{s_1 \dots s_Q} M$ is fixed. Let's denote by $T^{\alpha}_{\beta}(M)$ the set of all extended tensor fields of the type (α, β). According to the above definition 4.1 an extended tensor field is a function, its values are tensors. Hence, we can perform the algebraic operations with these values and thus define the algebraic operations with extended tensor fields:

(1) $T^\alpha_\beta(M) + T^\alpha_\beta(M) \longrightarrow T^\alpha_\beta(M)$;
(2) $T^0_0(M) \otimes T^\alpha_\beta(M) \longrightarrow T^\alpha_\beta(M)$;
(3) $T^\alpha_\beta(M) \otimes T^\sigma_\mu(M) \longrightarrow T^{\alpha+\sigma}_{\beta+\mu}(M)$;

According to the definition 4.1, each extended scalar field is a real-valued function in the bundle N. Such functions form a ring, we denote it $\mathfrak{F}(N)$. Then

$$T^0_0(M) = \mathfrak{F}(N). \tag{5.1}$$

Due to the properties (1) and (2) the set of extended tensor fields $T^\alpha_\beta(M)$ of the fixed type (α,β) is a module over the ring (5.1).

In general, one cannot add tensor fields of two different types $(\alpha,\beta) \neq (\sigma,\mu)$. However, one can consider formal sums

$$\mathbf{X} = \mathbf{X}[1] + \mathbf{X}[2] + \ldots + \mathbf{X}[K], \tag{5.2}$$

where $\mathbf{X}[1],\ \ldots\ ,\ \mathbf{X}[K]$ are taken from various modules $T^\alpha_\beta(M)$ over the ring $\mathfrak{F}(N)$. By definition, the set of all sums (5.2) is called the *direct sum*

$$\mathbf{T}(M) = \bigoplus_{\alpha=0}^{\infty}\bigoplus_{\beta=0}^{\infty} T^\alpha_\beta(M). \tag{5.3}$$

With respect to the algebraic operations (1), (2), and (3) the direct sum (5.3) is a *graded algebra* over the ring $T^0_0(M)$. This algebra is called the *algebra of extended tensor fields* (or the *extended algebra* for short).

The fourth class of algebraic operations in the extended algebra (5.3) is formed by the operations of contraction. They are performed with respect to some pair of indices one of which is an upper index and the other is a lower index (see details in [1] and [2]). For the sake of simplicity we denote these operations as follows:

(4) $C\colon T^\alpha_\beta(M) \longrightarrow T^{\alpha-1}_{\beta-1}(M)$ for $\alpha \geqslant 1$ and $\beta \geqslant 1$.

As it follows from the property (4), the contraction operations are concordant with the structure of graded algebra in $\mathbf{T}(M)$. The same is true for the operations of addition and tensor multiplication.

6. Differentiation of tensor fields.

Definition 6.1. An extended tensor field $\mathbf{X}$ of the type (α,β) associated with a composite tensor bundle $N = T^{r_1\ldots\,r_Q}_{s_1\ldots\,s_Q}M$ is called *smooth* if its components $X^{i_1\ldots\,i_\alpha}_{j_1\ldots\,j_\beta}$ are smooth function of their arguments (3.3) for any local chart U of M.

From now on we shall restrict our previous definition (5.3) of the extended algebra and denote by $\mathbf{T}(M)$ the algebra of smooth extended tensor fields. Similarly, by $\mathfrak{F}(N)$ below we denote the ring of smooth scalar functions in N. Then the equality (5.1) remains valid. The basic idea of considering smooth smooth extended tensor fields is to introduce the operation of differentiation in addition to the above four algebraic operations in $\mathbf{T}(M)$.

Definition 6.2. A mapping $D\colon \mathbf{T}(M) \to \mathbf{T}(M)$ is called a *differentiation* of the extended algebra of tensor fields if the following conditions are fulfilled:

(1) D is concordant with the grading: $D(T^{\alpha}_{\beta}(M)) \subset T^{\alpha}_{\beta}(M)$;
(2) D is $\mathbb{R}$-linear: $D(\mathbf{X}+\mathbf{Y}) = D(\mathbf{X}) + D(\mathbf{Y})$
and $D(\lambda\mathbf{X}) = \lambda D(\mathbf{X})$ for $\lambda \in \mathbb{R}$;
(3) D commutates with the contractions: $D(C(\mathbf{X})) = C(D(\mathbf{X}))$;
(4) D obeys the Leibniz rule: $D(\mathbf{X} \otimes \mathbf{Y}) = D(\mathbf{X}) \otimes \mathbf{Y} + \mathbf{X} \otimes D(\mathbf{Y})$.

Let's consider the set of all differentiations of the extended algebra $\mathbf{T}(M)$. We denote it $\mathfrak{D}(M)$. It is easy to check up that

(1) the sum of two differentiations is a differentiation of the algebra $\mathbf{T}(M)$;
(2) the product of a differentiation by a smooth function in N is a differentiation of the algebra $\mathbf{T}(M)$.

Now we see that $\mathfrak{D}(M)$ is equipped with the structure of a module over the ring of smooth functions $\mathfrak{F}(N) = T^0_0(M)$. The composition of two differentiations D_1 and D_2 is not a differentiation, but their commutator

$$[D_1,\, D_2] = D_1 \circ D_2 - D_2 \circ D_1 \tag{6.1}$$

is a differentiation. Therefore, $\mathfrak{D}(M)$ is a Lie algebra. Note, however, that $\mathfrak{D}(M)$ is not a Lie algebra over the ring of smooth functions $\mathfrak{F}(N)$. It is only a Lie algebra over the field of real numbers $\mathbb{R}$. For this reason the structure of Lie algebra in $\mathfrak{D}(M)$ is not of a primordial importance.

7. Localization.

Smooth extended tensor fields are global objects related to the tensor bundle N in whole, but they are functions — their values are local objects so that two different tensor fields $\mathbf{A} \neq \mathbf{B}$ can take the same values at some particular points. Whenever this happens, we write $\mathbf{A}_q = \mathbf{B}_q$, where $q \in N$.

Differentiations of the algebra $\mathbf{T}(M)$, as they are introduced in the definition 6.2, are global objects without any explicit subdivision into parts related to separate points of the bundle N. Below in this section we shall show that they also can be represented as functions taking their values in some linear spaces associated with separate points of the manifold N. Let $D \in \mathfrak{D}(M)$ be a differentiation of the algebra of extended tensor fields $\mathbf{T}(M)$. Let's denote by δ the restriction of the mapping $D\colon \mathbf{T}(M) \to \mathbf{T}(M)$ to the module $T^0_0(M)$ of extended scalar fields in (5.3):

$$\delta\colon\, T^0_0(M) \to T^0_0(M). \tag{7.1}$$

Since $T^0_0(M) = \mathfrak{F}(M)$, the mapping δ in (7.1) is a differentiation of the ring of smooth functions in N. It is known (see § 1 in Chapter I of [35]) that any differentiation of the ring of smooth functions of an arbitrary smooth manifold is determined by some vector field $\mathbf{Z}$ in this manifold. Applying this fact to N, we get

$$\delta = \mathbf{Z} = \sum_{i=1}^{n} Z^i\; \mathbf{U}_i + \sum_{P=1}^{Q}\, \sum^{\mathrm{n}}_{\substack{i_1,\,\ldots\,,\,i_r\\ j_1,\,\ldots\,,\,j_s}} \!\!\ldots\! \sum^{\mathrm{n}} Z^{i_1\ldots\, i_r}_{j_1\ldots\, j_s}[P]\;\, \mathbf{V}^{j_1\ldots\, j_s}_{i_1\ldots\, i_r}[P], \tag{7.2}$$

where $\mathbf{U}_i$ and $\mathbf{V}^{j_1\ldots\,j_s}_{i_1\ldots\,i_r}[P]$ are the differential operators (3.7). From the representation (7.2) for the mapping (7.1) we immediately derive the following lemma.

Lemma 7.1. *Let $\psi \in T^0_0(M) = \mathfrak{F}(N)$ an extended scalar field (a smooth function) identically constant within some open subset $O \subset N$ and let $\varphi = D(\psi)$ for some differentiation $D \in \mathfrak{D}(M)$. Then $\varphi = 0$ within the open set O.*

Proof. Since $D(\psi) = \delta(\psi)$, choosing some local chart U and applying the differential operators (3.7) to a constant, from (7.2) we derive that $\delta(\psi) = \mathbf{Z}\psi = 0$ at any point q of the open set O. □

Lemma 7.2. *Let $\mathbf{X}$ be an extended tensor field of the type (α, β). If $\mathbf{X} \equiv 0$, then $D(\mathbf{X})$ is also identically equal to zero for any differentiation $D \in \mathfrak{D}(M)$.*

The proof is trivial. Since $\mathbf{X} \equiv 0$, we can write $\mathbf{X} = \lambda\,\mathbf{X}$ with $\lambda \neq 1$. Then, applying the item (2) of the definition 6.2, we get $D(\mathbf{X}) = \lambda\,D(\mathbf{X})$. Since $\lambda \neq 1$, this yields the required equality $D(\mathbf{X}) \equiv 0$.

Lemma 7.3. *Let $\mathbf{X}$ be an extended tensor field of the type (α, β) identically zero within some open set $O \subset N$. If $\mathbf{Y} = D(\mathbf{X})$ for some differentiation $D \in \mathfrak{D}(M)$, then $\mathbf{Y}_q = 0$ at any point $q \in O$.*

Proof. Let's choose some arbitrary point $q \in O$ and take some smooth scalar function $\eta \in \mathfrak{F}(N)$ such that it is identically equal to the unity in some open neighborhood $O' \subset O$ of the point q and identically equal to zero outside the open set O. The existence of such a function η is easily proved by choosing some local chart U that covers the point q. The product $\eta\,\mathbf{X}$ is identically equal to zero:

$$\eta \otimes \mathbf{X} = \eta\,\mathbf{X} \equiv 0. \tag{7.3}$$

Applying the differentiation D to (7.3), then taking into account the lemma 7.2 and the item (4) of the definition 6.2, we obtain

$$0 = D(0) = D(\eta \otimes \mathbf{X}) = D(\eta) \otimes \mathbf{X} + \eta \otimes D(\mathbf{X}) = D(\eta)\,\mathbf{X} + \eta\,D(\mathbf{X}).$$

Note that $D(\eta) = 0$ at the point q due to the lemma 7.1. Moreover, $\mathbf{X}_q = 0$ and $\eta = 1$ at the point q. Therefore, by specifying the above equality to the point q we get $D(\mathbf{X}) = 0$ at the point q. The lemma is proved. □

Lemma 7.4. *If two extended tensor fields $\mathbf{X}$ and $\mathbf{Y}$ are equal within some open neighborhood O of a point $q \in N$, then for any differentiation $D \in \mathfrak{D}(M)$ their images $D(\mathbf{X})$ and $D(\mathbf{Y})$ are equal at the point q.*

The lemma 7.4 follows immediately from the lemma 7.3. This lemma is a basic tool for our purposes of localization in this section.

Let q be some point of N and let $\pi(q)$ be its projection in the base manifold M. Taking some local chart U that covers the point p in M, we can use its preimage $\pi^{-1}(U)$ as a local chart in N covering the point q. The variables (3.3) form the complete set of local coordinates in the chart $\pi^{-1}(U)$. Any extended tensor field $\mathbf{X}$ of the type (α, β) is represented by the formula

$$\mathbf{X} = \sum_{i_1=1}^{n}\ldots\sum_{i_\alpha=1}^{n}\sum_{j_1=1}^{n}\ldots\sum_{j_\beta=1}^{n} X^{i_1\ldots\,i_\alpha}_{j_1\ldots\,j_\beta}\;\mathbf{E}^{j_1\ldots\,j_\beta}_{i_1\ldots\,i_\alpha}, \tag{7.4}$$

which is identical to (1.11). The only difference is that the coefficients $X^{i_1\ldots\,i_\alpha}_{j_1\ldots\,j_\beta}$ in (7.4) depend not only on $x^1, \ldots, x^n$, but on the whole set of variables (3.3). Taking some differentiation $D \in \mathfrak{D}(M)$, we can apply it to the left hand side of the equality (7.4), but we cannot apply it to each summand in the right hand side of (7.4). The matter is that the scalars $X^{i_1\ldots\,i_\alpha}_{j_1\ldots\,j_\beta}$ and the tensors $\mathbf{E}^{j_1\ldots\,j_\beta}_{i_1\ldots\,i_\alpha}$ are defined locally only within the chart $\pi^{-1}(U)$, therefore they do not fit the definition 4.1. In order to convert them to global fields we choose some smooth function $\eta \in \mathfrak{F}(N)$ such that it is identically equal to the unity within some open neighborhood of the point q and is identically zero outside the chart $\pi^{-1}(U)$. Let's define

$$\hat{X}^{i_1\ldots\,i_\alpha}_{j_1\ldots\,j_\beta} = \begin{cases} \eta\ X^{i_1\ldots\,i_\alpha}_{j_1\ldots\,j_\beta} & \text{within } \ \pi^{-1}(U), \\ \ 0 & \text{outside } \ \pi^{-1}(U), \end{cases} \tag{7.5}$$

$$\hat{\mathbf{E}}_i = \begin{cases} \eta\ \mathbf{E}_i & \text{within } \ \pi^{-1}(U), \\ \ 0 & \text{outside } \ \pi^{-1}(U), \end{cases} \tag{7.6}$$

$$\hat{\mathbf{h}}^i = \begin{cases} \eta\ dx^i & \text{within } \ \pi^{-1}(U), \\ \ 0 & \text{outside } \ \pi^{-1}(U). \end{cases} \tag{7.7}$$

Then by analogy to (1.10) we write

$$\hat{\mathbf{E}}^{j_1\ldots\,j_\beta}_{i_1\ldots\,i_\alpha} = \hat{\mathbf{E}}_{i_1} \otimes \ldots \otimes \hat{\mathbf{E}}_{i_\alpha} \otimes \hat{\mathbf{h}}^{j_1} \otimes \ldots \otimes \hat{\mathbf{h}}^{j_\beta}. \tag{7.8}$$

Taking into account (7.5), (7.6), (7.7), and (7.8), from (7.4) we derive

$$\eta^m\ \mathbf{X} = \sum_{i_1=1}^{n} \ldots \sum_{i_\alpha=1}^{n} \sum_{j_1=1}^{n} \ldots \sum_{j_\beta=1}^{n} \hat{X}^{i_1\ldots\,i_\alpha}_{j_1\ldots\,j_\beta}\ \hat{\mathbf{E}}^{j_1\ldots\,j_\beta}_{i_1\ldots\,i_\alpha}, \tag{7.9}$$

where $m = \alpha + \beta + 1$. Now we can apply D to each summand in the right hand side of the equality (7.9). Using the item (4) of the definition 6.2, we get

$$\begin{aligned} D(\eta^m\ \mathbf{X}) = \sum_{i_1=1}^{n} \ldots \sum_{i_\alpha=1}^{n} \sum_{j_1=1}^{n} \ldots \sum_{j_\beta=1}^{n} D(\hat{X}^{i_1\ldots\,i_\alpha}_{j_1\ldots\,j_\beta})\ \hat{\mathbf{E}}^{j_1\ldots\,j_\beta}_{i_1\ldots\,i_\alpha} + \\ + \sum_{i_1=1}^{n} \ldots \sum_{i_\alpha=1}^{n} \sum_{j_1=1}^{n} \ldots \sum_{j_\beta=1}^{n} \hat{X}^{i_1\ldots\,i_\alpha}_{j_1\ldots\,j_\beta}\ D(\hat{\mathbf{E}}^{j_1\ldots\,j_\beta}_{i_1\ldots\,i_\alpha}). \end{aligned} \tag{7.10}$$

Due to the lemma 7.4 we have $D(\eta^m\ \mathbf{X}) = D(\mathbf{X})$ at the point q. Moreover, due to (7.5), (7.6), (7.7), (7.8) and since $\eta(q) = 1$, we have $\hat{\mathbf{E}}^{j_1\ldots\,j_\beta}_{i_1\ldots\,i_\alpha} = \mathbf{E}^{j_1\ldots\,j_\beta}_{i_1\ldots\,i_\alpha}$ and $\hat{X}^{i_1\ldots\,i_\alpha}_{j_1\ldots\,j_\beta} = X^{i_1\ldots\,i_\alpha}_{j_1\ldots\,j_\beta}$ at this point. As for the fields $D(\hat{X}^{i_1\ldots\,i_\alpha}_{j_1\ldots\,j_\beta})$ and $D(\hat{\mathbf{E}}^{j_1\ldots\,j_\beta}_{i_1\ldots\,i_\alpha})$ in (7.10), again due to the lemma 7.4 their values at the point q do not depend on a particular choice of the function η. Since $D(\hat{X}^{i_1\ldots\,i_\alpha}_{j_1\ldots\,j_\beta}) = \delta(\hat{X}^{i_1\ldots\,i_\alpha}_{j_1\ldots\,j_\beta})$, from the formulas (7.2) and (3.7) for the value of $D(\hat{X}^{i_1\ldots\,i_\alpha}_{j_1\ldots\,j_\beta})$ at the point q we derive

$$D(\hat{X}^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}) = \sum_{i=1}^{n} Z^i\ \frac{\partial X^{i_1\ldots\,i_\alpha}_{j_1\ldots\,j_\beta}}{\partial x^i} + \sum_{P=1}^{Q} Z^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}[P]\ \frac{\partial X^{i_1\ldots\,i_\alpha}_{j_1\ldots\,j_\beta}}{\partial T^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}[P]}. \tag{7.11}$$

In order to evaluate $D(\hat{\mathbf{E}}^{j_1\ldots\,j_\beta}_{i_1\ldots\,i_\alpha})$ at the point q let's apply D to (7.8). Using the item (4) of the definition 6.2, we obtain the following equality:

$$\begin{aligned} D(\hat{\mathbf{E}}^{j_1\ldots\,j_\beta}_{i_1\ldots\,i_\alpha})&=\sum_{v=1}^{\alpha}\mathbf{E}_{i_1}\otimes\ldots\otimes D(\,\hat{\mathbf{E}}_{i_v})\otimes\ldots\otimes\mathbf{E}_{i_\alpha}\otimes\\ &\otimes\,dx^{j_1}\otimes\ldots\otimes dx^{j_\beta}+\sum_{w=1}^{\beta}\mathbf{E}_{i_1}\otimes\ldots\otimes\mathbf{E}_{i_\alpha}\otimes\\ &\otimes\,dx^{j_1}\otimes\ldots\otimes D(\,\hat{\mathbf{h}}^{j_w})\otimes\ldots\otimes dx^{j_\beta}.\end{aligned}\tag{7.12}$$

Summarizing the above three formulas (7.10), (7.11), (7.12), and the equality $D(\eta^m\,\mathbf{X})=D(\mathbf{X})$, we can formulate the following lemma.

Lemma 7.5. *Any differentiation $D\in\mathfrak{D}(M)$ is uniquely fixed by its restrictions to the modules $T^0_0(M)$, $T^1_0(M)$, $T^0_1(M)$ in the direct sum (5.3).*

The value of the extended vector field $D(\hat{\mathbf{E}}_i)$ at the the point q is a vector of the tangent space $T_{\pi(q)}(M)$. We can write the following expansion for this vector:

$$D(\hat{\mathbf{E}}_i)=\sum_{k=1}^{n}\Gamma^k_i\,\mathbf{E}_k.\tag{7.13}$$

Due to the lemma 7.4 the left hand side of the equality (7.13) does not depend on a particular choice of the function η in (7.6). Therefore, the coefficients Γ^k_i in (7.13) represent the differentiation D at the point q for a given local chart U in M. The same is true for Z^i and $Z^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}[P]$ in (7.11). Being dependent on q, all these coefficients Z^i, $Z^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}[P]$, and Γ^k_i are some smooth functions of the variables (3.3). However, if q is fixed, they all are constants.

Under a change of a local chart the coefficients Z^i, $Z^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}[P]$, and Γ^k_i obey some definite transformation rules. The transformation rules for Z^i and $Z^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}[P]$ are derived from the formulas (3.13), (3.14), and (7.2):

$$\begin{cases} Z^i=\displaystyle\sum_{j=1}^{n}S^i_j\,\tilde{Z}^j,\\[2ex] Z^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}[P]=\displaystyle\sum^{\mathrm{n}}_{\substack{h_1,\,\ldots\,,\,h_r\\k_1,\,\ldots\,,\,k_s}}\!\!\ldots\!\sum^{\mathrm{n}}S^{i_1}_{h_1}\,\ldots\,S^{i_r}_{h_r}\;T^{k_1}_{j_1}\,\ldots\,T^{k_s}_{j_s}\;\tilde{Z}^{h_1\ldots\,h_r}_{k_1\ldots\,k_s}[P]\,-\\ \qquad-\displaystyle\sum_{m=1}^{r}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{v_m=1}^{n}\theta^{i_m}_{i\,v_m}\;T^{i_1\ldots\,v_m\ldots\,i_r}_{j_1\ldots\ldots\ldots\,j_s}[P]\;S^i_j\;\tilde{Z}^j\,+\\ \qquad+\displaystyle\sum_{m=1}^{s}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{w_m=1}^{n}\theta^{w_m}_{i\,j_m}\;T^{i_1\ldots\ldots\ldots\,i_r}_{j_1\ldots\,w_m\ldots\,j_s}[P]\;S^i_j\;\tilde{Z}^j.\end{cases}\tag{7.14}$$

Here the components of the direct and inverse transition matrices S and T are taken from (1.2), while the θ-parameters are given by the formulas (3.15). The transformation formulas (7.14) can be inverted in the following way:

$$\begin{cases} \tilde{Z}^i = \sum_{j=1}^{n} T^i_j\, Z^j, \\ \\ \tilde{Z}^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}[P] = \sum_{\substack{h_1,\,\ldots\,,\,h_r\\ k_1,\,\ldots\,,\,k_s}}^{\mathrm{n}}\!\!\ldots\!\!\sum^{\mathrm{n}} T^{i_1}_{h_1}\ldots\,T^{i_r}_{h_r}\;S^{k_1}_{j_1}\ldots\,S^{k_s}_{j_s}\;Z^{h_1\ldots\,h_r}_{k_1\ldots\,k_s}[P]\,- \\ \qquad -\sum_{m=1}^{r}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{v_m=1}^{n}\tilde{\theta}^{i_m}_{i\,v_m}\;\tilde{T}^{i_1\ldots\,v_m\ldots\,i_r}_{j_1\ldots\,\ldots\,\ldots\,j_s}[P]\;T^i_j\;Z^j\,+ \\ \qquad +\sum_{m=1}^{s}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{w_m=1}^{n}\tilde{\theta}^{w_m}_{i\,j_m}\;\tilde{T}^{i_1\ldots\,\ldots\,\ldots\,i_r}_{j_1\ldots\,w_m\ldots\,j_s}[P]\;T^i_j\;Z^j. \end{cases} \tag{7.15}$$

The θ-parameters for (7.15) are taken from (3.10) or from (3.12). The transformation formulas (7.14) and (7.15) should be completed with the analogous formulas for the coefficients Γ^k_i in the expansion (7.13):

$$\Gamma^k_i = \sum_{b=1}^{n}\sum_{a=1}^{n} S^k_a\,T^b_i\;\tilde{\Gamma}^a_b + \sum_{a=1}^{n} Z^a\,\theta^k_{ai}, \tag{7.16}$$

$$\tilde{\Gamma}^k_i = \sum_{b=1}^{n}\sum_{a=1}^{n} T^k_a\;S^b_i\;\Gamma^a_b + \sum_{a=1}^{n} \tilde{Z}^a\,\tilde{\theta}^k_{ai}. \tag{7.17}$$

The formula (7.16) is derived immediately from (7.13) and (1.4), taking into account (3.15). The formula (7.17) then is written by analogy.

Let's return back to (7.12). In order to calculate $\mathbf{D}(\,\hat{\mathbf{h}}_j)$ in this formula we need to remember (1.9). This equality can be written as follows:

$$C(dx^i \otimes \mathbf{E}_j) = \delta^i_j. \tag{7.18}$$

Multiplying the equality (7.18) by η^2, we transform it to the following one:

$$C(\hat{\mathbf{h}}^i \otimes \hat{\mathbf{E}}_j) = \delta^i_j\,\eta^2. \tag{7.19}$$

Here η is the same function as in (7.6) and (7.7). Now one can apply the differentiation D to both sides of (7.19). Taking into account the item (3) in the definition 6.2 and taking into account the lemma 7.4, at the fixed point q we obtain

$$C(D(\hat{\mathbf{h}}^i) \otimes \hat{\mathbf{E}}_j) + C(\hat{\mathbf{h}}^i \otimes D(\hat{\mathbf{E}}_j)) = D(\delta^i_j\,\eta^2) = 0. \tag{7.20}$$

Since $\eta = 1$ at the point q, from (7.13), (7.19), and (7.20) we derive

$$C(D(\hat{\mathbf{h}}^i) \otimes \hat{\mathbf{E}}_j) = -\sum_{k=1}^{n}\Gamma^k_j\;C(\hat{\mathbf{h}}^i \otimes \hat{\mathbf{E}}_k) = -\Gamma^i_j. \tag{7.21}$$

By the definition of a differentiation the value of $D(\hat{\mathbf{h}}^i)$ at the point q is a covector from the cotangent space $T^*_q(M)$. One can expand it in the basis of differentials $dx^1, \ldots, dx^n$. Due to (7.21) this expansion looks like

$$D(\hat{\mathbf{h}}^i) = -\sum_{j=1}^{n} \Gamma^i_j \, dx^j. \tag{7.22}$$

The formula (7.22) means that we can strengthen the lemma 7.5 as follows.

Theorem 7.1. *Any differentiation $D \in \mathfrak{D}(M)$ is uniquely fixed by its restrictions to the modules $T^0_0(M)$, $T^1_0(M)$ in the direct sum (5.3).*

Indeed, let's denote by ξ and ζ the restrictions of the differentiation D to the modules $T^1_0(M)$ and $T^0_1(M)$ respectively. Its restriction to the module $T^0_0(M)$ was already considered. It was denoted by δ (see formula (7.1) above):

$$\xi\colon\, T^1_0(M) \to T^1_0(M), \tag{7.23}$$

$$\zeta\colon\, T^0_1(M) \to T^0_1(M). \tag{7.24}$$

The formulas (7.13) and (7.22) mean that the mapping (7.24) is completely determined if the mapping (7.23) is known. Hence, D is completely determined by the mappings δ and ξ. Then, using (7.10), (7.11), and (7.12), one can reconstruct the mapping D itself. Let $\mathbf{Y} = D(\mathbf{X})$ and assume that $\mathbf{X}$ is given by the formula (7.4) in local coordinates. Then $\mathbf{Y}$ is given by a similar expansion

$$\mathbf{Y} = \sum_{i_1=1}^{n} \ldots \sum_{i_\alpha=1}^{n} \sum_{j_1=1}^{n} \ldots \sum_{j_\beta=1}^{n} Y^{i_1\ldots\, i_\alpha}_{j_1\ldots\, j_\beta}\; \mathbf{E}^{j_1\ldots\, j_\beta}_{i_1\ldots\, i_\alpha}, \tag{7.25}$$

where its components $Y^{i_1\ldots\, i_\alpha}_{j_1\ldots\, j_\beta}$ are calculated as follows:

$$\begin{gathered} Y^{i_1\ldots\, i_\alpha}_{j_1\ldots\, j_\beta} = \sum_{i=1}^{n} Z^i\, \frac{\partial X^{i_1\ldots\, i_\alpha}_{j_1\ldots\, j_\beta}}{\partial x^i} + \sum_{P=1}^{Q} \sum_{\substack{h_1,\,\ldots\,,\,h_r\\ k_1,\,\ldots\,,\,k_s}}^{\mathrm{n}} \!\!\ldots\!\! \sum^{\mathrm{n}} Z^{h_1\ldots\, h_r}_{k_1\ldots\, k_s}[P]\; \frac{\partial X^{i_1\ldots\, i_\alpha}_{j_1\ldots\, j_\beta}}{\partial T^{h_1\ldots\, h_r}_{k_1\ldots\, k_s}[P]} \;+ \\ + \sum_{m=1}^{\alpha} \sum_{v_m=1}^{n} \Gamma^{i_m}_{v_m}\; X^{i_1\ldots\, v_m\ldots\, i_\alpha}_{j_1\ldots\ldots\ldots\, j_\beta} - \sum_{m=1}^{\beta} \sum_{w_m=1}^{n} \Gamma^{w_m}_{j_m}\; X^{i_1\ldots\ldots\ldots\, i_\alpha}_{j_1\ldots\, w_m\ldots\, j_\beta}. \end{gathered} \tag{7.26}$$

The formulas (7.25) and (7.26) prove the theorem 7.1. As for the mappings (7.23) and (7.24), one can formulate the following theorem for them.

Theorem 7.2. *Defining a differentiation D of the algebra of extended tensor fields $\mathbf{T}(M)$ is equivalent to defining two $\mathbb{R}$-linear mappings (7.23) and (7.24) such that*

$$\delta(\varphi\,\psi) = \delta(\varphi)\,\psi + \varphi\,\delta(\psi) \quad \textit{for any} \quad \varphi, \psi \in \mathfrak{F}(N) = T^0_0(M), \tag{7.27}$$

$$\xi(\varphi\,\mathbf{X}) = \delta(\varphi)\;\mathbf{X} + \varphi\;\xi(\mathbf{X}) \quad \textit{for any} \quad \varphi \in \mathfrak{F}(N) \quad \textit{and} \quad \mathbf{X} \in T^1_0(M). \tag{7.28}$$

The formula (7.27) is immediate from the definition 6.2. It means that δ is a differentiation of the ring $\mathfrak{F}(N)$. This fact was used for writing (7.2). The formula (7.28) is also immediate from the definition 6.2.

Looking at the formula (7.26), we see that the differentiation D acts as a first order linear differential operator upon the components of a tensor field $\mathbf{X}$. The coefficients Z^i, $Z^{h_1\dots\,h_r}_{k_1\dots\,k_s}[P]$, Γ^k_i of this linear operator are not differentiated in (7.26). Therefore, fixing some point $q \in N$ and taking the values of the coefficients Z^i, $Z^{h_1\dots\,h_r}_{k_1\dots\,k_s}[P]$, Γ^k_i, we can say that we know this linear operator at that particular point q. In order to formalize this idea we need a coordinate-free definition of a first order linear differential operator at a point q acting upon tensors.

Definition 7.1. Two smooth extended tensor fields $\mathbf{X}_1$ and $\mathbf{X}_2$ defined in some open neighborhoods O_1 and O_2 of a point $q \in N$ are called *q-equivalent* if they take the same values within some smaller neighborhood $O \subset O_1 \cap \vee_2$ of the point q.

Definition 7.2. A class of q-equivalent smooth extended tensor fields is called a *stalk* of a smooth extended tensor field at the point q.

The stalks of various extended tensor fields at a fixed point $q \in N$ form a graded algebra over the real numbers $\mathbb{R}$ (compare to (5.3) above):

$$\mathfrak{T}(q, M) = \bigoplus_{\alpha=0}^{\infty}\bigoplus_{\beta=0}^{\infty}\mathfrak{T}^{\alpha}_{\beta}(q, M). \tag{7.29}$$

Each summand $\mathfrak{T}^{\alpha}_{\beta}(q, M)$ in (7.29) is a linear space over $\mathbb{R}$. It is composed by stalks of extended tensor fields of some fixed type (α, β).

A stalk of a tensor field is somewhat like its value at a fixed point q. However, they do not coincide since the stalk comprises much more information:

$$\mathfrak{T}^{\alpha}_{\beta}(q, M) \neq T^{\alpha}_{\beta}(q, M).$$

Informally speaking, a stalk is the restriction of a tensor field to the infinitesimal neighborhood of a point q. The information stored in a stalk of a tensor field is sufficient for to apply a differential operator to it. As a result we get a tensor (not a stalk) at a fixed point.

Definition 7.3. A *tensorial first order differential operator* D_q at a point $q \in N$ is a mapping $D_q\colon \mathfrak{T}(q, M) \to T(q, M)$ such that

(1) D_q is concordant with the grading: $D_q(\mathfrak{T}^{\alpha}_{\beta}(q, M)) \subset T^{\alpha}_{\beta}(q, M)$;
(2) D_q is $\mathbb{R}$-linear: $D_q(\mathbf{X} + \mathbf{Y}) = D_q(\mathbf{X}) + D_q(\mathbf{Y})$
and $D_q(\lambda\,\mathbf{X}) = \lambda\, D_q(\mathbf{X})$ for $\lambda \in \mathbb{R}$;
(3) D_q commutates with the contractions: $D_q(C(\mathbf{X})) = C(D_q(\mathbf{X}))$;
(4) D_q obeys the Leibniz rule: $D_q(\mathbf{X} \otimes \mathbf{Y}) = D_q(\mathbf{X}) \otimes \mathbf{Y}_q + \mathbf{X}_q \otimes D_q(\mathbf{Y})$.

The formulas (7.25) and (7.26) yield the representation of a tensorial differential operator in a local chart. In the case of a differential operator D_q the coefficients Z^i, $Z^{h_1\dots\,h_r}_{k_1\dots\,k_s}[P]$, Γ^k_i are constants related to a point q. They are called the *components* of D_q. The operators D_q at a fixed point q form a finite-dimensional linear space, we denote it $\mathfrak{D}(q, M)$. The lemma 7.4 provides the following result.

Theorem 7.3. *Any differentiation D of the algebra of extended tensor fields $\mathbf{T}(M)$ is represented as a field of differential operators $D_q \in \mathfrak{D}(q, M)$, one per each point $q \in N$. Conversely, each smooth field of tensorial first order differential operators is a differentiation of the algebra $\mathbf{T}(M)$.*

The theorem 7.3 solves the problem of localization announced in the very beginning of this section. Relying on this theorem, one can formulate the following definition for a differentiation of the algebra of extended tensor fields $\mathbf{T}(M)$.

Definition 7.4. Let $N = T^{r_1\ldots\,r_Q}_{s_1\ldots\,s_Q}M$ be a composite tensor bundle over a smooth real manifold M. A differentiation D of the algebra $\mathbf{T}(M)$ is a smooth operator-valued function in N such that it takes each point $q \in N$ to some differential operator $D_q \in \mathfrak{D}(q, M)$.

Smoothness in both cases — in theorem 7.3 and in definition 7.4 means that the components Z^i, $Z^{h_1\ldots\,h_r}_{k_1\ldots\,k_s}[P]$, Γ^k_i of the differential operator D_q are smooth functions of the variables (3.3) in any local chart.

Let's compare the definitions 7.4 and 4.1. They are very similar. Therefore, by analogy to the definition 1.3, one can formulate the following definition.

Definition 7.5. A differentiation D of the algebra $\mathbf{T}(M)$ is a geometric object in each local chart represented by its components Z^i, $Z^{h_1\ldots\,h_r}_{k_1\ldots\,k_s}[P]$, Γ^k_i and such that its components are smooth functions transformed according to the formulas (7.14), (7.15), (7.16), and (7.17) under a change of local coordinates.

The definition 7.5 is a coordinate form of the definition 7.4, and conversely, the definition 7.4 is a coordinate-free form of the definition 7.5.

8. Degenerate differentiations.

Definition 8.1. A differentiation D of the algebra of extended tensor fields $\mathbf{T}(M)$ is called a *degenerate differentiation* if its restriction $\delta\colon\, T^0_0(M) \to T^0_0(M)$ to the module $T^0_0(M)$ is identically zero.

For a degenerate differentiation the corresponding vector field (7.2) is identically equal to zero. Then the equality (7.27) is obviously fulfilled, while the equality (7.28) is reduced to the following one:

$$\xi(\varphi\,\mathbf{X}) = \varphi\;\xi(\mathbf{X}). \tag{8.1}$$

From the equality (8.1) we conclude that the map $\xi:\, T^1_0(M) \to T^1_0(M)$ is an endomorphism of the module $T^1_0(M)$ over the ring $\mathfrak{F}(N)$.

Definition 8.2. Let A be a module over the ring of smooth real-valued functions $\mathfrak{F}(M)$ in some manifold M. We say that the module A admits a *localization* if it is isomorphic to a functional module so that each element $\mathbf{a} \in A$ is represented as some function $\mathbf{a}(q) = \mathbf{a}_q$ in M taking its values in some $\mathbb{R}$-linear spaces A_q associated with each point q of the manifold M.

Definition 8.3. Let A be a module that admits a localization in the sense of the definition 8.2. We say that the localization of A is a *complete localization* if the following two conditions are fulfilled:

1. for any point $q \in M$ and for any vector $\mathbf{v} \in A_q$ there exists an element $\mathbf{a} \in A$ such that $\mathbf{a}_q = \mathbf{v}$;
2. if $\mathbf{a}_q = 0$ at some point $q \in M$, then there exist some finite set of elements $\mathbf{E}_0, \ldots, \mathbf{E}_n$ in A and some smooth functions $\alpha_0\ \ldots, \alpha_n$ vanishing at the point q such that $\mathbf{a} = \alpha_0\,\mathbf{E}_0 + \ldots + \alpha_n\,\mathbf{E}_n$.

Theorem 8.1. *Let A and B be two $\mathfrak{F}(M)$-modules that admit localizations. If the localization of A is a complete localization, then each homomorphism $f\colon A \to B$ is represented by a family of $\mathbb{R}$-linear mappings*

$$F_q\colon A_q \to B_q \tag{8.2}$$

so that if $\mathbf{a} \in A$ and $\mathbf{b} = f(\mathbf{a})$, then $\mathbf{b}_q = F_q(\mathbf{a}_q)$ for each point $q \in M$.

Proof. First of all we should construct the mappings (8.2). Let q be some arbitrary point of the manifold M and let $\mathbf{v}$ be some arbitrary vector of the $\mathbb{R}$-linear space A_q. Then, according to the item (1) of the definition 8.3, we have an element $\mathbf{a} \in A$ such that $\mathbf{v} = \mathbf{a}_q$. Applying the homomorphism f to $\mathbf{a}$ we get $\mathbf{b} = f(\mathbf{a}) \in B$. Let's define the mapping (8.2) as follows:

$$F_q(\mathbf{v}) = \mathbf{b}_q, \ \text{ where } \ \mathbf{b} = f(\mathbf{a}) \ \text{ for some } \ \mathbf{a} \in A \ \text{ such that } \ \mathbf{v} = \mathbf{a}_q. \tag{8.3}$$

The choice of an element $\mathbf{a} \in A$ such that $\mathbf{v} = \mathbf{a}_q$ is not unique. Therefore, one should prove the consistence of the definition (8.3). Suppose that $\mathbf{a}$ and $\tilde{\mathbf{a}}$ are two elements of the module A such that $\mathbf{v} = \mathbf{a}_q$ and $\mathbf{v} = \tilde{\mathbf{a}}_q$. Then for the element $\mathbf{c} = \mathbf{a} - \tilde{\mathbf{a}}$ we get $\mathbf{c}_q = 0$. Applying the item (2) of the definition 8.3 to $\mathbf{c}$, we get

$$\mathbf{c} = \alpha_0\,\mathbf{E}_0 + \ldots + \alpha_n\,\mathbf{E}_n, \tag{8.4}$$

where $\alpha_0, \ldots, \alpha_n$ are smooth functions vanishing at the point $q \in M$. Applying the homomorphism f to (8.4), we derive

$$\mathbf{d} = f(\mathbf{a}) - f(\tilde{\mathbf{a}}) = f(\mathbf{c}) = \alpha_0\,f(\mathbf{E}_0) + \ldots + \alpha_n\,f(\mathbf{E}_n). \tag{8.5}$$

Since $\alpha_0(q) = \ldots = \alpha_n(q) = 0$, from (8.5) we obtain $\mathbf{d}_q = 0$. This means that two elements $\mathbf{b} = f(\mathbf{a})$ and $\tilde{\mathbf{b}} = f(\tilde{\mathbf{a}})$ determine the same vector $\mathbf{b}_q = \tilde{\mathbf{b}}_q$ in (8.3). So, (8.3) is a consistent definition of a mapping $F_q\colon A_q \to B_q$.

The mapping $F_q\colon A_q \to B_q$ consistently defined by the equality (8.3) is $\mathbb{R}$-linear. This fact is a trivial consequence of the equalities

$$\begin{aligned} &f(\mathbf{a}_1 + \mathbf{a}_2) = f(\mathbf{a}_1) + f(\mathbf{a}_2) &&\text{for any } \ \mathbf{a}_1, \mathbf{a}_2 \in A,\\ &f(\lambda\,\mathbf{a}) = \lambda\,f(\mathbf{a}) &&\text{for any } \ \mathbf{a}_2 \in A \ \text{ and } \ \lambda \in \mathfrak{F}(M), \end{aligned}$$

which mean that f is a homomorphism of $\mathfrak{F}(M)$-modules. And finally, from the equality (8.3) it follows that for any $\mathbf{a} \in A$ its image $\mathbf{b} = f(\mathbf{a})$ is represented by a function $\mathbf{b}$ whose values are obtained by applying the mappings (8.2) to the values of $\mathbf{a}$. The theorem is proved. $\square$

Theorem 8.2. *Let $\pi\colon VM \to M$ be a smooth n-dimensional vector bundle over some base manifold M and let A be the set of all global smooth sections[1] of this bundle. Then A admits complete localization in the sense of the definition 8.3.*

Proof. The module structure of A and its localization are obvious. By definition, each section $\mathbf{a}$ of the bundle $\pi\colon VM \to M$ is a function taking its values in fibers $V_q = \pi^{-1}(q)$ of the bundle VM. We need to prove that this natural localization of

[1] See the definition in [31].

A is complete. Let's begin with the item (1) in the definition 8.3. Let q be some fixed point of the base manifold M and let $\mathbf{v}$ be some vector in the fiber V_q over this point. Each vector bundle is locally trivial. This means that there is some open neighborhood U of our point q such that $\pi^{-1}(U)$ is isomorphic to the trivial vector bundle $U\times\mathbb{R}^n$. This fact is expressed by the following diagram:

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\ \psi\ } & U\times\mathbb{R}^n \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow \\ U & \xrightarrow{\ \mathbf{id}\ } & U. \end{array} \tag{8.6}$$

The isomorphism ψ in the diagram (8.6) is linear within each fiber of the bundle VM. Let's denote $\mathbf{r}_0=\psi(\mathbf{v})\in\mathbb{R}^n$ and let's choose the constant section $\mathbf{r}(q)\equiv\mathbf{r}_0$ of the trivial bundle $U\times\mathbb{R}^n$. Its preimage $\mathbf{b}=\psi^{-1}(\mathbf{r})$ is a smooth section of VM over the open set U and $\mathbf{b}_q=\mathbf{v}$. However, $\mathbf{b}$ is a local section. In order to convert it to a global section let's choose some smooth function η in M such that $\eta(q)=1$ at the point q and such that $\eta\equiv 0$ outside the open set U. Then let's define

$$\mathbf{a}=\begin{cases} \eta\,\mathbf{b} & \text{within } U,\\ 0 & \text{outside } U.\end{cases} \tag{8.7}$$

It is easy to see that $\mathbf{a}$ in (8.7) is a smooth global section of the bundle VM. From $\eta(q)=1$ and froom $\mathbf{b}_q=\mathbf{v}$ we derive that $\mathbf{a}_q=\mathbf{v}$. Thus, we have proved that the module A fits the item (1) in the definition 8.3.

Now let's proceed with the item (2) in the definition 8.3. Suppose that $\mathbf{a}$ is a smooth section of the bundle VM such that $\mathbf{a}_q=0$ at the point $q\in M$. Applying the isomorphism ψ taken from the diagram (8.6) to the restriction of $\mathbf{a}$ to the open set U, we get the following smooth section of the trivial bundle $U\times\mathbb{R}^n$:

$$\mathbf{r}=\psi(\mathbf{a})=\begin{Vmatrix}\beta_1\\ \vdots\\ \beta_n\end{Vmatrix}=\sum_{i=1}^{n}\beta_i\,\mathbf{e}_i. \tag{8.8}$$

Here $\mathbf{e}_1,\,\ldots\,,\,\mathbf{e}_n$ are constant unit vectors in $\mathbb{R}^n$:

$$\mathbf{e}_1=\begin{Vmatrix}1\\0\\ \vdots\\0\\0\end{Vmatrix},\quad \mathbf{e}_2=\begin{Vmatrix}0\\1\\ \vdots\\0\\0\end{Vmatrix},\quad \ldots\,,\quad \mathbf{e}_{n-1}=\begin{Vmatrix}0\\0\\ \vdots\\1\\0\end{Vmatrix},\quad \mathbf{e}_n=\begin{Vmatrix}0\\0\\ \vdots\\0\\1\end{Vmatrix}. \tag{8.9}$$

From $\mathbf{a}_q=0$ and from (8.8) we derive that the smooth functions $\beta_1,\,\ldots\,,\,\beta_n$ should vanish at the point q, i.e. $\beta_i(q)=0$. In (8.8) the vectors (8.9) represent some constant smooth sections of the trivial bundle $U\times\mathbb{R}^n$. Let's denote $\hat{\mathbf{E}}_i=\psi^{-1}(\mathbf{e}_i)$. Then $\hat{\mathbf{E}}_1,\,\ldots\,,\,\hat{\mathbf{E}}_n$ are smooth local sections of the bundle VM over the open set U. From (8.8) we derive the following expansion:

$$\mathbf{a}=\sum_{i=1}^{n}\beta_i\,\hat{\mathbf{E}}_i. \tag{8.10}$$

The expansion (8.10) is a local expansion, it is not the expansion of $\mathbf{a}$, but of its restriction to the open set U. In order to make it global let's define

$$\alpha_i = \begin{cases} \eta\,\beta_i & \text{within } U,\\ 0 & \text{outside } U,\end{cases} \qquad \mathbf{E}_i = \begin{cases} \eta\,\hat{\mathbf{E}}_i & \text{within } U,\\ 0 & \text{outside } U.\end{cases} \tag{8.11}$$

Here η again is a smooth function in M such that $\eta(q) = 1$ and such that it is identically zero outside U. Multiplying both sides of the expansion (8.10) by η^2 and taking into account (8.11), we derive

$$\mathbf{a}\,\eta^2 = \sum_{i=1}^{n} \alpha_i\,\mathbf{E}_i. \tag{8.12}$$

Then, for the original section $\mathbf{a}$ due to (8.12) we get

$$\mathbf{a} = (1-\eta^2)\,\mathbf{a} + \mathbf{a}\,\eta^2 = (1-\eta^2)\,\mathbf{a} + \sum_{i=1}^{n} \alpha_i\,\mathbf{E}_i.$$

Let's denote $\alpha_0 = 1 - \eta$ and $\mathbf{E}_0 = (1+\eta)\,\mathbf{a}$. Then the above local expansion (8.10) is transformed to the following global one:

$$\mathbf{a} = \sum_{i=0}^{n} \alpha_i\,\mathbf{E}_i. \tag{8.13}$$

Since $\beta_i(q) = 0$ for $i = 1, \ldots, n$ and since $\eta(q) = 1$, we find that all coefficients α_i in (8.13) do vanish at the point q. Comparing (8.13) with the expansion in the item (2) of the definition 8.3, we finally conclude that the module A admits a complete localization. The theorem 8.2 is proved. $\square$

Now let's return to the definition 8.1 and consider the map $\xi\colon T^1_0(M) \to T^1_0(M)$ associated with some differentiation D. The extended vector fields are naturally interpreted as sections of a vector bundle. Indeed, any tensor bundle is a vector bundle in the sense that any tensor space $T^\alpha_\beta(p, M)$ is a linear vector space over the real numbers $\mathbb{R}$. Therefore, one can apply the theorem 8.2 to the module $T^1_0(M)$. As it was mentioned above, the mapping $\xi\colon T^1_0(M) \to T^1_0(M)$ is an endomorphism. Applying the theorem 8.1 to it, we find that ξ is given by some extended tensor field $\mathbf{S}$ of the type $(1,1)$ acting as a linear operator at each point $q \in N$:

$$\xi(\mathbf{X}) = C(\mathbf{S} \otimes \mathbf{X}) \quad \text{for any} \quad \mathbf{X} \in T^1_0(M). \tag{8.14}$$

Theorem 8.3. *Defining a degenerate differentiation D of the algebra $\mathbf{T}(M)$ is equivalent to defining some extended tensor field $\mathbf{S}$ of the type $(1,1)$.*

Apart from (8.14), the theorem 8.3 can be understood in a coordinate form. Indeed, for a degenerate differentiation D from (7.2) and from the definition 8.1 we derive $Z^i = 0$ and $Z^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}[P] = 0$. Since $Z^i = 0$, the transformation formulas (7.16) and (7.17) now are written as follows:

$$\Gamma^k_i = \sum_{b=1}^{n}\sum_{a=1}^{n} S^k_a\,T^b_i\,\tilde{\Gamma}^a_b, \qquad \tilde{\Gamma}^k_i = \sum_{b=1}^{n}\sum_{a=1}^{n} T^k_a\,S^b_i\,\Gamma^a_b. \tag{8.15}$$

The formulas (8.15) mean that Γ^k_i and $\tilde{\Gamma}^a_b$ are the components of some extended tensor field of the type $(1,1)$, see (1.12) and (1.13) for comparison. Applying (7.26) to (8.15), we find that Γ^k_i are the components of the tensor field $\mathbf{S}$ in some local chart. This is a coordinate proof for the theorem 8.3.

9. Covariant differentiations.

The set of differentiations of the extended algebra $\mathbf{T}(M)$ possesses the structure of a module over the ring of smooth functions $\mathfrak{F}(N)$. The set of extended vector fields $T^1_0(M)$ is also a module over the same ring $\mathfrak{F}(N)$. Therefore, the following definition is consistent.

Definition 9.1. Say that in a manifold M a covariant differentiation of the algebra of extended tensor fields $\mathbf{T}(M)$ is given if some homomorphism of $\mathfrak{F}(N)$-modules $\nabla\colon\, T^1_0(M) \to \mathfrak{D}(M)$ is given. The image of a vector field $\mathbf{Y}$ under such homomorphism is denoted by $\nabla_{\mathbf{Y}}$. The differentiation $D = \nabla_{\mathbf{Y}} \in \mathfrak{D}(M)$ is called the *covariant differentiation along the vector field* $\mathbf{Y}$.

Let's remember that the module $\mathfrak{D}(M)$ admits a localization. Indeed, according to the theorem 7.3 each differentiation D is a field of differential operators. As for the differential operators themselves, they form finite-dimensional $\mathbb{R}$-linear spaces $\mathfrak{D}(q, M)$, one per each point $q \in N$. The coefficients Z^i, $Z^{i_1\ldots\, i_r}_{j_1\ldots\, j_s}[P]$, and Γ^k_i from (7.26) are coordinates within $\mathfrak{D}(q, M)$. Therefore, we have

$$\dim(\mathfrak{D}(q, M)) = n + n^2 + \sum_{P=1}^{Q} n^{r_P + s_P} = n^2 + \dim N. \tag{9.1}$$

Under a change of a local chart the coefficients Z^i, $Z^{i_1\ldots\, i_r}_{j_1\ldots\, j_s}[P]$, Γ^k_i are transformed according to the formulas (7.14), (7.15), (7.16), and (7.17). These formulas are linear with respect to Z^i, $Z^{i_1\ldots\, i_r}_{j_1\ldots\, j_s}[P]$, Γ^k_i and with respect to transformed coordinates $\tilde{Z}^i$, $\tilde{Z}^{i_1\ldots\, i_r}_{j_1\ldots\, j_s}[P]$, $\tilde{\Gamma}^k_i$. Therefore, the spaces $\mathfrak{D}(q, M)$ are glued into a vector bundle of the dimension (9.1) for which N is a base manifold. This fact means that one can apply the theorem 8.2 to the module $\mathfrak{D}(M)$.

Let ∇ be some covariant differentiation of the algebra of extended tensor fields. Then, applying the theorem 8.1 to the homomorphism $\nabla\colon\, T^1_0(M) \to \mathfrak{D}(M)$, we find that this homomorphism is composed by R-linear maps $T_{\pi(q)}(M) \to \mathfrak{D}(q, M)$ specific to each point $q \in N$. This fact is expressed by the following formula:

$$\nabla_{\mathbf{Y}}\mathbf{X} = C(\mathbf{Y} \otimes \nabla\mathbf{X}) = \sum_{j=1}^{n} \sum_{\substack{i_1,\,\ldots\,,\,i_\alpha\\ j_1,\,\ldots\,,\,j_\beta}}^{\mathrm{n}} \!\!\ldots\! \sum^{\mathrm{n}} Y^j\ \nabla_{\!j} X^{i_1\ldots\, i_\alpha}_{j_1\ldots\, j_\beta}\ \mathbf{E}^{j_1\ldots\, j_\beta}_{i_1\ldots\, i_\alpha}. \tag{9.2}$$

Looking at (9.2), we see that each covariant differentiation ∇ can be treated as an operator producing the extended tensor field $\nabla\mathbf{X}$ of the type $(\alpha, \beta + 1)$ from any given extended tensor field $\mathbf{X}$ of the type (α, β). This operator increases by one the number of covariant indices of a tensor field $\mathbf{X}$. It is called the operator of *covariant differential* associated with the covariant differentiation ∇.

Let's consider the linear map $T_{\pi(q)}(M) \to \mathfrak{D}(q, M)$ produced by some covariant differentiation ∇ at some particular point $q \in N$. In a local chart this map is given by some linear functions expressing the components of the differential operator D_q, where $D = \nabla_{\mathbf{Y}}$, through the components of the vector $\mathbf{Y}_q$:

$$Z^{i_1\ldots\, i_r}_{j_1\ldots\, j_s}[P] = \sum_{j=1}^{n} Z_{j\,j_1\ldots\, j_s}^{\ i_1\ldots\, i_r}[P]\ Y^j, \tag{9.3}$$

$$Z^i = \sum_{j=1}^{n} Z^i_j\ Y^j, \qquad \Gamma^k_i = \sum_{j=1}^{n} \Gamma^k_{j\,i}\ Y^j. \tag{9.4}$$

Substituting (9.3) and (9.4) into the formula (7.26), then taking into account (9.2) and (7.25), we derive the following formula:

$$\begin{gathered}\nabla_{\!j} X^{i_1\ldots\, i_\alpha}_{j_1\ldots\, j_\beta} = \sum_{i=1}^{n} Z^i_j\ \frac{\partial X^{i_1\ldots\, i_\alpha}_{j_1\ldots\, j_\beta}}{\partial x^i} + \sum_{P=1}^{Q}\sum_{\substack{h_1,\,\ldots\,,\,h_r\\ k_1,\,\ldots\,,\,k_s}}^{\mathrm{n}}\!\!\ldots\!\!\sum^{\mathrm{n}} Z_{j\,k_1\ldots\, k_s}^{\ h_1\ldots\, h_r}[P]\ \frac{\partial X^{i_1\ldots\, i_\alpha}_{j_1\ldots\, j_\beta}}{\partial T^{h_1\ldots\, h_r}_{k_1\ldots\, k_s}[P]}\ + \\ +\sum_{m=1}^{\alpha}\sum_{v_m=1}^{n}\Gamma^{i_m}_{j\,v_m}\ X^{i_1\ldots\, v_m\ldots\, i_\alpha}_{j_1\ldots\ldots\ldots\, j_\beta} - \sum_{m=1}^{\beta}\sum_{w_m=1}^{n}\Gamma^{w_m}_{j\,j_m}\ X^{i_1\ldots\ldots\ldots\, i_\alpha}_{j_1\ldots\, w_m\ldots\, j_\beta},\end{gathered} \tag{9.5}$$

From (7.14) and (7.15) we derive the following transformation formulas for the quantities Z^i_j, $Z_{j\,j_1\ldots\, j_s}^{\ i_1\ldots\, i_r}[P]$ in (9.3) and (9.4):

$$\left\{\begin{aligned} &Z^i_j = \sum_{h=1}^{n}\sum_{k=1}^{n} S^i_h\, T^k_j\ \tilde Z^h_k,\\ &Z_{j\,j_1\ldots\, j_s}^{\ i_1\ldots\, i_r}[P] = \sum_{k=1}^{n}\sum_{\substack{h_1,\,\ldots\,,\,h_r\\ k_1,\,\ldots\,,\,k_s}}^{\mathrm{n}}\!\!\ldots\!\!\sum^{\mathrm{n}} S^{i_1}_{h_1}\ldots\, S^{i_r}_{h_r}\ T^{k_1}_{j_1}\ldots\, T^{k_s}_{j_s}\, T^k_j\ \tilde Z_{k\,k_1\ldots\, k_s}^{\ h_1\ldots\, h_r}[P]\ -\\ &\qquad -\sum_{m=1}^{r}\sum_{i=1}^{n}\sum_{h=1}^{n}\sum_{k=1}^{n}\sum_{v_m=1}^{n}\theta^{i_m}_{i\,v_m}\ T^{i_1\ldots\, v_m\ldots\, i_r}_{j_1\ldots\ldots\ldots\, j_s}[P]\ S^i_h\, T^k_j\ \tilde Z^h_k\ +\\ &\qquad +\sum_{m=1}^{s}\sum_{i=1}^{n}\sum_{h=1}^{n}\sum_{k=1}^{n}\sum_{w_m=1}^{n}\theta^{w_m}_{i\,j_m}\ T^{i_1\ldots\ldots\ldots\, i_r}_{j_1\ldots\, w_m\ldots\, j_s}[P]\ S^i_h\, T^k_j\ \tilde Z^h_k, \end{aligned}\right. \tag{9.6}$$

$$\left\{\begin{aligned} &\tilde Z^i_j = \sum_{h=1}^{n}\sum_{k=1}^{n} T^i_h\, S^k_j\ Z^h_k,\\ &\tilde Z_{j\,j_1\ldots\, j_s}^{\ i_1\ldots\, i_r}[P] = \sum_{k=1}^{n}\sum_{\substack{h_1,\,\ldots\,,\,h_r\\ k_1,\,\ldots\,,\,k_s}}^{\mathrm{n}}\!\!\ldots\!\!\sum^{\mathrm{n}} T^{i_1}_{h_1}\ldots\, T^{i_r}_{h_r}\ S^{k_1}_{j_1}\ldots\, S^{k_s}_{j_s}\, S^k_j\ Z_{k\,k_1\ldots\, k_s}^{\ h_1\ldots\, h_r}[P]\ -\\ &\qquad -\sum_{m=1}^{r}\sum_{i=1}^{n}\sum_{h=1}^{n}\sum_{k=1}^{n}\sum_{v_m=1}^{n}\tilde\theta^{i_m}_{i\,v_m}\ T^{i_1\ldots\, v_m\ldots\, i_r}_{j_1\ldots\ldots\ldots\, j_s}[P]\ T^i_h\, S^k_j\ Z^h_k\ +\\ &\qquad +\sum_{m=1}^{s}\sum_{i=1}^{n}\sum_{h=1}^{n}\sum_{k=1}^{n}\sum_{w_m=1}^{n}\tilde\theta^{w_m}_{i\,j_m}\ T^{i_1\ldots\ldots\ldots\, i_r}_{j_1\ldots\, w_m\ldots\, j_s}[P]\ T^i_h\, S^k_j\ Z^h_k. \end{aligned}\right. \tag{9.7}$$

Similarly, from (7.16) and (7.17) we derive the transformation formulas for the quantities $\Gamma^k_{j\,i}$ in the formula (9.4):

$$\Gamma^k_{j\,i}=\sum_{b=1}^{n}\sum_{a=1}^{n}\sum_{c=1}^{n}S^k_a\,T^b_i\,T^c_j\;\tilde\Gamma^a_{c\,b}+\sum_{a=1}^{n}Z^a_j\,\theta^k_{ai},\tag{9.8}$$

$$\tilde\Gamma^k_{j\,i}=\sum_{b=1}^{n}\sum_{a=1}^{n}\sum_{c=1}^{n}T^k_a\,S^b_i\,S^c_j\;\Gamma^a_{c\,b}+\sum_{a=1}^{n}\tilde Z^a_j\,\tilde\theta^k_{ai}.\tag{9.9}$$

The formula (9.5) yields the explicit expression for an arbitrary covariant derivative in general case. However, below we consider some specializations of this formula which appear to be more valuable than the formula (9.5) itself.

10. Degenerate covariant differentiations.

Definition 10.1. A covariant differentiation ∇ is said to be *degenerate* if $\nabla_{\mathbf{Y}}\psi=0$ for any extended scalar field ψ and for any extended vector field $\mathbf{Y}$.

This definition is concordant with the definition 8.1. If ∇ is a degenerate covariant differentiation, then $D=\nabla_{\mathbf{Y}}$ is a degenerate differentiation for any extended vector field $\mathbf{Y}$. According to the theorem 8.3 and formula (8.14), D is associated with some extended tensor field $\mathbf{S}$ of the type $(1,1)$. In the present case this field should depend of $\mathbf{Y}$, so the homomorphism $\nabla:\ T^1_0(M)\to\mathfrak{D}(M)$ in the definition 9.1 reduces to the homomorphism

$$T^1_0(M)\to T^1_1(M).\tag{10.1}$$

Applying the theorem 8.1 to the homomorphism (10.1) we derive the following theorem for degenerate covariant differentiations.

Theorem 10.1. *Defining a degenerate covariant differentiation ∇ of the algebra $\mathbf{T}(M)$ is equivalent to defining some extended tensor field $\mathbf{S}$ of the type $(1,2)$.*

Like the theorem 8.3, this theorem can be understood in a coordinate form. Indeed, if ∇ is degenerate, the vector field (7.2) associated with the differentiation $D=\nabla_{\mathbf{Y}}$ should be identically zero for any extended vector field $\mathbf{Y}$. This means that the coefficients $Z^{\,i_1\ldots\,i_r}_{j\,j_1\ldots\,j_s}[P]$ and Z^i_j in (9.3) and (9.4) are equal to zero. Then the transformation formulas (9.8) and (9.9) are written as follows:

$$\Gamma^k_{j\,i}=\sum_{b=1}^{n}\sum_{a=1}^{n}\sum_{c=1}^{n}S^k_a\,T^b_i\,T^c_j\;\tilde\Gamma^a_{c\,b},\tag{10.2}$$

$$\tilde\Gamma^k_{j\,i}=\sum_{b=1}^{n}\sum_{a=1}^{n}\sum_{c=1}^{n}T^k_a\,S^b_i\,S^c_j\;\Gamma^a_{c\,b}.\tag{10.3}$$

Comparing (10.2) and (10.3) with (1.12) and (1.13), we see that $\Gamma^k_{j\,i}$ behave like the components of a tensor of the type $(1,2)$. They are that very quantities that define the extended tensor field $\mathbf{S}$ in a local chart.

11. Horizontal and vertical covariant differentiations.

Suppose again that some composite tensor bundle $N = T^{r_1 \dots r_Q}_{s_1 \dots s_Q} M$ over a base manifold M is fixed. Let ∇ be a covariant differentiation of the algebra of extended tensor fields $\mathbf{T}(M)$. Then $D = \nabla_{\mathbf{Y}}$ is a differentiation of $\mathbf{T}(M)$, its restriction to the set of scalar fields is given by some vector field $\mathbf{Z} = \mathbf{Z}(\mathbf{Y})$ in N (see formula (7.2) above). In other words, we have a homomorphism

$$T^1_0(M) \to T^1_0(N) \tag{11.1}$$

that maps an extended vector field $\mathbf{Y}$ of M to some regular vector field of N. Applying the localization theorem 8.1 to the homomorphism (11.1), we come to the following definition and the theorem after it.

Definition 11.1. Suppose that for each point q of the composite tensor bundle N over the base M some $\mathbb{R}$-linear map of the vector spaces

$$f\colon\ T_{\pi(q)}(M) \to T_q(N) \tag{11.2}$$

is given. Then we say that a *lift* of vectors from M to the bundle N is defined.

Theorem 11.1. *Any homomorphism of $\mathfrak{F}(N)$-modules (11.1) is uniquely associated with some smooth lift of vectors from M to N. It is represented by this lift as a collection of $\mathbb{R}$-linear maps $T_{\pi(q)}(M) \to T_q(N)$ specific to each point $q \in N$.*

Now let's consider the canonical projection $\pi\colon\ N \to M$. The differential of this map acts in the direction opposite to the lift of vectors (11.2) introduced in the definition 11.1. Indeed, we have $\pi_*:\ T_q(N) \to T_{\pi(q)}(M)$ at each point $q \in N$. Therefore, the composition $f \circ \pi_*$ acts from $T_{\pi(q)}(M)$ to $T_{\pi(q)}(M)$. This composite map determines an extended operator field (a tensor field of the type $(1, 1)$).

Definition 11.2. A lift of vectors f from M to N is called *vertical* if $\pi_* \circ f = 0$.

Definition 11.3. A lift of vectors f from M to N is called *horizontal* if $\pi_* \circ f = \mathbf{id}$, i. e. if the composition $\pi_* \circ f$ coincides with the field of identical operators.

Like any other bundle, the composite tensor bundle N naturally subdivides into fibers over the points of the base manifold M. The set of vectors tangent to the fiber at a point q is a linear subspace within the tangent space $T_q(N)$. This subspace coincides with the kernel of the mapping π_*. We denote this subspace

$$V_q(N) = \operatorname{Ker} \pi_* \subset T_q(N) \tag{11.3}$$

and call it the *vertical subspace*. Any vertical lift of vectors determines a set linear maps from $T_{\pi(q)}$ to the vertical subspace (11.3) for each point $q \in N$.

Lemma 11.1. *The difference of two horizontal lifts is a vertical lift of vectors from the base manifold M to the bundle N.*

Indeed, if one takes two horizontal lifts of vectors f_1 and f_2, then $\pi_* \circ (f_1 - f_2) = = \pi_* \circ f_1 - \pi_* \circ f_2 = \mathbf{id} - \mathbf{id} = 0$. This means that the difference $f_1 - f_2$ is a vertical lift according to the definition 11.2.

Each covariant differentiation ∇ is associated with some lift of vectors (see the definition 11.1, and the theorem 11.1 above).

Definition 11.4. A covariant differentiation ∇ of the algebra of extended tensor fields $\mathbf{T}(M)$ is called a *horizontal differentiation* (or a *vertical differentiation*) if the corresponding lift of vectors is *horizontal* (or *vertical*).

Lemma 11.2. *The difference of two horizontal covariant differentiations is a vertical covariant differentiation of the algebra of extended tensor fields* $\mathbf{T}(M)$*.*

The lemma 11.2 is an immediate consequence of the lemma 11.1. Unlike [4], here we shall not pay much attention to vertical covariant differentiations. In the present more general theory they are replaced by a more general construct.

12. Native extended tensor fields and vertical multivariate differentiations.

Let $N = T^{r_1\ldots\, r_Q}_{s_1\ldots\, s_Q}M$ be a composite tensor bundle over a base manifold M. Then each its point q is represented by a list $q = (p,\, \mathbf{T}[1],\, \ldots\,,\, \mathbf{T}[Q])$, where $p \in M$ and $\mathbf{T}[1],\, \ldots\,,\, \mathbf{T}[Q]$ are some tensors at the point p (see formula (3.2) above). Let's consider the map that takes q to the P-th tensor $\mathbf{T}[P]$ in the list. According to the definition 4.1, this map is an extended tensor field of the type (r_P, s_P). It is canonically associated with the bundle N. Therefore, it is called a *native extended tensor field*. Totally, we have Q native extended tensor fields associated with the composite tensor bundle $N = T^{r_1\ldots\, r_Q}_{s_1\ldots\, s_Q}M$, we denote them $\mathbf{T}[1],\, \ldots\,,\, \mathbf{T}[Q]$.

Definition 12.1. A *multivariate differentiation* of the type (β, α) in the algebra of extended tensor fields $\mathbf{T}(M)$ is a homomorphism of $\mathfrak{F}(N)$-modules

$$\nabla\colon\, T^{\alpha}_{\beta}(M) \to \mathfrak{D}(M). \tag{12.1}$$

If $\mathbf{Y}$ is an extended tensor field of the type (α, β), then we can apply the homomorphism (12.1) to it. As a result we get the differentiation $D = \nabla_{\mathbf{Y}}$ of the algebra $\mathbf{T}(M)$. It is called the *multivariate differentiation along the tensor field* $\mathbf{Y}$.

Note that the type of a multivariate differentiation (β, α) in the above definition 12.1 is dual to the type of the module $T^{\alpha}_{\beta}(M)$ in the formula (12.1). If $\alpha = 1$ and $\beta = 0$, the definition 12.1 reduces to the definition 9.1. This means that a covariant differentiation is a special multivariate differentiation whose type is $(0, 1)$. Similarly, a multivariate differentiation of the type $(1, 0)$ is called a *contravariant differentiation*. Covariant differentiations of extended tensor fields appear to be a useful tool in describing Newtonian dynamical systems in Riemannian manifolds (see [4–23]). The same is true for contravariant differentiations in the case of Hamiltonian dynamical systems (see [24–30]). As for general multivariate differentiations introduced in the above definition 11.1, I think they will find their proper place in theories of continuous media (see [34] and [36–39]) and in field theories.

A remark. Let's consider the special case, where the tensor field $\mathbf{Y}$ of the type (α, β) is constructed as a tensor product:

$$\mathbf{Y} = \mathbf{Y}[1] \otimes \ldots \otimes \mathbf{Y}[\alpha] \otimes \mathbf{H}[1] \otimes \ldots \otimes \mathbf{H}[\beta]. \tag{12.2}$$

Here $\mathbf{Y}[1],\, \ldots\,,\, \mathbf{Y}[\alpha]$ are some vector fields and $\mathbf{H}[1],\, \ldots\,,\, \mathbf{H}[\beta]$ are some covector fields. Substituting (12.2) into $\nabla_{\mathbf{Y}}$, we find that $\nabla_{\mathbf{Y}}$ is a differentiation depending on α vectorial variables and β covectorial variables. Keeping in mind this special case, we used the term «multivariate differentiation» for $\nabla_{\mathbf{Y}}$ in the definition 12.1.

Let ∇ be some multivariate differentiation of the algebra of extended tensor fields $\mathbf{T}(M)$. Then, applying the localization theorem 8.1 to the homomorphism $\nabla\colon\, T^{\alpha}_{\beta}(M) \to \mathfrak{D}(M)$, we find that this homomorphism is composed by $\mathbb{R}$-linear maps $T^{\alpha}_{\beta}(\pi(q), M) \to \mathfrak{D}(q, M)$ specific to each point $q \in N$. This fact is expressed by the following formula similar to the formula (9.2) above:

$$\begin{aligned}\mathbf{Y} \mapsto \nabla_{\mathbf{Y}}\mathbf{X} = \sum^{\mathrm{n}}_{\substack{i_1,\,\ldots\,,\,i_\alpha\\ j_1,\,\ldots\,,\,j_\beta}}\!\!\ldots\!\!\sum^{\mathrm{n}} \sum^{\mathrm{n}}_{\substack{h_1,\,\ldots\,,\,h_r\\ k_1,\,\ldots\,,\,k_s}}\!\!\ldots\!\!\sum^{\mathrm{n}} Y^{h_1\ldots\,h_r}_{k_1\ldots\,k_s}\ \nabla^{k_1\ldots\,k_s}_{h_1\ldots\,h_r} X^{i_1\ldots\,i_\alpha}_{j_1\ldots\,j_\beta}\ \times\\ \times\ \mathbf{E}^{j_1\ldots\,j_\beta}_{i_1\ldots\,i_\alpha} = C(\mathbf{Y} \otimes \nabla\mathbf{X}).\end{aligned}\tag{12.3}$$

Looking at (12.3), we see that each multivariate differentiation ∇ of the type (s, r) can be treated as an operator producing the extended tensor field $\nabla\mathbf{X}$ of the type $(\alpha + s, \beta + r)$ from any given extended tensor field $\mathbf{X}$ of the type (α, β). This operator is called the operator of *multivariate differential* of the type (s, r).

Let P be an integer number such that $1 \leqslant P \leqslant Q$ and let $\mathbf{Y}$ be an extended tensor field of the type (r_P, s_P). Remember that each point q of the composite tensor bundle $N = T^{r_1\ldots\,r_Q}_{s_1\ldots\,s_Q}M$ is a list of the form (3.2):

$$q = (p,\, \mathbf{T}[1],\, \ldots\,,\, \mathbf{T}[Q]).\tag{12.4}$$

Note that the P-th tensor $\mathbf{T}[P]$ in the list (12.4) has the same type as the tensor $\mathbf{Y} = \mathbf{Y}_q$ (the value of the extended tensor field $\mathbf{Y}$ at the point q). They both belong to the same tensor space $T^{r_P}_{s_P}(p, M)$, therefore we can add them. Then

$$q(t) = (p,\, \mathbf{T}[1],\, \ldots\,,\, \mathbf{T}[P] + t\,\mathbf{Y}_q,\, \ldots\,,\, \mathbf{T}[Q])\tag{12.5}$$

is a one-parametric set of points in N, the scalar variable t being its parameter. In other words, in (12.5) we have a line (a straight line) passing through the initial point $q \in N$ and lying completely within the fiber over the point $p = \pi(q) \in M$. Suppose that $\mathbf{X}$ is some extended tensor field of the type (α, β). Denote by $\mathbf{X}(t)$ the values of this tensor field at the points of the above parametric line (12.5):

$$\mathbf{X}(t) = \mathbf{X}_{q(t)}.\tag{12.6}$$

Since $\pi(q(t)) = p = \mathrm{const}$ for any t, the values of the tensor-valued function (12.6) all belong to the same tensor space $T^{\alpha}_{\beta}(p, M)$. Hence, we can add and subtract them, and, since $\mathbf{X}$ is a smooth field, we can take the following limit of the ratio:

$$\dot{X}(t) = \lim_{\tau \to 0} \frac{\mathbf{X}(t + \tau) - \mathbf{X}(t)}{\tau}.\tag{12.7}$$

Let's denote by $\mathbf{Z}_q$ the value of the derivative (12.7) for $t = 0$:

$$\mathbf{Z}_q = \dot{X}(0) = \left.\frac{d\mathbf{X}_{q(t)}}{dt}\right|_{t=0}.\tag{12.8}$$

It is easy to understand that, when q is fixed, $\mathbf{Z}_q$ is a tensor from the tensor space

$T^\alpha_\beta(p, M)$ at the point $p = \pi(q)$. By varying $q \in N$, we find that the tensors $\mathbf{Z}_q$ constitute a smooth extended tensor field $\mathbf{Z}$. So, we have constructed a map

$$D\colon\ \mathbf{T}(M) \to \mathbf{T}(M). \tag{12.9}$$

It is easy to check up that the map (12.9) defined by means of the formulas (12.5), (12.6), (12.7), and (12.8) is a differentiation of the algebra of extended tensor fields $\mathbf{T}(M)$, i.e. $D \in \mathfrak{D}(M)$ (see definition 6.2 above). Moreover, due to the formula (12.5) this differentiation D depends on the extended tensor field $\mathbf{Y}$. This dependence $D = D(\mathbf{Y})$ is a homomorphism $T^{r_P}_{s_P}(M) \to \mathfrak{D}(M)$ fitting the definition 12.1. The easiest way to prove this fact is to write the equality (12.8) in a local chart, i.e. in some local coordinates (3.3):

$$\mathbf{Z} = \sum^{\mathrm{n}}_{\substack{i_1,\,\ldots\,,\,i_\alpha\\ j_1,\,\ldots\,,\,j_\beta}}\ldots\sum^{\mathrm{n}}\ \sum^{\mathrm{n}}_{\substack{h_1,\,\ldots\,,\,h_r\\ k_1,\,\ldots\,,\,k_s}}\ldots\sum^{\mathrm{n}} Y^{h_1\ldots\,h_r}_{k_1\ldots\,k_s}\ \frac{\partial X^{i_1\ldots\,i_\alpha}_{j_1\ldots\,j_\beta}}{\partial T^{h_1\ldots\,h_r}_{k_1\ldots\,k_s}[P]}\ \mathbf{E}^{j_1\ldots\,j_\beta}_{i_1\ldots\,i_\alpha}. \tag{12.10}$$

Here $r = r_P$ and $s = s_P$. Now, comparing (12.10) with the formula (12.3), we can write $\mathbf{Z} = \nabla\!\!\!\!\nabla_{\mathbf{Y}}[P]\mathbf{X}$, where $\nabla\!\!\!\!\nabla$ is a special sign, the «double bar nabla», that we shall use for denoting the multivariate differentiations defined through the formulas (12.5), (12.6), (12.7), and (12.8). In a local chart $\nabla\!\!\!\!\nabla[P]$ is represented by the formula

$$\nabla\!\!\!\!\nabla^{k_1\ldots\,k_s}_{h_1\ldots\,h_r}[P] = \frac{\partial}{\partial T^{h_1\ldots\,h_r}_{k_1\ldots\,k_s}[P]}, \tag{12.11}$$

where $r = r_P$ and $s = s_P$. This formula (12.11) is a short version of the formula (12.10). Following the tradition, we shall use the term *multivariate derivative* for the differential operator representing the differentiation $\nabla\!\!\!\!\nabla[P]$ in local coordinates.

Definition 12.2. The multivariate differentiation $\nabla\!\!\!\!\nabla[P]$ defined through the formulas (12.5), (12.6), (12.7), (12.8) and represented by the formula (12.11) in local coordinates is called the *P-th canonical*[1] *vertical multivariate differentiation* associated with the composite tensor bundle $N = T^{r_1\ldots\,r_Q}_{s_1\ldots\,s_Q}M$.

Let $\mathbf{T}[R]$ be R-th native extended tensor field associated with the tensor bundle $N = T^{r_1\ldots\,r_Q}_{s_1\ldots\,s_Q}M$ and let $\mathbf{Y}$ be some arbitrary extended tensor field of the type (r_P, s_P). Then we can apply $\nabla\!\!\!\!\nabla_{\mathbf{Y}}[P]$ to $\mathbf{T}[R]$. By means of the direct calculations using the explicit formula (12.11) in local coordinates we find that

$$\nabla\!\!\!\!\nabla_{\mathbf{Y}}[P]\mathbf{T}[R] = \begin{cases} \mathbf{Y} & \text{for}\quad P = R,\\ 0 & \text{for}\quad P \neq R.\end{cases}$$

Like covariant differentiations (see theorem 11.1 and definition 11.4), multivariate differentiations are associated with some lifts. However, unlike covariant differentiations, they lift not vectors, but tensors, though converting them into tangent vectors of the bundle N. In the case of the canonical multivariate differentiation

[1] Note that $\nabla\!\!\!\!\nabla[P]$ is canonically associated with the bundle N, its definition does not require any auxiliary structures like metrics and connections.

$\nabla[P]$ for each point $q \in N$ we have some $\mathbb{R}$-linear map

$$f[P]\colon\, T^r_s(\pi(q), M) \to T_q(N), \tag{12.12}$$

where $r = r_P$ and $s = s_P$. The map (12.12) takes a tensor $\mathbf{Y} \in T^r_s(\pi(q), M)$ to the following vector in the tangent space $T_q(N)$ of the manifold N at the point q:

$$f[P](\mathbf{Y}) = \sum^{\mathrm{n}}_{\substack{h_1,\,\ldots\,,\,h_r\\k_1,\,\ldots\,,\,k_s}}\ldots\sum^{\mathrm{n}} Y^{h_1\ldots\,h_r}_{k_1\ldots\,k_s}\ \mathbf{V}^{k_1\ldots\,k_s}_{h_1\ldots\,h_r}[P]. \tag{12.13}$$

Here again $r = r_P$, $s = s_P$, and the vectors $\mathbf{V}^{k_1\ldots\,k_s}_{h_1\ldots\,h_r}[P]$ are given by the second formula (3.7). In a coordinate-free form the formula (12.13) can be interpreted as follows: the vector $f[P](\mathbf{Y})$ in (12.13) is the tangent vector of the parametric curve (12.5) at its initial point $q = q(0)$.

Let's consider the image of the $\mathbb{R}$-linear map (12.12). We denote it $V_q[P](N)$. Then from (12.13) one easily derives that $V_q[P](N)$ is a subspace within the vertical subspace $V_q(N)$ of the tangent space $T_q(N)$. Moreover, we have

$$V_q(N) = V_q[1](N) \oplus \ldots \oplus V_q[Q](N). \tag{12.14}$$

The formula (12.14) is a well-known fact, it follows from (3.1). Due to the inclusion

$$\operatorname{Im} f[P] = V_q[P](N) \subset V_q(N)$$

the multivariate differentiation (12.11) is a vertical differentiation.

13. Horizontal covariant differentiations and extended connections.

Let ∇ be some horizontal covariant differentiation of the algebra of extended tensor fields $\mathbf{T}(M)$ and let f be the horizontal lift of vectors associated with it (see definition 11.4). The horizontality of f means that the image of the linear map (11.2) is some n-dimensional subspace $H_q(N)$ within the tangent space $T_q(N)$. It is called a *horizontal subspace*. Due to $\pi_* \circ f = \mathbf{id}$ the mappings

$$f\colon T_{\pi(q)}(M) \to H_q(N), \qquad\qquad \pi_*\colon H_q(N) \to T_{\pi(q)}(M) \tag{13.1}$$

are inverse to each other. Due to the same equality $\pi_* \circ f = \mathbf{id}$ the sum of the vertical and horizontal subspaces is a direct sum:

$$H_q(N) \oplus V_q(N) = T_q(N). \tag{13.2}$$

Theorem 13.1. *Defining a horizontal lift of vectors from M to N is equivalent to fixing some direct complement $H_q(N)$ of the vertical subspace $V_q(N)$ within the tangent space $T_q(N)$ at each point $q \in N$.*

Proof. Suppose that some horizontal lift of vectors f is given. Then the subspace

$H_q(N)$ at the point q is determined as the image of the mapping (11.2), while the relationship (13.2) is derived from $\pi_*\circ f=\mathbf{id}$ and from (11.3).

Conversely, assume that at each point $q\in N$ we have a subspace $H_q(N)$ complementary to $V_q(N)$. Then at each point $q\in N$ the relationship (13.2) is fulfilled. The kernel of the mapping $\pi_*\colon T_q(N)\to T_{\pi(q)}(M)$ coincides with $V_q(N)$, therefore the restriction of π_* to the horizontal subspace $H_q(N)$ is a bijection. The lift of vectors f from M to N then can be defined as the inverse mapping for $\pi_*\colon H_q(N)\to T_{\pi(q)}(M)$. If f is defined in this way, then the mappings (13.1) appear to be inverse to each other and we get the equality $\pi_*\circ f=\mathbf{id}$. According to the definition 11.3, it means that f is a horizontal mapping. The theorem is completely proved. $\square$

Let's study a horizontal lift of vectors f in a coordinate form. Upon choosing some local chart in M we can consider the coordinate vector fields $\mathbf{E}_1,\,\ldots\,,\,\mathbf{E}_n$ in this chart (see (1.3)). Applying the lift f to them, we get

$$f(\mathbf{E}_j)=\mathbf{U}_j-\sum^{Q}_{P=1}\sum^{\mathrm{n}}_{\substack{i_1,\,\ldots\,,\,i_r\\ j_1,\,\ldots\,,\,j_s}}\!\!\ldots\!\!\sum^{\mathrm{n}}\Gamma^{\,i_1\ldots\,i_r}_{j\,j_1\ldots\,j_s}[P]\ \mathbf{V}^{j_1\ldots\,j_s}_{i_1\ldots\,i_r}[P]. \tag{13.3}$$

Here $r=r_P$ and $s=s_P$, while $\mathbf{U}_j$ and $\mathbf{V}^{j_1\ldots\,j_s}_{i_1\ldots\,i_r}[P]$ are determined by (3.7) This formula for $f(\mathbf{E}_j)$ follows from $\pi_*\circ f=\mathbf{id}$ due to the equalities

$$\pi_*(\mathbf{U}_j)=\mathbf{E}_j, \qquad\qquad \pi_*\Bigl(\mathbf{V}^{j_1\ldots\,j_s}_{i_1\ldots\,i_r}[P]\Bigr)=0. \tag{13.4}$$

The quantities $\Gamma^{\,i_1\ldots\,i_r}_{j\,j_1\ldots\,j_s}[P]$ in (13.3) are called the *components* of a horizontal lift of vectors in a local chart U. If the lift f is induced by some horizontal covariant differentiation ∇, then for its components in(13.4) we have

$$\Gamma^{\,i_1\ldots\,i_r}_{j\,j_1\ldots\,j_s}[P]=-Z^{\,i_1\ldots\,i_r}_{j\,j_1\ldots\,j_s}[P]. \tag{13.5}$$

The quantities $Z^{\,i_1\ldots\,i_r}_{j\,j_1\ldots\,j_s}[P]$ in (13.5) are the same as in (9.3), (9.5), (9.6), and in (9.7). As for the quantities Z^i_j in (9.4), in the case of a horizontal covariant differentiation they are given by the Kronecker's delta-symbol: $Z^i_j=\delta^i_j$. Substituting $Z^i_j=\tilde Z^i_j=\delta^i_j$ into (9.6) and taking into account (13.5), we derive

$$\begin{cases}\displaystyle\Gamma^{\,i_1\ldots\,i_r}_{j\,j_1\ldots\,j_s}[P]=\sum^{n}_{k=1}\sum^{\mathrm{n}}_{\substack{h_1,\,\ldots\,,\,h_r\\ k_1,\,\ldots\,,\,k_s}}\!\!\ldots\!\!\sum^{\mathrm{n}}S^{i_1}_{h_1}\ldots\,S^{i_r}_{h_r}\,T^{k_1}_{j_1}\ldots\,T^{k_s}_{j_s}\,T^k_j\,\tilde\Gamma^{\,h_1\ldots\,h_r}_{k\,k_1\ldots\,k_s}[P]\,+\\[2ex] \displaystyle+\sum^{r}_{m=1}\sum^{n}_{v_m=1}\theta^{i_m}_{j\,v_m}\,T^{\,i_1\ldots\,v_m\ldots\,i_r}_{j_1\ldots\ldots\ldots\,j_s}[P]-\sum^{s}_{m=1}\sum^{n}_{w_m=1}\theta^{w_m}_{j\,j_m}\,T^{\,i_1\ldots\ldots\ldots\,i_r}_{j_1\ldots\,w_m\ldots\,j_s}[P],\end{cases} \tag{13.6}$$

$$\begin{cases}\displaystyle\tilde\Gamma^{\,i_1\ldots\,i_r}_{j\,j_1\ldots\,j_s}[P]=\sum^{n}_{k=1}\sum^{\mathrm{n}}_{\substack{h_1,\,\ldots\,,\,h_r\\ k_1,\,\ldots\,,\,k_s}}\!\!\ldots\!\!\sum^{\mathrm{n}}T^{i_1}_{h_1}\ldots\,T^{i_r}_{h_r}\,S^{k_1}_{j_1}\ldots\,S^{k_s}_{j_s}\,S^k_j\,\Gamma^{\,h_1\ldots\,h_r}_{k\,k_1\ldots\,k_s}[P]\,+\\[2ex] \displaystyle+\sum^{r}_{m=1}\sum^{n}_{v_m=1}\tilde\theta^{i_m}_{j\,v_m}\,\tilde T^{\,i_1\ldots\,v_m\ldots\,i_r}_{j_1\ldots\ldots\ldots\,j_s}[P]-\sum^{s}_{m=1}\sum^{n}_{w_m=1}\tilde\theta^{w_m}_{j\,j_m}\,\tilde T^{\,i_1\ldots\ldots\ldots\,i_r}_{j_1\ldots\,w_m\ldots\,j_s}[P],\end{cases} \tag{13.7}$$

Here $r = r_P$, $s = s_P$. The θ-parameters are taken from (3.15), (3.10), and (3.12). The formulas (13.6) and (13.7) express the transformation rules for the components of a horizontal lift of vectors in (13.3).

Another geometric structure associated with a horizontal covariant differentiation ∇ reveals when we apply $D = \nabla_{\mathbf{Y}}$ to the module $T^1_0(M)$. This produces the mapping (7.23). In a local chart it is described by the formula (7.13). In the present case we can take $\mathbf{Y} = \hat{\mathbf{E}}_j$ and write this formula as

$$\nabla_{\hat{\mathbf{E}}_j} \hat{\mathbf{E}}_i = \sum_{k=1}^{n} \Gamma^k_{j\,i}\,\mathbf{E}_k. \tag{13.8}$$

Here $\hat{\mathbf{E}}_i$ and $\hat{\mathbf{E}}_j$ are determined by the formula (7.13). Relying on the lemma 7.4 and on the localization theorem 8.1, we can write (13.8) as follows:

$$\nabla_{\mathbf{E}_j} \mathbf{E}_i = \sum_{k=1}^{n} \Gamma^k_{j\,i}\,\mathbf{E}_k. \tag{13.9}$$

The coefficients $\Gamma^k_{j\,i}$ are the same as in (9.4). Since $Z^i_j = \delta^i_j$ for a horizontal covariant differentiation, the transformation formulas (9.8) and (9.9) for the coefficients $\Gamma^k_{j\,i}$ in (13.8) and (13.9) now reduce to the following ones:

$$\Gamma^k_{j\,i} = \sum_{b=1}^{n}\sum_{a=1}^{n}\sum_{c=1}^{n} S^k_a\,T^b_i\,T^c_j\;\tilde{\Gamma}^a_{c\,b} + \sum_{a=1}^{n} \theta^k_{j\,i}, \tag{13.10}$$

$$\tilde{\Gamma}^k_{j\,i} = \sum_{b=1}^{n}\sum_{a=1}^{n}\sum_{c=1}^{n} T^k_a\,S^b_i\,S^c_j\;\Gamma^a_{c\,b} + \sum_{a=1}^{n} \tilde{\theta}^k_{j\,i}. \tag{13.11}$$

Definition 13.1. Let M be a smooth manifold and let $N = T^{r_1 \dots r_Q}_{s_1 \dots s_Q}M$ be a composite tensor bundle over M. An extended affine connection Γ is a geometric object in each local chart of M represented by its components $\Gamma^k_{j\,i}$ and such that its components are smooth functions of the variables (3.3) transformed according to the formulas (13.10) and (13.11) under a change of a local chart.

Theorem 13.2. *On any smooth paracompact manifold M equipped with a composite tensor bundle $N = T^{r_1 \dots r_Q}_{s_1 \dots s_Q}M$ there is at least one extended affine connection.*

We shall not prove this theorem here. Its proof for the spacial case, where $N = TM$, is given in Chapter III of the thesis [4]. This proof can be easily transformed for the present more general case. Note also that any traditional affine connection fits the above definition 13.1 being a special case for this more general concept of an extended connection.

Definition 13.2. A horizontal covariant differentiation ∇ of the algebra of extended tensor fields $\mathbf{T}(M)$ associated with some composite tensor bundle $T^{r_1 \dots r_Q}_{s_1 \dots s_Q}M$ is called a *spatial covariant differentiation* or a *spatial gradient* if

$$\nabla\mathbf{T}[P] = 0 \quad \text{for all} \quad P = 1, \dots, Q, \tag{13.12}$$

i. e. if the operator ∇ annuls all native extended tensor fields $\mathbf{T}[1], \dots, \mathbf{T}[Q]$.

Let's study the equality (13.12) specifying spatial covariant differentiations. For this purpose we use the formula (9.5) substituting $Z^i_j = \delta^i_j$ and (13.5) into it:

$$
\begin{gathered}
\nabla_{\!j} X^{i_1\ldots\,i_\alpha}_{j_1\ldots\,j_\beta} = \frac{\partial X^{i_1\ldots\,i_\alpha}_{j_1\ldots\,j_\beta}}{\partial x^j} - \sum^{Q}_{R=1}\sum^{\mathrm{n}}_{\substack{h_1,\,\ldots\,,\,h_r\\ k_1,\,\ldots\,,\,k_s}}\!\!\ldots\sum^{\mathrm{n}}\Gamma^{\,h_1\ldots\,h_r}_{\,j\,k_1\ldots\,k_s}[R]\;\frac{\partial X^{i_1\ldots\,i_\alpha}_{j_1\ldots\,j_\beta}}{\partial T^{h_1\ldots\,h_r}_{k_1\ldots\,k_s}[R]} \;+\\
+\sum^{\alpha}_{m=1}\sum^{n}_{v_m=1}\Gamma^{i_m}_{j\,v_m}\;X^{i_1\ldots\,v_m\ldots\,i_\alpha}_{j_1\ldots\ldots\ldots\,j_\beta} - \sum^{\beta}_{m=1}\sum^{n}_{w_m=1}\Gamma^{w_m}_{j\,j_m}\;X^{i_1\ldots\ldots\ldots\,i_\alpha}_{j_1\ldots\,w_m\ldots\,j_\beta},
\end{gathered}
\tag{13.13}
$$

According to the formula (13.12), we should substitute $\alpha = r = r_P$, $\beta = s = s_P$, and $X^{i_1\ldots\,i_\alpha}_{j_1\ldots\,j_\beta} = T^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}[P]$ into the formula (13.13). Recall that the quantities $T^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}[P]$ in (9.5) and (13.13) are treated as independent variables. Therefore, we get

$$
\begin{gathered}
\nabla_{\!j} T^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}[P] = -\Gamma^{\,i_1\ldots\,i_r}_{\,j\,j_1\ldots\,j_s}[P] \;+\\
+\sum^{r}_{m=1}\sum^{n}_{v_m=1}\Gamma^{i_m}_{j\,v_m}\;T^{i_1\ldots\,v_m\ldots\,i_r}_{j_1\ldots\ldots\ldots\,j_s}[P] - \sum^{s}_{m=1}\sum^{n}_{w_m=1}\Gamma^{w_m}_{j\,j_m}\;T^{i_1\ldots\ldots\ldots\,i_r}_{j_1\ldots\,w_m\ldots\,j_s}[P] = 0.
\end{gathered}
$$

This formula can be rewritten in the following form:

$$
\begin{gathered}
\Gamma^{\,i_1\ldots\,i_r}_{\,j\,j_1\ldots\,j_s}[P] = \sum^{r}_{m=1}\sum^{n}_{v_m=1}\Gamma^{i_m}_{j\,v_m}\;T^{i_1\ldots\,v_m\ldots\,i_r}_{j_1\ldots\ldots\ldots\,j_s}[P] \;-\\
-\sum^{s}_{m=1}\sum^{n}_{w_m=1}\Gamma^{w_m}_{j\,j_m}\;T^{i_1\ldots\ldots\ldots\,i_r}_{j_1\ldots\,w_m\ldots\,j_s}[P].
\end{gathered}
\tag{13.14}
$$

Substituting (13.14) back into (13.13), we derive

$$
\begin{gathered}
\nabla_{\!j} X^{i_1\ldots\,i_\alpha}_{j_1\ldots\,j_\beta} = \frac{\partial X^{i_1\ldots\,i_\alpha}_{j_1\ldots\,j_\beta}}{\partial x^j} + \sum^{\alpha}_{m=1}\sum^{n}_{v_m=1}\Gamma^{i_m}_{j\,v_m}\;X^{i_1\ldots\,v_m\ldots\,i_\alpha}_{j_1\ldots\ldots\ldots\,j_\beta} \;-\\
-\sum^{Q}_{R=1}\sum^{\mathrm{n}}_{\substack{h_1,\,\ldots\,,\,h_r\\ k_1,\,\ldots\,,\,k_s}}\!\!\ldots\sum^{\mathrm{n}}\sum^{r}_{m=1}\sum^{n}_{v_m=1}\Gamma^{h_m}_{j\,v_m}\;T^{\,h_1\ldots\,v_m\ldots\,h_r}_{\,k_1\ldots\ldots\ldots\,k_s}[R]\;\frac{\partial X^{i_1\ldots\,i_\alpha}_{j_1\ldots\,j_\beta}}{\partial T^{h_1\ldots\,h_r}_{k_1\ldots\,k_s}[R]} \;-\\
-\sum^{\beta}_{m=1}\sum^{n}_{w_m=1}\Gamma^{w_m}_{j\,j_m}\;X^{i_1\ldots\ldots\ldots\,i_\alpha}_{j_1\ldots\,w_m\ldots\,j_\beta} \;+\\
+\sum^{Q}_{R=1}\sum^{\mathrm{n}}_{\substack{h_1,\,\ldots\,,\,h_r\\ k_1,\,\ldots\,,\,k_s}}\!\!\ldots\sum^{\mathrm{n}}\sum^{s}_{m=1}\sum^{n}_{w_m=1}\Gamma^{w_m}_{j\,k_m}\;T^{h_1\ldots\ldots\ldots\,h_r}_{k_1\ldots\,w_m\ldots\,k_s}[R]\;\frac{\partial X^{i_1\ldots\,i_\alpha}_{j_1\ldots\,j_\beta}}{\partial T^{h_1\ldots\,h_r}_{k_1\ldots\,k_s}[R]}.
\end{gathered}
\tag{13.15}
$$

The theorem 7.2 applied to a horizontal covariant differentiation says that any such differentiation is defined by two independent geometric structures:

(1) a horizontal lift of vectors from M to N;
(2) an extended connection.

The formula (13.14) relates these two structures. It expresses the components of the horizontal lift in (13.3) through the components of an extended connection Γ in (13.9). This result is formulated as the following theorem.

Theorem 13.3. *Defining a spacial covariant differentiation in the algebra of extended tensor fields $\mathbf{T}(M)$ is equivalent to defining an extended connection Γ.*

14. The structural theorem for differentiations.

Theorem 14.1. *Let M be a smooth manifold and let $N = T^{r_1 \ldots r_Q}_{s_1 \ldots s_Q} M$ be a composite tensor bundle over M. If M is equipped with some extended affine connection Γ, then each differentiation D of the algebra of extended tensor fields $\mathbf{T}(M)$ in this manifold M is uniquely expanded into a sum*

$$D = \nabla_{\mathbf{X}} + \sum_{P=1}^{Q} \tilde{\nabla}_{\mathbf{Y}_P}[P] + S, \tag{14.1}$$

where $\nabla_{\mathbf{X}}$ is the spacial covariant differentiation along some extended vector field $\mathbf{X}$, $\tilde{\nabla}_{\mathbf{Y}_P}[P]$ is the P-th canonical vertical multivariate differentiation along some extended tensor field $\mathbf{Y}_P$ of the type (r_P, s_P), and S is a degenerate differentiation given by some extended tensor field $\mathbf{S}$ of the type $(1, 1)$.

Proof. Let $D \in \mathfrak{D}(M)$. Then its restriction to $T^0_0(M)$ is given by some vector field $\mathbf{Z}$ in N. The extended affine connection Γ in M determines some horizontal lift of vectors f from M to N. Its components in a local chart are given by the formula (13.14). According to the theorem 13.1, this lift of vectors determines the expansion of the tangent space $T_q(N)$ into a direct sum (13.2) at each point $q \in N$. The vertical subspace $V_q(N)$ in (13.2) has its own expansion (12.14) into a direct sum. Combining (13.2) and (12.14), we obtain

$$T_q(N) = H_q(N) \oplus V_q[1](N) \oplus \ldots \oplus V_q[Q](N). \tag{14.2}$$

Then the vector field $\mathbf{Z}$ is expanded into a sum of vector fields

$$\mathbf{Z} = \mathbf{H} + \mathbf{V}_1 + \ldots + \mathbf{V}_Q \tag{14.3}$$

uniquely determined by the expansion (14.2). Due to the maps (12.12) each vector field $\mathbf{V}_P$ in (14.3) is uniquely associated with some extended tensor field $\mathbf{Y}_P$ of the type (r_P, s_P). Similarly, the vector field $\mathbf{H}$ is uniquely associated with the the extended vector field $\mathbf{X}$ such that $\mathbf{H}_q = f(\mathbf{X}_q)$. Then we can consider the sum

$$\tilde{D} = \nabla_{\mathbf{X}} + \sum_{P=1}^{Q} \tilde{\nabla}_{\mathbf{Y}_P}[P]. \tag{14.4}$$

The sum (14.4) is a differentiation of $\mathbf{T}(M)$ such that its restriction to $T^0_0(M)$ is given by the vector (14.3). Hence, $D-\tilde{D}$ is a differentiation of $\mathbf{T}(M)$ with identically zero restriction to $T^0_0(M)$. This means that $D - \tilde{D}$ is a degenerate differentiation (see definition 8.1). Applying the theorem 8.3, we find that $S = D - \tilde{D}$ is given by some extended tensor field $\mathbf{S}$ of the type $(1, 1)$. Thus, the expansion (14.1) and the theorem 14.1 in whole are proved. □

The theorem 14.1 is the structural theorem for differentiations in the algebra of extended tensor fields $\mathbf{T}(M)$. It approves our previous efforts in studying the three basic types of differentiations which are used in the formula (14.1).

§ 15. Commutation relationships and curvature tensors.

Let's remember that the set of all differentiations $\mathfrak{D}(M)$ is an infinite-dimensional Lie algebra (see formula (6.1)). Using the above structural theorem 14.1, one can give a more detailed description of this Lie algebra. Let's begin with degenerate differentiations. Assume that S_1 and S_2 are two degenerate differentiations given by two extended tensor fields $\mathbf{S}_1$ and $\mathbf{S}_2$ of the type $(1, 1)$. Then

$$[S_1,\, S_2] = S_3, \ \text{ wehere } \ \mathbf{S}_3 = C(\mathbf{S}_1 \otimes \mathbf{S}_2 - \mathbf{S}_2 \otimes \mathbf{S}_1). \tag{15.1}$$

The formula (15.1) means that the commutator of two degenerate differentiations is a degenerate differentiation given by the pointwise commutator of the corresponding extended tensor fields $\mathbf{S}_1$ and $\mathbf{S}_2$.

Assume that M is equipped with an extended affine connection Γ. Then we can consider the commutators of some degenerate differentiation $\mathbf{S}$ with the spacial covariant differentiation $\nabla_{\mathbf{X}}$ and with the P-th canonical vertical multivariate differentiation $\tilde{\nabla}_{\mathbf{Y}}[P]$. These commutators are given by the formula

$$\begin{aligned} &[\nabla_{\mathbf{X}},\, S] = S_1, &&\text{where } \ \mathbf{S}_1 = \nabla_{\mathbf{X}}\mathbf{S};\\ &[\tilde{\nabla}_{\mathbf{Y}}[P],\, \mathbf{S}] = S_2, &&\text{where } \ \mathbf{S}_2 = \tilde{\nabla}_{\mathbf{Y}}[P]\mathbf{S}. \end{aligned} \tag{15.2}$$

The formulas (15.2) mean that both commutators are again degenerate differentiations. They are given by the extended tensor fields $\nabla_{\mathbf{X}}\mathbf{S}$ and $\tilde{\nabla}_{\mathbf{Y}}[P]\mathbf{S}$ respectively.

The commutator of two canonical vertical multivariate differentiations $\tilde{\nabla}_{\mathbf{X}}[P]$ and $\tilde{\nabla}_{\mathbf{Y}}[R]$ is composed by other two such differentiations. Indeed, we have

$$[\tilde{\nabla}_{\mathbf{X}}[P],\, \tilde{\nabla}_{\mathbf{Y}}[R]] = \tilde{\nabla}_{\mathbf{U}}[R] - \tilde{\nabla}_{\mathbf{V}}[P]. \tag{15.3}$$

where $\mathbf{U}$ and $\mathbf{V}$ are determined as follows:

$$\mathbf{U} = \tilde{\nabla}_{\mathbf{X}}[P]\mathbf{Y}, \qquad\qquad \mathbf{V} = \tilde{\nabla}_{\mathbf{Y}}[R]\mathbf{X}. \tag{15.4}$$

Similarly, for the commutator of the spatial covariant differentiation $\nabla_{\mathbf{X}}$ with the canonical vertical multivariate differentiation $\tilde{\nabla}_{\mathbf{Y}}[P]$ we get

$$[\nabla_{\mathbf{X}},\, \tilde{\nabla}_{\mathbf{Y}}[P]] = \tilde{\nabla}_{\mathbf{U}}[P] + \sum_{R=1}^{Q} \tilde{\nabla}_{\mathbf{U}[R]}[R] - \tilde{\nabla}_{\mathbf{V}} + S, \tag{15.5}$$

where $\mathbf{U}$, $\mathbf{U}[R]$, and $\mathbf{V}$ are determined as follows:

$$\mathbf{U} = \nabla_{\mathbf{X}}\mathbf{Y}, \qquad \mathbf{U}[R] = -S\,\mathbf{T}[R], \qquad \mathbf{V} = \tilde{\nabla}_{\mathbf{Y}}[P]\mathbf{X}. \tag{15.6}$$

As for S in (15.5) and (15.6), it is a degenerate differentiation determined by some definite extended tensor field $\mathbf{S}$ of the type $(1, 1)$ depending on $\mathbf{X}$ and on $\mathbf{Y}$:

$$\mathbf{S} = \mathbf{D}[P](\mathbf{X}, \mathbf{Y}) = C(\mathbf{D}[P] \otimes \mathbf{X} \otimes \mathbf{Y}). \tag{15.7}$$

Similarly, $\mathbf{U}[R]$ in (15.6) is some definite extended tensor field of the type (r_P, s_P)

depending on $\mathbf{X}$, on $\mathbf{Y}$, and on the indices P and R:

$$\mathbf{U}[R] = \boldsymbol{\Theta}[P,R](\mathbf{X},\mathbf{Y}) = C(\boldsymbol{\Theta}[P,R] \otimes \mathbf{X} \otimes \mathbf{Y}). \tag{15.8}$$

The basic object in the series of notations (15.6), (15.7), (15.8) is $\mathbf{D}[P]$. It is called the *P-th dynamic curvature tensor*. This is an extended tensor field of the type $(s_P + 1, r_P + 2)$. Its components in a local chart are given by the formula

$$D^{k\,k_1 \ldots\, k_s}_{ij\,h_1 \ldots\, h_r}[P] = -\frac{\partial \Gamma^k_{j\,i}}{\partial T^{h_1 \ldots\, h_r}_{k_1 \ldots\, k_s}[P]}, \tag{15.9}$$

where $r = r_P$ and $s = s_P$. Then in (15.8) we have the extended tensor field $\boldsymbol{\Theta}[P,R]$ of the type $(r_R + s_P, s_R + r_P + 1)$. Its components are expressed through the components of $\mathbf{D}[P]$ in (15.9) according to the formula

$$\begin{aligned}\Theta^{i_1 \ldots\, i_\alpha k_1 \ldots\, k_s}_{j_1 \ldots\, j_\beta\, j\, h_1 \ldots\, h_r}[P,R] = \sum_{m=1}^{\beta} \sum_{w_m=1}^{n} D^{w_m\, h_1 \ldots\, h_r}_{j_m\, j\, k_1 \ldots\, k_s}[P]\; T^{i_1 \ldots \ldots \ldots\, i_\alpha}_{j_1 \ldots\, w_m \ldots\, j_\beta}[R] \,-\\ -\sum_{m=1}^{\alpha} \sum_{v_m=1}^{n} D^{i_m\, h_1 \ldots\, h_r}_{v_m\, j\, k_1 \ldots\, k_s}[P]\; T^{i_1 \ldots\, v_m \ldots\, i_\alpha}_{j_1 \ldots \ldots \ldots\, j_\beta}[R],\end{aligned} \tag{15.10}$$

where $r = r_P$, $s = s_P$, $\alpha = r_R$, and $\beta = s_R$. The components of the tensor $\mathbf{U}[R] = \boldsymbol{\Theta}[P,R](\mathbf{X},\mathbf{Y})$ in (15.8) are expressed through (15.10) as follows:

$$U^{i_1 \ldots\, i_\alpha}_{j_1 \ldots\, j_\beta}[R] = \sum_{j=1}^{n} \sum_{\substack{h_1,\, \ldots\,,\, h_r\\ k_1,\, \ldots\,,\, k_s}}^{\mathrm{n}} \!\!\ldots\! \sum^{\mathrm{n}} \Theta^{i_1 \ldots\, i_\alpha k_1 \ldots\, k_s}_{j_1 \ldots\, j_\beta\, j\, h_1 \ldots\, h_r}[P,R]\; X^j\; Y^{h_1 \ldots\, h_r}_{k_1 \ldots\, k_s}. \tag{15.11}$$

Similarly, the components of the tensor $\mathbf{S} = \mathbf{D}[P](\mathbf{X},\mathbf{Y})$ in (15.7) are expressed through (15.9) according to the following formula:

$$S^k_i = \sum_{j=1}^{n} \sum_{\substack{h_1,\, \ldots\,,\, h_r\\ k_1,\, \ldots\,,\, k_s}}^{\mathrm{n}} \!\!\ldots\! \sum^{\mathrm{n}} D^{k\,k_1 \ldots\, k_s}_{ij\,h_1 \ldots\, h_r}[P]\; X^j\; Y^{h_1 \ldots\, h_r}_{k_1 \ldots\, k_s}. \tag{15.12}$$

The formula (15.10) is derived from the second formula (15.6) due to (15.7) and (15.8). The formulas (15.11) and (15.12) are rather obvious. They complete the series of equalities which are used in order to make certain the right hand side of the commutation relationship (15.5).

In the last step now we consider the commutator of two spatial covariant differentiations $\nabla_{\mathbf{X}}$ and $\nabla_{\mathbf{Y}}$. The formula for this commutator is written as

$$[\nabla_{\mathbf{X}},\, \nabla_{\mathbf{Y}}] = \nabla_{\mathbf{U}} + \sum_{R=1}^{Q} \tilde{\nabla}_{\mathbf{U}[P]}[P] + S, \tag{15.13}$$

where $\mathbf{U}$ and $\mathbf{U}[R]$ are determined in the following way:

$$\mathbf{U} = \nabla_{\mathbf{X}}\mathbf{Y} - \nabla_{\mathbf{Y}}\mathbf{X} - \mathbf{V}, \qquad\qquad \mathbf{U}[R] = -S\,\mathbf{T}[R]. \tag{15.14}$$

Like in (15.5), by S in (15.13) and (15.14) we denote a degenerate differentiation determined by some definite extended tensor field $\mathbf{S}$ of the type $(1,1)$ depending on both extended vector fields $\mathbf{X}$ and $\mathbf{Y}$:

$$\mathbf{S} = \mathbf{R}(\mathbf{X},\mathbf{Y}) = C(\mathbf{R}\otimes\mathbf{X}\otimes\mathbf{Y}). \tag{15.15}$$

Similarly, $\mathbf{V}$ in (15.6) is some definite extended vector field depending on $\mathbf{X}$ and on $\mathbf{Y}$. It is expressed through the *torsion tensor* $\mathbf{T}$:

$$\mathbf{V} = \mathbf{T}(\mathbf{X},\mathbf{Y}) = C(\mathbf{T}\otimes\mathbf{X}\otimes\mathbf{Y}). \tag{15.16}$$

The components of the torsion tensor in a local chart are given by the formula

$$T^k_{ij} = \Gamma^k_{ij} - \Gamma^k_{j\,i}. \tag{15.17}$$

This formula (15.17) coincides with the standard formula for torsion (see [35]). The only difference here is that Γ is assumed to be an extended connection, therefore $\mathbf{T}$ is an extended tensor field of the type $(1,2)$.

For the parameter $\mathbf{U}[R]$ in (15.14) we write the formula analogous to (15.8) since this is also some definite extended tensor field depending on $\mathbf{X}$ and $\mathbf{Y}$:

$$\mathbf{U}[R] = \mathbf{\Omega}[R](\mathbf{X},\mathbf{Y}) = C(\mathbf{\Omega}[R]\otimes\mathbf{X}\otimes\mathbf{Y}). \tag{15.18}$$

The basic object in the series of notations (15.14), (15.15), (15.16), and (15.18) is the *curvature tensor* $\mathbf{R}$. In contrast to $\mathbf{D}[P]$ in (15.9), we call it the *static curvature tensor*. For the components of the static curvature tensor we have the formula

$$\begin{aligned}
&R^k_{h\,ij} = \frac{\partial\Gamma^k_{jh}}{\partial x^i} - \frac{\partial\Gamma^k_{ih}}{\partial x^j} + \sum^n_{a=1}\Gamma^a_{jh}\,\Gamma^k_{i\,a} - \sum^n_{a=1}\Gamma^a_{ih}\,\Gamma^k_{j\,a} - \\
&-\sum^Q_{P=1}\sum^{\mathrm{n}}_{\substack{h_1,\,\ldots,\,h_r\\ k_1,\,\ldots,\,k_s}}\!\!\ldots\!\!\sum^{\mathrm{n}}\sum^r_{m=1}\sum^n_{v_m=1}\Gamma^{h_m}_{i\,v_m}\;T^{\,h_1\ldots\,v_m\ldots\,h_r}_{\,k_1\ldots\ldots\ldots\,k_s}[P]\;\frac{\partial\Gamma^k_{jh}}{\partial T^{h_1\ldots\,h_r}_{k_1\ldots\,k_s}[P]} + \\
&+\sum^Q_{P=1}\sum^{\mathrm{n}}_{\substack{h_1,\,\ldots,\,h_r\\ k_1,\,\ldots,\,k_s}}\!\!\ldots\!\!\sum^{\mathrm{n}}\sum^r_{m=1}\sum^n_{v_m=1}\Gamma^{h_m}_{j\,v_m}\;T^{\,h_1\ldots\,v_m\ldots\,h_r}_{\,k_1\ldots\ldots\ldots\,k_s}[P]\;\frac{\partial\Gamma^k_{ih}}{\partial T^{h_1\ldots\,h_r}_{k_1\ldots\,k_s}[P]} + \\
&+\sum^Q_{P=1}\sum^{\mathrm{n}}_{\substack{h_1,\,\ldots,\,h_r\\ k_1,\,\ldots,\,k_s}}\!\!\ldots\!\!\sum^{\mathrm{n}}\sum^s_{m=1}\sum^n_{w_m=1}\Gamma^{w_m}_{i\,k_m}\;T^{h_1\ldots\ldots\ldots\,h_r}_{k_1\ldots\,w_m\ldots\,k_s}[P]\;\frac{\partial\Gamma^k_{jh}}{\partial T^{h_1\ldots\,h_r}_{k_1\ldots\,k_s}[P]} - \\
&-\sum^Q_{P=1}\sum^{\mathrm{n}}_{\substack{h_1,\,\ldots,\,h_r\\ k_1,\,\ldots,\,k_s}}\!\!\ldots\!\!\sum^{\mathrm{n}}\sum^s_{m=1}\sum^n_{w_m=1}\Gamma^{w_m}_{j\,k_m}\;T^{h_1\ldots\ldots\ldots\,h_r}_{k_1\ldots\,w_m\ldots\,k_s}[P]\;\frac{\partial\Gamma^k_{ih}}{\partial T^{h_1\ldots\,h_r}_{k_1\ldots\,k_s}[P]}.
\end{aligned} \tag{15.19}$$

In the case of non-extended connection Γ the formula (15.19) reduces to the standard formula for the curvature tensor (see [35]).

Returning back to the equality (15.18), we need to write the formula for the components of the tensor $\mathbf{\Omega}[R]$. Here is this formula:

$$\begin{gathered}\Omega^{\,i_1\dots\,i_\alpha}_{\,ij\,j_1\dots\,j_\beta}[R]=\sum_{m=1}^{\beta}\sum_{w_m=1}^{n}R^{w_m}_{j_m ij}\;T^{i_1\dots\dots\dots\,i_\alpha}_{j_1\dots\,w_m\dots\,j_\beta}[R]\;-\\ -\sum_{m=1}^{\alpha}\sum_{v_m=1}^{n}R^{i_m}_{v_m\,ij}\;T^{i_1\dots\,v_m\dots\,i_\alpha}_{j_1\dots\dots\dots\,j_\beta}[R].\end{gathered}\tag{15.20}$$

It is similar to (15.10). The formula (15.20) is derived from (15.18), (15.15), and from the second formula (15.14). The analogs of the formulas (15.11) and (15.12) in this case are written as follows:

$$U^{i_1\dots\,i_\alpha}_{j_1\dots\,j_\beta}[R]=\sum_{i=1}^{n}\sum_{j=1}^{n}\Omega^{\,i_1\dots\,i_\alpha}_{\,ij\,j_1\dots\,j_\beta}[R]\;X^i\;Y^j,\tag{15.21}$$

$$V^k=\sum_{i=1}^{n}\sum_{j=1}^{n}T^k_{ij}\;X^i\;Y^j,\tag{15.22}$$

$$S^k_h=\sum_{i=1}^{n}\sum_{j=1}^{n}R^k_{hij}\;X^i\;Y^j.\tag{15.23}$$

The formulas (15.21), (15.22), (15.23) complete the series of equalities which are written in order to make certain the right hand side of the commutation relationship (15.13). As for the commutation relationships themselves, they can be derived by direct calculations on the base of the formulas (12.11) and (13.15).

16. Coordinate representation of commutation relationships.

The first commutation relationship (15.1) is trivial. In coordinate form, i. e. in a local chart, it means that the matrix of the tensor $\mathbf{S}_3$ is the matrix commutator produced from the matrices of $\mathbf{S}_1$ and $\mathbf{S}_2$.

The next two commutator relationships (15.2) are also rather simple. They mean that the components of $\mathbf{S}_1$ and $\mathbf{S}_2$ are derived from the components of $\mathbf{S}$ by means of the formulas (12.11) and (13.15).

The fourth commutation relationship (15.3) is not so simple, but in a coordinate form it reduces to the following one:

$$[\nabla^{j_1\dots\,j_s}_{i_1\dots\,i_r}[P],\;\nabla^{k_1\dots\,k_\beta}_{h_1\dots\,h_\alpha}[R]]=0.\tag{16.1}$$

The relationship (16.1) is easily derived from (12.11).

Now let's proceed with the fifth commutation relationship (15.5). In a local chart we should consider the commutator of ∇_i and $\nabla^{j_1\dots\,j_s}_{i_1\dots\,i_r}[P]$. From (15.5) we derive

$$\begin{gathered}[\nabla_i,\,\nabla^{j_1\dots\,j_s}_{i_1\dots\,i_r}[P]]X^k=\sum_{h=1}^{n}D^{\,k\,j_1\dots\,j_s}_{\,h\,i\,i_1\dots\,i_r}[P]\;X^h\;+\\ +\sum_{R=1}^{Q}\sum_{\substack{h_1,\,\dots\,,\,h_\alpha\\ k_1,\,\dots\,,\,k_\beta}}^{\text{n}}\!\!\dots\!\sum^{\text{n}}\Theta^{\,h_1\dots\,h_\alpha\,j_1\dots\,j_s}_{\,k_1\dots\,k_\beta\,i\,i_1\dots\,i_r}[P,R]\;\nabla^{k_1\dots\,k_\beta}_{h_1\dots\,h_\alpha}[R]\;X^k.\end{gathered}\tag{16.2}$$

Here $X^1, \ldots, X^n$ are the components of some extended vector field $\mathbf{X}$. When applied to an extended scalar field φ the same commutator is written as follows:

$$[\nabla_i,\, \nabla^{j_1\ldots\, j_s}_{i_1\ldots\, i_r}[P]]\,\varphi = \sum^{Q}_{R=1}\sum^{\mathrm{n}}_{\substack{h_1,\,\ldots\,,\,h_\alpha\\ k_1,\,\ldots\,,\,k_\beta}}\!\!\!\cdots\!\sum^{\mathrm{n}}\Theta^{h_1\ldots\, h_\alpha\, j_1\ldots\, j_s}_{k_1\ldots\, k_\beta\, i\, i_1\ldots\, i_r}[P,R]\ \nabla^{k_1\ldots\, k_\beta}_{h_1\ldots\, h_\alpha}[R]\,\varphi. \tag{16.3}$$

And finally, in the case of an extended covector field $\mathbf{X}$ one should write

$$\begin{gathered}[\nabla_i,\, \nabla^{j_1\ldots\, j_s}_{i_1\ldots\, i_r}[P]]X_k = -\sum^{n}_{h=1} D^{h\, j_1\ldots\, j_s}_{k\, i\, i_1\ldots\, i_r}[P]\ X_h\, +\\ +\sum^{Q}_{R=1}\sum^{\mathrm{n}}_{\substack{h_1,\,\ldots\,,\,h_\alpha\\ k_1,\,\ldots\,,\,k_\beta}}\!\!\!\cdots\!\sum^{\mathrm{n}}\Theta^{h_1\ldots\, h_\alpha\, j_1\ldots\, j_s}_{k_1\ldots\, k_\beta\, i\, i_1\ldots\, i_r}[P,R]\ \nabla^{k_1\ldots\, k_\beta}_{h_1\ldots\, h_\alpha}[R]\,X_k.\end{gathered} \tag{16.4}$$

The components of $\mathbf{D}[P]$ and $\boldsymbol{\Theta}[P,R]$ in the above three formulas (16.2), (16.3), (16.4) are taken from (15.9) and (15.10) respectively.

The last commutation relationship is (15.13). In order to write it in a local chart one should consider the commutator of two covariant derivatives ∇_i and ∇_j:

$$\begin{gathered}[\nabla_i,\, \nabla_j]X^k = -\sum^{n}_{h=1} T^h_{ij}\ \nabla_h X^k + \sum^{n}_{h=1} R^k_{h\,ij}\ X^h\, +\\ +\sum^{\mathrm{n}}_{\substack{h_1,\,\ldots\,,\,h_\alpha\\ k_1,\,\ldots\,,\,k_\beta}}\!\!\!\cdots\!\sum^{\mathrm{n}}\Omega^{\ \ h_1\ldots\, h_\alpha}_{ij\,k_1\ldots\, k_\beta}[R]\ \nabla^{k_1\ldots\, k_\beta}_{h_1\ldots\, h_\alpha}[R]\,X^k,\end{gathered} \tag{16.5}$$

$$[\nabla_i,\, \nabla_j]\varphi = -\sum^{n}_{h=1} T^h_{ij}\ \nabla_h\,\varphi + \sum^{\mathrm{n}}_{\substack{h_1,\,\ldots\,,\,h_\alpha\\ k_1,\,\ldots\,,\,k_\beta}}\!\!\!\cdots\!\sum^{\mathrm{n}}\Omega^{\ \ h_1\ldots\, h_\alpha}_{ij\,k_1\ldots\, k_\beta}[R]\ \nabla^{k_1\ldots\, k_\beta}_{h_1\ldots\, h_\alpha}[R]\,\varphi, \tag{16.6}$$

$$\begin{gathered}[\nabla_i,\, \nabla_j]X_k = -\sum^{n}_{h=1} T^h_{ij}\ \nabla_h X^k - \sum^{n}_{h=1} R^h_{k\,ij}\ X_h\, +\\ +\sum^{\mathrm{n}}_{\substack{h_1,\,\ldots\,,\,h_\alpha\\ k_1,\,\ldots\,,\,k_\beta}}\!\!\!\cdots\!\sum^{\mathrm{n}}\Omega^{\ \ h_1\ldots\, h_\alpha}_{ij\,k_1\ldots\, k_\beta}[R]\ \nabla^{k_1\ldots\, k_\beta}_{h_1\ldots\, h_\alpha}[R]\,X_k,\end{gathered} \tag{16.7}$$

The components of the torsion tensor $\mathbf{T}$, the components of the curvature tensor $\mathbf{R}$, and the components of the tensor $\boldsymbol{\Omega}[R]$ in (16.5), (16.6), (16.7) are given by the formulas (15.17), (15.19), and (15.20) respectively.

The formulas (16.2), (16.3), (16.4) and (16.5), (16.6), (16.7) are written for the cases of vectorial, covectorial, and scalar fields. However, the lemma 7.5 and the theorem 7.1 say that they are sufficient for to write the analogous formulas in the case where the commutators $[\nabla_i,\, \nabla^{j_1\ldots\, j_s}_{i_1\ldots\, i_r}[P]]$ and $[\nabla_i,\, \nabla_j]$ are applied to the components of an arbitrary extended tensor field $\mathbf{X}$.

17. Tensor functions of tensors and the chain rule in tensorial form.

Tensor-valued functions with tensorial arguments appear rather often in applications. The most simple examples are the following ones:

— the force field $\mathbf{F}(x^1, x^2, x^3, v^1, v^2, v^3)$ acting upon a point mass that moves according the Newton's second law;

— the Lagrange function $L(x^1, x^2, x^3, v^1, v^2, v^3)$ of such a point mass;

— the Hamilton function $H(x^1, x^2, x^3, p_1, p_2, p_3)$ of such a point mass.

These examples in a little bit more general form were mentioned in section 2 (see comment to the formula (2.2)) and in section 4. Our next example is from the field theory. The action integral of the electromagnetic field in vacuum is written as

$$S = -\frac{1}{16\,\pi\,c}\int\sum_{i=0}^{3}\sum_{j=0}^{3}\sum_{\alpha=0}^{3}\sum_{\beta=0}^{3} g_{ij}\ g_{\alpha\beta}\ F^{i\alpha}\,F^{j\beta}\sqrt{-\det g}\ d^4x$$

(see [40] for details). The term under integration in this formula is a scalar function

$$L = -\frac{1}{16\,\pi\,c}\sum_{i=0}^{3}\sum_{j=0}^{3}\sum_{\alpha=0}^{3}\sum_{\beta=0}^{3} g_{ij}\ g_{\alpha\beta}\ F^{i\alpha}\,F^{j\beta}. \tag{17.1}$$

However, its value is determined by the tensor of the electromagnetic field $\mathbf{F}$:

$$F^{i\alpha} = \begin{Vmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -H^3 & H^2 \\ E^2 & H^3 & 0 & -H^1 \\ E^3 & -H^2 & H^1 & 0 \end{Vmatrix}. \tag{17.2}$$

Apart from (17.2), in (17.1) we have the components of the Minkowski metric:

$$g_{ij} = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{Vmatrix}. \tag{17.3}$$

In Cartesian coordinates the Minkowski metric is represented by the matrix (17.3). If we use some curvilinear coordinate system, the matrix components g_{ij} become depending on the coordinates x^0, x^1, x^2, x^3 of a point in the Minkowski space. In special relativity the role of the Minkowski metric is not so significant as in general relativity. For this reason, writing (17.2) formally, we can indicate the presence of $\mathbf{g}$ as an additional dependence on the spatial variables x^0, x^1, x^2, x^3 in L:

$$L = L(x^0, \ldots, x^3, F^{00}, F^{01}, F^{02}, \ldots, F^{33}). \tag{17.4}$$

For each particular configuration of the electromagnetic field F^{ij} in (17.4) are some particular functions of x^0, x^1, x^2, x^3. However, in some cases, e. g. in deriving the field equations from the variational principle in form of the Euler-Lagrange equations, the quantities F^{ij} are treated as independent variables.

The example of the electromagnetic field, i. e. the function (17.4), can be considered as a background for various generalizations of the electromagnetism. Such theories could include several tensorial fields $\mathbf{T}[1], \ldots, \mathbf{T}[Q]$. Therefore, for the density in the action integral of such theories one should choose some function L depending on the variables (3.3):

$$L = L(x^1, \ldots, x^n, T^{1\,\ldots\,1}_{1\,\ldots\,1}[1], \ldots, T^{n\,\ldots\,n}_{n\,\ldots\,n}[Q]). \tag{17.5}$$

This means that L in (17.5) is an extended scalar field associated with some composite tensor bundle $N = T^{r_1\ldots\,r_Q}_{s_1\ldots\,s_Q}M$. If the whole scenario is performed in the Minkowski space or in some space M equipped with a metric $\mathbf{g}$ and with some connection Γ, then the differentiations introduced in the definition 12.2 and in the definition 13.2 are applicable to L. On the other hand, if some particular configuration of the fields $\mathbf{T}[1], \ldots, \mathbf{T}[Q]$ is given, then

$$\begin{cases} T^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}[1] = T^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}[1](x^1, \ldots, x^n), \ \text{ where } \ r = r_1, \ s = s_1, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ T^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}[Q] = T^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}[Q](x^1, \ldots, x^n), \ \text{ where } \ r = r_Q, \ s = s_Q. \end{cases} \tag{17.6}$$

Substituting (17.6) into (17.5), we obtain

$$\widetilde{L} = \widetilde{L}(x^1, \ldots, x^n). \tag{17.7}$$

The function $\widetilde{L}$ in (17.7) represents a standard (not extended) scalar field. This means that we can differentiate L in two ways: as an extended field in its original form (17.4) and as a standard field upon substituting some particular fields (17.6) into its arguments. The same is true for an arbitrary extended tensor field $\mathbf{X}$.

Theorem 17.1. *Let $\mathbf{X}$ be an extended tensor field of the type (α, β) associated with a composite tensor bundle $N = T^{r_1\ldots\,r_Q}_{s_1\ldots\,s_Q}M$ and let $\mathbf{T}[1], \ldots, \mathbf{T}[Q]$ be some non-extended tensor fields that determine some particular section $q = q(p)$ of the bundle N. Denote by $\widetilde{\mathbf{X}}$ the non-extended tensor field obtained from $\mathbf{X}$ by substituting $\mathbf{T}[1], \ldots, \mathbf{T}[Q]$ into its arguments. Then*

$$\nabla_{\mathbf{Y}}\widetilde{\mathbf{X}} = \nabla_{\mathbf{Y}}\mathbf{X} + \sum_{P=1}^{Q} C(\nabla_{\mathbf{Y}}\mathbf{T}[P] \otimes \tilde{\nabla}[P]\,\mathbf{X}), \tag{17.8}$$

where $\mathbf{Y}$ is some non-extended vector field in M, $\nabla_{\mathbf{Y}}\widetilde{\mathbf{X}}$ is the standard covariant differentiation[1]*, $\nabla_{\mathbf{Y}}\mathbf{X}$ is the spacial covariant differentiation, and $\nabla_{\mathbf{Y}}\mathbf{T}[P]$ is again the standard covariant differentiation.*

The equality (17.8) in the theorem 17.1 is a tensorial form of the well-known chain rule for differentiating composite functions. Its proof is pure calculations. First of all one should write the equality (17.8) in local coordinates. Here covariant

[1] Writing covariant differentiations we assume that M is equipped with some connection Γ. This can be either a standard connection or an extended connection. In the latter case it is converted to the standard connection by means of the section $q = q(p)$. In other words, one should substitute (17.6) into the arguments of $\Gamma^k_{ij}(x^1, \ldots, x^n, T^{1\,\ldots\,1}_{1\,\ldots\,1}[1], \ldots, T^{n\,\ldots\,n}_{n\,\ldots\,n}[Q])$.

differentiations are replaced by covariant derivatives. As for the vector field $\mathbf{Y}$, it can be dropped at all. As a result (17.8) is written as

$$\begin{gathered}\nabla_i \widetilde{X}^{i_1\ldots\,i_\alpha}_{j_1\ldots\,j_\beta}=\nabla_i X^{i_1\ldots\,i_\alpha}_{j_1\ldots\,j_\beta}+\\+\sum^{Q}_{P=1}\sum^{\mathrm{n}}_{\substack{h_1,\,\ldots\,,\,h_r\\k_1,\,\ldots\,,\,k_s}}\!\!\ldots\sum^{\mathrm{n}}\nabla_i T^{h_1\ldots\,h_r}_{k_1\ldots\,k_s}[P]\ \nabla^{k_1\ldots\,k_s}_{h_1\ldots\,h_r}[P]X^{i_1\ldots\,i_\alpha}_{j_1\ldots\,j_\beta}.\end{gathered}\tag{17.9}$$

The equality (17.9) is derived by direct calculations based on the formulas (13.15) and (12.11). The equality (17.8) then is derived by multiplying both sides of (17.9) by Y^i and summing over the index i.

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[3] Russian journal **Mathematical Collection** (Математический Сборник), see the web-pages http://math.ras.ru/journals/Mat._Sbornik/ and http://turpion.ioc.ac.ru/main/pa_sm.html in Russian and in English respectively.

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