ON THE SUBSET OF NORMALITY EQUATIONS describing generalized Legendre transformation.

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Abstract. Normality equations describe Newtonian dynamical systems admitting normal shift of hypersurfaces. They were first derived in Euclidean geometry, then in Riemannian geometry. Recently they were rederived in more general case, when geometry of manifold is given by generalized Legendre transformation. As appears, in this case some part of normality equations describe generalized Legendre transformation itself irrespective to that Newtonian dynamical system, for which others are written. In present paper this smaller part of normality equations is studied.

1. Newtonian dynamical systems and generalized Legendre transformation.

Let M be smooth manifold of dimension n. We say that the motion of a point $p = p(t)$ of this manifold obeys Newton's second low if in local chart it is described by the following ordinary differential equations:

$$\dot{x}^i = v^i, \qquad\qquad \dot{v}^i = \Phi^i(x^1, \dots, x^n, v^1, \dots, v^n). \tag{1.1}$$

Here $v^1, \dots, v^n$ are components of velocity vector $\mathbf{v}$ of moving point. Its mass is assumed to be equal to unity: $m = 1$. Therefore functions $\Phi^1, \dots, \Phi^n$ in (1.1) play the role of force vector, though, unlike $v^1, \dots, v^n$, they are not components of tangent vector to M.

Not always, but very often differential equations (1.1) are associated with some extremal principle and hence are given implicitly by Euler-Lagrange equations:

$$\dot{x}^i = v^i, \qquad\qquad \frac{d}{dt}\left(\frac{\partial L}{\partial v^i}\right) = \frac{\partial L}{\partial x^i}$$

In this case they can be transformed to Hamiltonian form

$$\dot{x}^i = \frac{\partial H}{\partial p_i}, \qquad\qquad \dot{p}_i = -\frac{\partial H}{\partial x^i}$$

by means of classical Legendre transformation that relates velocity vector $\mathbf{v}$ and momentum covector $\mathbf{p}$ according to the following formula:

$$p_i = \frac{\partial L}{\partial v^i}. \tag{1.2}$$

1991 *Mathematics Subject Classification.* 53D50, 70G10, 70G45.

Key words and phrases. Normality Equations, Generalized Legendre Transformation.

In [1] and [2] more general transformation was considered. It is given by functions

$$\begin{cases} p_1 = L_1(x^1, \ldots, x^n, v^1, \ldots, v^n), \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ p_n = L_n(x^1, \ldots, x^n, v^1, \ldots, v^n). \end{cases} \tag{1.3}$$

From geometric point of view generalized Legendre transformation (1.3) is a smooth **fiber-preserving** map from tangent bundle to cotangent bundle:

$$\lambda\colon TM \to T^*M. \tag{1.4}$$

Fiber-preserving means that each fixed fiber of tangent bundle TM is mapped into a fiber of T^*M over the same base point of M. For the sake of simplicity we shall assume generalized Legendre map (1.4) to be diffeomorphic. Then inverse map

$$\lambda^{-1}\colon T^*M \to TM \tag{1.5}$$

is also fiber-preserving. In local chart it is given by functions

$$\begin{cases} v^1 = V^1(x^1, \ldots, x^n, p_1, \ldots, p_n), \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ v^n = V^n(x^1, \ldots, x^n, p_1, \ldots, p_n). \end{cases} \tag{1.6}$$

In paper [1] generalized Legendre maps (1.4) and (1.5) were used in order to transform dynamical system (1.1) to **p**-representation. Here it looks like

$$\dot{x}^i = V^i \qquad\qquad \dot{p}_i = \Theta_i, \tag{1.7}$$

where functions $V^1, \ldots, V^n$ are given by (1.6), while $\Theta_1, \ldots, \Theta_n$ are similar functions playing the same role as function $\Phi^1, \ldots, \Phi^n$ in (1.1). Then in paper [1] shift of hypersurfaces along trajectories of dynamical system (1.7) was studied and **theory of Newtonian dynamical systems admitting normal shift of hypersurfaces** was generalized to present non-metric geometry given by maps (1.4) and (1.5). Previous stage of development of this theory is reflected in paper [3] and in theses [4] and [5] (see also recent papers [6–13]).

Main result of theory constructed in paper [1] is a set of **normality equations**. This is rather huge system of partial differential equations with respect to functions $V^1, \ldots, V^n$ and $\Theta_1, \ldots, \Theta_n$. In paper [2] normality equations were transformed back to **v**-representation. Here they form a system of partial differential equations with respect to functions $\Phi^1, \ldots, \Phi^n$ in (1.1) and functions $L_1, \ldots, L_n$ in (1.3). Total set of normality equations is divided into two parts: **weak normality equations** written for $n \geqslant 2$ and **additional normality equations**, which are present only in multidimensional case $n \geqslant 3$. Additional normality equations in turn are subdivided into **three** parts. It is remarkable that equations in the **first part** have no entries of functions $\Phi^1, \ldots, \Phi^n$ in them. They form a system of partial differential equations with respect to functions $L_1, \ldots, L_n$ that define generalized Legendre transformation (1.4). Further we shall call them **normality equations for generalized Legendre transformation**. Main goal of present paper is to study these equations and describe generalized Legendre transformations determined by their solutions.

2. Normality equations for generalized Legendre transformation.

Values of functions $L_1, \ldots, L_n$ in (1.3) form components of covector $\mathbf{p} \in T^*_p(M)$ when their arguments are fixed. However, they do not form components of traditional covector field. They form so called **extended covector field**.

Definition 2.1. Extended tensor field $\mathbf{X}$ of type (r, s) in $\mathbf{v}$-representation is a tensor-valued function $\mathbf{X} = \mathbf{X}(q)$ with argument $q = (p, \mathbf{v})$ in tangent bundle TM and with values in the following tensor space:

$$T^r_s(p, M) = \overbrace{T_p(M) \otimes \ldots \otimes T_p(M)}^{r \text{ times}} \otimes \underbrace{T^*_p(M) \otimes \ldots \otimes T^*_p(M)}_{s \text{ times}}.$$

Extended covector field is a special case of extended tensor field, when $r = 0$ and $s = 1$. Now we shall not discuss theory of extended tensor fields, referring reader to Chapters II, III, and IV of thesis [4]. However, we should note that if

$$X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s} = X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}(x^1, \ldots, x^n, v^1, \ldots, v^n)$$

are components of extended tensor field $\mathbf{X}$, then partial derivatives

$$\tilde{\nabla}_k X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s} = \frac{\partial X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}}{\partial v^k} \tag{2.1}$$

are components of another extended tensor field $\tilde{\nabla}\mathbf{X}$. Therefore in (2.1) we use symbol of covariant derivative $\tilde{\nabla}_k$ for partial derivative $\partial/\partial v^k$.

Let's apply covariant differentiation $\tilde{\nabla}$ to extended covector field $\mathbf{L}$ with components (1.3). As a result we get extended tensor field $\mathbf{g}$ of type $(0, 2)$ with components

$$g_{qk} = \tilde{\nabla}_k L_q. \tag{2.2}$$

Matrix g_{qk} in (2.2) is non-degenerate since it coincides with Jacobi matrix for diffeomorphic map (1.4). Hence we can consider inverse matrix with components g^{qk}. It defines extended tensor field of type $(2, 0)$, we denote it by the same symbol $\mathbf{g}$. Though being non-symmetric, tensor field $\mathbf{g}$ with components (2.2) and its dual field with components g^{qk} here play the same role as metric tensor and dual metric tensor in Riemannian geometry.

Now, according to paper [2], we define extended scalar field Ω and operator-valued extended tensor field $\mathbf{P}$. They are determined as follows:

$$\Omega = \sum_{s=1}^{n} L_s\, L^s = |\mathbf{L}|^2, \qquad\qquad P^i_j = \delta^i_j - \frac{L^i\, L_j}{|\mathbf{L}|^2}. \tag{2.3}$$

Here L^s and L^i are components of extended vector field $\mathbf{L}$ dual to covector field $\mathbf{L}$ with components (1.3) with respect to non-symmetric metric (2.2):

$$L^i = \sum_{s=1}^{n} L_s\, g^{si}. \tag{2.4}$$

Being more accurate, we should say that (2.4) are components of vector field right-dual to covector field $\mathbf{L}$. One can also define left-dual vector field with components

$$\check{L}^i = \sum_{s=1}^{n} g^{is}\, L_s. \tag{2.5}$$

In (2.3) we denoted $\Omega = |\mathbf{L}|^2$, it is positive if non-symmetric metric (2.2) is positive. However, this is not obligatory. We shall only require that $\Omega \neq 0$ since it is in denominator in second formula (2.3).

Now we are ready to write normality equations for generalized Legendre transformation (1.4). In local chart they are written as follows:

$$\sum_{r=1}^{n}\sum_{s=1}^{n}(A^{rs} - A^{sr})\, P^i_r\, P^j_s = 0. \tag{2.6}$$

Here A^{rs} are components of extended tensor field $\mathbf{A}$. According to paper [2], in $\mathbf{v}$-representation they are given by formula

$$A^{rs} = \sum_{q=1}^{n} g^{qr}\, \tilde{\nabla}_q L^s. \tag{2.7}$$

Note that metric tensor $\mathbf{g}$ in (2.2), projector field $\mathbf{P}$, and tensor field $\mathbf{A}$ in (2.7) are completely determined by covector field $\mathbf{L}$. Therefore (2.6) form a system of partial differential equations with respect to functions $L_1, \ldots, L_n$. Further steps are intended to study these equations. Note also that equations (2.6) are written only for multidimensional case $n \geqslant 3$. In two-dimensional case $n = 2$ we have no restrictions for generalized Legendre transformation (1.4).

3. Preliminary transformation of normality equations.

Let's consider formula (2.7). Applying formula (2.4) to L^s in it, we derive the following expression for covariant derivative $\tilde{\nabla}_q L^s$:

$$\begin{gathered}\tilde{\nabla}_q L^s = \tilde{\nabla}_q \left(\sum_{i=1}^{n} L_i\, g^{is}\right) = \sum_{i=1}^{n} \tilde{\nabla}_q L_i\, g^{is} + \sum_{i=1}^{n} L_i\, \tilde{\nabla}_q g^{is} = \sum_{i=1}^{n} g_{iq}\, g^{is} - \\ - \sum_{i=1}^{n}\sum_{a=1}^{n}\sum_{k=1}^{n} L_i\, g^{ia}\, \tilde{\nabla}_q g_{ak}\, g^{ks} = \sum_{i=1}^{n} g_{iq}\, g^{is} - \sum_{i=1}^{n}\sum_{a=1}^{n}\sum_{k=1}^{n} L_i\, g^{ia}\, \tilde{\nabla}_q \tilde{\nabla}_k L_a\, g^{ks}.\end{gathered}$$

Upon substituting this expression into (2.7) for A^{rs} we obtain

$$A^{rs} = g^{rs} - \sum_{a=1}^{n}\sum_{q=1}^{n}\sum_{k=1}^{n} g^{qr}\, g^{ks}\, L^a\, \tilde{\nabla}_q \tilde{\nabla}_k L_a. \tag{3.1}$$

It is obvious that last term in (3.1) is symmetric with respect to indices r and s. Therefore it makes no contribution to ultimate form of normality equations when we substitute (3.1) into (2.6). Thus from (2.6) we derive

$$\sum_{r=1}^{n}\sum_{s=1}^{n}(g^{rs} - g^{sr})\, P^i_r\, P^j_s = 0. \tag{3.2}$$

Now let's apply formula (2.3) to components of projector field P^i_r and P^j_s in (3.2):

$$\sum_{r=1}^{n}\sum_{s=1}^{n}(g^{rs}-g^{sr})\,P^i_r\,P^j_s=\sum_{r=1}^{n}\sum_{s=1}^{n}(g^{rs}-g^{sr})\left(\delta^i_r-\frac{L^i\,L_r}{|\mathbf{L}|^2}\right)\left(\delta^j_s-\frac{L^j\,L_s}{|\mathbf{L}|^2}\right)=$$
$$=g^{ij}-g^{ji}-\frac{(\check{L}^i-L^i)\,L^j}{|\mathbf{L}|^2}-\frac{(L^j-\check{L}^j)\,L^i}{|\mathbf{L}|^2}=g^{ij}-g^{ji}-\frac{\check{L}^i\,L^j-L^i\,\check{L}^j}{|\mathbf{L}|^2}=0.$$

Here L^i, L^j, $\check{L}^i$, and $\check{L}^j$ are determined by formulas (2.4) and (2.5). As a result of the above calculations normality equations (2.6) are written as

$$g^{ij}-\frac{\check{L}^i\,L^j}{|\mathbf{L}|^2}=g^{ji}-\frac{\check{L}^j\,L^i}{|\mathbf{L}|^2}.\tag{3.3}$$

If we denote by u^{ij} left hand side of the equality (3.3), then g^{ij} is given by formula

$$g^{ij}=u^{ij}+\frac{\check{L}^i\,L^j}{|\mathbf{L}|^2},\tag{3.4}$$

while normality equations (3.3) themselves are equivalent to symmetry of tensor $\mathbf{u}$ with components u^{ij}. Thus, non-symmetric metric $\mathbf{g}$ is expressed through symmetric tensor $\mathbf{u}$ by formula (3.4). This is basic observation for the next step.

4. Fine structure of metric tensor.

Let's fix some point $q=(p,\mathbf{v})$ of TM such that $|\mathbf{L}|\neq 0$. This means that we fix arguments of extended tensor fields in (3.4). Then values of $\mathbf{g}$ and $\mathbf{u}$ for that fixed argument q are tensors from $T^2_0(p,M)$, while values of $\mathbf{L}$ and $\check{\mathbf{L}}$ are vectors from tangent space $T_p(M)$. Tensors $\mathbf{g}$ and $\mathbf{u}$ of type $(2,0)$ can be treated as bilinear forms (bilinear functions) with arguments in cotangent space $T^*_p(M)$:

$$\mathbf{g}=\mathbf{g}(\mathbf{x},\mathbf{y}),\qquad\qquad \mathbf{u}=\mathbf{u}(\mathbf{x},\mathbf{y}).\tag{4.1}$$

Due to symmetry $u^{ij}=u^{ji}$ bilinear form $\mathbf{u}$ in (4.1) is symmetric, i. e.

$$\mathbf{u}(\mathbf{x},\mathbf{y})=\mathbf{u}(\mathbf{y},\mathbf{x}).$$

It is well known fact from linear algebra (see [14]) that each symmetric bilinear form can be diagonalized. This means that one can choose some special base in $T_p(M)$ and its dual base in $T^*_p(M)$ such that matrix u^{ij} is diagonal

$$u^{ij}=\begin{Vmatrix}\varepsilon_1 & 0 & \dots & 0\\ 0 & \varepsilon_2 & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & \varepsilon_n\end{Vmatrix}.\tag{4.2}$$

Here it is important to note that tensor field $\mathbf{u}$ is diagonalized at one fixed point $q=(p,\mathbf{v})$, not in whole neighborhood of that point.

Lemma 4.1. *For each point $q \in TM$, where $|\mathbf{L}| \neq 0$, tensor field $\mathbf{u}$ and its matrix (4.2) are degenerate, i. e. at least one number among $\varepsilon_1, \ldots, \varepsilon_n$ is equal to zero.*

Proof. Let's multiply both sides of (3.4) by L_j and then sum up with respect to index j. As a result we get the following equality:

$$\check{L}^i = \sum_{j=1}^{n} g^{ij}\,L_j = \sum_{j=1}^{n}\left(u^{ij} + \frac{\check{L}^i\,L^j}{|\mathbf{L}|^2}\right)L_j = \sum_{j=1}^{n} u^{ij}\,L_j + \check{L}^i. \tag{4.3}$$

Comparing left and right hand sides of the equality (4.3), we derive

$$\sum_{j=1}^{n} u^{ij}\,L_j = 0. \tag{4.4}$$

If $\mathbf{L} \neq 0$, then the equality (4.4) means that $\det\mathbf{u} = 0$. This proves lemma for all points $q = (p, \mathbf{v})$, where $\mathbf{L} \neq 0$. But $|\mathbf{L}| \neq 0$ implies that covector $\mathbf{L}$ is non-zero. Thus, lemma 4.1 is proved. $\square$

Remark. Normality equation (2.6) is derived only for those points, where $|\mathbf{L}| \neq 0$ (see [1] and [2]). Indeed, $|\mathbf{L}|$ is in denominator in formula (2.3) for P^i_j. Hence lemma 4.1 is sufficient result for our further purposes.

Lemma 4.1 means that bilinear form $\mathbf{u}$ has nonzero kernel. This is linear subspace in cotangent space $T^*_p(M)$ defined as follows:

$$\operatorname{Ker}\mathbf{u} = \{\mathbf{x} \in T^*_p(M)\colon\ \mathbf{u}(\mathbf{x},\mathbf{y}) = 0\ \forall\mathbf{y} \in T^*_p(M)\}. \tag{4.5}$$

In terms of kernel (4.5) the equality (4.4) now can be written as

$$\mathbf{L} \in \operatorname{Ker}\mathbf{u} \neq \{0\}. \tag{4.6}$$

Lemma 4.2. *For symmetric bilinear form $\mathbf{u}$ in $T^*_p(M)$ defined by (3.4) its rank is $n-1$ and the dimension of its kernel is equal to unity, i. e.*

$$\operatorname{rank}\mathbf{u} = n - 1, \qquad\qquad \dim\operatorname{Ker}\mathbf{u} = 1. \tag{4.7}$$

Proof. Let's multiply both sides of (3.4) by P^s_j and sum up with respect to double index j. As a result we obtain the following equality

$$\sum_{j=1}^{n} P^s_j\,g^{ij} = \sum_{j=1}^{n} u^{ij}\,P^s_j = \sum_{j=1}^{n} u^{ij}\left(\delta^s_j - \frac{L_j\,L^s}{|\mathbf{L}|^2}\right) = u^{is}. \tag{4.8}$$

Here in the above calculations we used (4.4). Sum in left hand side of (4.8) represents matrix product of two matrices: P^s_j and g^{ij} transposed. Matrix g^{ij} is non-degenerate, while rank of projection operator $\mathbf{P}$ is equal to $n-1$. This proves the equalities (4.7) and lemma 4.2 in whole. $\square$

Lemma 4.3. *Matrix equality $g^{ij} = u^{ij} + A^i\,L^j$ is equivalent to normality equations (3.2) if and only if matrix u^{ij} is symmetric and degenerate.*

Proof. Above we have derived the equality (3.4) from normality equation (3.2) and we have proved that matrix u^{ij} in (3.4) is degenerate (see lemma 4.1 and lemma 4.2). Denoting $A^i = \check{L}^i / |\mathbf{L}|^2$ we get the equality $g^{ij} = u^{ij} + A^i\,L^j$. Thus, direct proposition of lemma 4.3 is proved.

Let's prove converse proposition. Suppose that metric tensor is given by the equality $g^{ij} = u^{ij} + A^i\,L^j$, where L^j are determined by formula (2.4), A^i are components of some vector, while matrix u^{ij} is symmetric and degenerate. Then there exists some covector $\mathbf{x} \neq 0$ with components $x_1, \ldots, x_n$ such that

$$\sum_{j=1}^{n} u^{ij}\,x_j = 0, \qquad\qquad \sum_{i=1}^{n} x_i\,u^{ij} = 0. \tag{4.9}$$

Applying relationships (4.9) to the equality $g^{ij} = u^{ij} + A^i\,L^j$, we get

$$\begin{aligned} x^j &= \sum_{i=1}^{n} x_i\,g^{ij} = \sum_{i=1}^{n} x_i\,A^i\,L^j = \langle \mathbf{x}\,|\,\mathbf{A}\rangle \cdot L^j,\\ \check{x}^i &= \sum_{j=1}^{n} g^{ij}\,x_j = \sum_{j=1}^{n} A^i\,L^j\,x_j = \langle \mathbf{x}\,|\,\mathbf{L}\rangle \cdot A^i. \end{aligned} \tag{4.10}$$

From first equality (4.10) we derive that covectors $\mathbf{x}$ and $\mathbf{L}$ are collinear:

$$x_i = \sum_{j=1}^{n} x^j\,g_{ji} = \sum_{j=1}^{n} \langle \mathbf{x}\,|\,\mathbf{A}\rangle\;L^j\,g_{ji} = \langle \mathbf{x}\,|\,\mathbf{A}\rangle \cdot L_i. \tag{4.11}$$

Note that $\mathbf{x} \neq 0$ and $\mathbf{L} \neq 0$. Hence $\langle \mathbf{x}\,|\,\mathbf{A}\rangle \neq 0$. Substituting formula (4.11) for x_j into both sides of second equality (4.10), we obtain

$$\langle \mathbf{x}\,|\,\mathbf{A}\rangle \cdot \check{L}^i = \langle \mathbf{x}\,|\,\mathbf{A}\rangle \cdot |\mathbf{L}|^2 \cdot A^i. \tag{4.12}$$

Since $\langle \mathbf{x}\,|\,\mathbf{A}\rangle \neq 0$, we can cancel this factor in (4.12). Then we get formula for A^i:

$$A^i = \frac{\check{L}^i}{|\mathbf{L}|^2}. \tag{4.13}$$

Substituting (4.13) back into the equality $g^{ij} = u^{ij} + A^i\,L^j$, we get formula coinciding with (3.4). Using symmetry of u^{ij}, we can transform it to (3.3). Then multiplying (3.3) by $P^r_i\,P^s_j$, upon summation with respect to double indices r and s we rederive normality equations (3.2). Lemma 4.3 is proved. $\square$

Now let's multiply (3.4) by $g_{ir}\,g_{js}$ and let's sum resulting equality with respect to double indices i and j. Then we introduce the following notations:

$$u_{sr} = \sum_{i=1}^{n}\sum_{j=1}^{n} u^{ij}\,g_{ir}\,g_{js}, \qquad\qquad \check{L}_r = \sum_{i=1}^{n} \check{L}^i\,g_{ir}. \tag{4.14}$$

Here $\check{L}_1, \ldots, \check{L}_n$ are components of extended covector field left dual to vector field

$\check{\mathbf{L}}$, while vector field $\check{\mathbf{L}}$ is right dual to initial covector field $\mathbf{L}$. In terms of these newly introduced notations (4.14) transformed equality (3.4) is written as

$$g_{sr} = u_{sr} + \frac{L_s\,\check{L}_r}{|\mathbf{L}|^2}. \tag{4.15}$$

Matrix u_{sr} in (4.15) is symmetric. This matrix is degenerate, its rank is equal to $n-1$. This follows from (4.6) due to (4.14). Moreover, g_{ir} and g_{js} in (4.14) are components of non-degenerate matrix, therefore (4.15) is equivalent to (3.4).

Lemma 4.4. *Matrix equality $g_{sr} = u_{sr}+L_s\,A_r$ is equivalent to normality equations* (3.2) *if and only if matrix u_{sr} is symmetric and degenerate.*

Proof. Note that matrix equality (4.15) with symmetric degenerate matrix u_{sr}, which was derived above from normality equation (3.4), is particular form of the equality $g_{sr} = u_{sr}+L_s\,A_r$, where $A_r = \check{L}_r/|\mathbf{L}|^2$. This means that direct proposition of lemma 4.4 is proved.

Let's prove converse proposition. Suppose that metric tensor is given by the equality $g_{sr} = u_{sr} + L_s\,A_r$, where matrix u^{ij} is symmetric and degenerate. Then there exists some vector $\mathbf{X} \neq 0$ with components $X^1,\,\ldots\,,X^n$ such that

$$\sum_{r=1}^{n} u_{sr}\,X^r = 0, \qquad\qquad \sum_{s=1}^{n} X^s\,u_{sr} = 0. \tag{4.16}$$

Applying relationships (4.16) to the equality $g_{sr} = u_{sr} + L_s\,A_r$, we get

$$\begin{aligned} \check{X}_s &= \sum_{r=1}^{n} g_{sr}\,X^r = \sum_{r=1}^{n} L_s\,A_r\,X^r = L_s\cdot\langle\mathbf{A}\,|\,\mathbf{X}\rangle\,,\\ X_r &= \sum_{s=1}^{n} X^s\,g_{sr} = \sum_{s=1}^{n} X^s\,L_s\,A_r = A_r\cdot\langle\mathbf{L}\,|\,\mathbf{X}\rangle\,. \end{aligned} \tag{4.17}$$

From first equality (4.17) we derive that vectors $\mathbf{X}$ and $\check{\mathbf{L}}$ are collinear:

$$X^r = \sum_{s=1}^{n} g^{rs}\,\check{X}_s = \sum_{s=1}^{n} \langle\mathbf{A}\,|\,\mathbf{X}\rangle\;g^{rs}\,L_s = \langle\mathbf{A}\,|\,\mathbf{X}\rangle\cdot\check{L}^r. \tag{4.18}$$

Note that $\mathbf{X} \neq 0$ and $\check{\mathbf{L}} \neq 0$. Hence $\langle\mathbf{A}\,|\,\mathbf{X}\rangle \neq 0$. Substituting formula (4.18) for X^s into both sides of second equality (4.17) and taking into account (4.14), we get

$$\langle\mathbf{A}\,|\,\mathbf{X}\rangle\cdot\check{L}_r = \langle\mathbf{A}\,|\,\mathbf{X}\rangle\cdot|\mathbf{L}|^2\cdot A_r. \tag{4.19}$$

Since $\langle\mathbf{A}\,|\,\mathbf{X}\rangle \neq 0$, we can cancel this factor in (4.19). As a result we obtain

$$A_r = \frac{\check{L}_r}{|\mathbf{L}|^2}. \tag{4.20}$$

Substituting (4.20) back into the equality $g_{sr} = u_{sr}+L_s\,A_r$, we get formula coinciding with (4.15). Remember that (4.15) is equivalent to (3.4) (see above). Further from (3.4) we can rederive normality equations (3.2). This step is the same as in proving previous lemma 4.3. Thus, lemma 4.4 is proved. □

5. Skew symmetry and differential forms.

Now we shall draw some conclusions from lemma 4.4. Lemma 4.4 asserts that functions $L_1, \ldots, L_n$ of the form (1.3) define generalized Legendre transformation (1.4) satisfying normality equations (2.6) if and only if their partial derivatives $g_{sr} = \tilde{\nabla}_r L_s$ are related to them by means of the equality

$$\frac{\partial L_s}{\partial v^r} = u_{sr} + L_s\,A_r, \tag{5.1}$$

where u_{sr} are components of some symmetric degenerate extended tensor field $\mathbf{u}$, which is not initially predefined, and A_r are components of some extended covector field $\mathbf{A}$, which also is not initially predefined. Alternating (5.1), we get

$$\frac{\partial L_s}{\partial v^r} - \frac{\partial L_r}{\partial v^s} = L_s\,A_r - L_r\,A_s. \tag{5.2}$$

For matrix u_{sr} due to its symmetry $u_{sr} = u_{rs}$ from (5.1) we derive

$$u_{sr} = \frac{1}{2}\left(\frac{\partial L_s}{\partial v^r} + \frac{\partial L_s}{\partial v^r}\right) - \frac{L_s\,A_r + L_r\,A_s}{2}. \tag{5.3}$$

If functions $A_1, \ldots, A_n$ are given, then (5.2) can be treated as differential equations for functions $L_1, \ldots, L_n$. Suppose we take some covector field $\mathbf{A}$ and solve differential equations (5.2). Does it mean that we can reconstruct the equality (5.1) and further get the solution of normality equations (2.6)? Indeed, we could define matrix u_{rs} by formula (5.3) and then derive (5.1) from (5.2) and (5.3). Anyway, matrix u_{rs} determined by formula (5.3) is symmetric, but it could be non-degenerate. In this case lemma 4.4 is not applicable and further thread of reasoning is torn.

However, thing are not so bad. Note that partial differential equations (5.2) admit gauge transformations of the following form:

$$L_r \to L_r, \qquad\qquad A_r \to A_r - \lambda\,L_r. \tag{5.4}$$

Here λ is some scalar factor, i. e. some extended scalar field in M. Applying gauge transformation (5.4) we get new fields $\mathbf{A}'$ and $\mathbf{u}'$ from initial ones:

$$A'_r = A_r - \lambda\,L_r. \qquad\qquad u'_{sr} = u_{sr} + \lambda\,L_s\,L_r. \tag{5.5}$$

If matrix u_{sr} in (5.5) is non-degenerate, then we can calculate determinant of u'_{sr}:

$$\det(u'_{sr}) = \det(u'_{sr})\left(1 + \lambda\sum_{s=1}^{n}\sum_{r=1}^{n} w^{rs}\,L_r\,L_s\right) = 0. \tag{5.6}$$

Here w^{rs} is inverse matrix for u_{rs}. Looking at characteristic equation (5.6), we see that it is linear with respect to scalar factor λ. This means that it is solvable if and only if double sum in round brackets is nonzero:

$$\|\mathbf{L}\|_{\mathbf{u}}=\sum_{s=1}^{n}\sum_{r=1}^{n}w^{rs}\,L_r\,L_s\neq 0. \tag{5.7}$$

Now we shall leave inequality (5.7) for separate study in separate paper and we shall formulate main result of this section in the following theorem.

Theorem 5.1. *Any solution of differential equations* (5.2) *defines locally diffeomorphic generalized Legendre map* (1.4) *if metric tensor* (2.2) *is non-degenerate and if one of the following two conditions is fulfilled: matrix* (5.3) *is degenerate or* $\|\mathbf{L}\|_{\mathbf{u}}\neq 0$, *if matrix* (5.3) *is non-degenerate.*

Note that differential equations (5.2) have no partial derivatives with respect to $x^1,\,\ldots\,,\,x^n$. This means that we can fix some arbitrary point $p\in M$ and consider partial differential equations (5.2) within fixed fiber of tangent bundle. Then extended covector fields $\mathbf{L}$ and $\mathbf{A}$ can be treated as differential 1-forms:

$$\mathbf{L}=\sum_{i=1}^{n}L_i\,dv^i, \qquad\qquad \mathbf{A}=\sum_{i=1}^{n}A_i\,dv^i. \tag{5.8}$$

In terms of differential forms (5.8) differential equations (5.2) are written as

$$d\,\mathbf{L}=\mathbf{L}\wedge\mathbf{A}. \tag{5.9}$$

Remark. Here we should especially emphasize that differential forms (5.8) are defined only within separate fibers of tangent bundle TM. They cannot be canonically extended as 1-forms in TM in whole.

6. Compatibility conditions.

Initial normality equations (2.6), as well as their transformed counterparts (5.9), form overdetermined system of partial differential equations for the functions (1.3). They should be studied for compatibility. Let's apply external differentiation operator d to both sides of (5.9). As a result we get

$$0=d\,(d\,\mathbf{L})=d\,\mathbf{L}\wedge\mathbf{A}-\mathbf{L}\wedge d\,\mathbf{A}=\mathbf{L}\wedge\mathbf{A}\wedge\mathbf{A}-\mathbf{L}\wedge d\,\mathbf{A}=-\mathbf{L}\wedge d\,\mathbf{A}.$$

This means that external product $\mathbf{L}\wedge d\,\mathbf{A}$ is equal to zero:

$$\mathbf{L}\wedge d\,\mathbf{A}=0 \tag{6.1}$$

Lemma 6.1. *For* 1*-form* $\mathbf{L}\neq 0$ *and differential* m*-form* Ω *the equality* $\mathbf{L}\wedge\Omega=0$ *is equivalent to the equality* $\Omega=\mathbf{L}\wedge\mathbf{B}$ *for some differential* $(m-1)$*-form* $\mathbf{B}$.

Lemma 6.1 is special case of division theorem by E. Cartan, see proof in [15]. Applying lemma 6.1 to $\Omega=d\,\mathbf{A}$ in (6.1), we get the equality

$$d\,\mathbf{A}=\mathbf{L}\wedge\mathbf{B}, \tag{6.2}$$

where $\mathbf{B}$ is some differential 1-form within separate fibers of tangent bundle TM. Differential equations (6.2) form compatibility condition for equations (5.9). They have almost the same shape as (5.9). Therefore we shall treat them similarly:

$$0 = d\,(d\,\mathbf{A}) = d\,\mathbf{L}\wedge\mathbf{B} - \mathbf{L}\wedge d\,\mathbf{B} = \mathbf{L}\wedge\mathbf{A}\wedge\mathbf{B} - \mathbf{L}\wedge d\,\mathbf{B} = \mathbf{L}\wedge(\mathbf{A}\wedge\mathbf{B} - d\,\mathbf{B}).$$

Applying lemma 6.1 to the above equality, we get differential equations for $\mathbf{B}$:

$$d\,\mathbf{B} = \mathbf{A}\wedge\mathbf{B} + \mathbf{L}\wedge\mathbf{C}. \tag{6.3}$$

Here $\mathbf{C}$ is some other 1-form. Differential equations (6.3) are a little bit more complicated than (5.9) and (6.1). But nevertheless we apply operator d to them:

$$\begin{gathered} d\,(d\,\mathbf{B}) = d\,\mathbf{A}\wedge\mathbf{B} - \mathbf{A}\wedge d\,\mathbf{B} + d\,\mathbf{L}\wedge\mathbf{C} - \mathbf{L}\wedge d\,\mathbf{C} = \mathbf{L}\wedge\mathbf{B}\wedge\mathbf{B} -\\ -\mathbf{A}\wedge\mathbf{A}\wedge\mathbf{B} - \mathbf{A}\wedge\mathbf{L}\wedge\mathbf{C} + \mathbf{L}\wedge\mathbf{A}\wedge\mathbf{C} - \mathbf{L}\wedge d\,\mathbf{C}. \end{gathered}$$

Applying lemma 6.1 to the above equality, we get differential equations for $\mathbf{C}$:

$$d\,\mathbf{C} = 2\,\mathbf{A}\wedge\mathbf{C} + \mathbf{L}\wedge\mathbf{D}. \tag{6.4}$$

Now again, we apply external differentiation d to the equations (6.4) and we get

$$\begin{gathered} d\,(d\,\mathbf{C}) = 2\,d\,\mathbf{A}\wedge\mathbf{C} - 2\,\mathbf{A}\wedge d\,\mathbf{C} + d\,\mathbf{L}\wedge\mathbf{D} - \mathbf{L}\wedge d\,\mathbf{D} = 2\,\mathbf{L}\wedge\mathbf{B}\wedge\mathbf{C} -\\ -4\,\mathbf{A}\wedge\mathbf{A}\wedge\mathbf{C} - 2\,\mathbf{A}\wedge\mathbf{L}\wedge\mathbf{D} + \mathbf{L}\wedge\mathbf{A}\wedge\mathbf{D} - \mathbf{L}\wedge d\,\mathbf{D}. \end{gathered}$$

Applying lemma 6.1 to this equality, we derive differential equations for $\mathbf{D}$:

$$d\,\mathbf{D} = 3\,\mathbf{A}\wedge\mathbf{D} + 2\,\mathbf{B}\wedge\mathbf{C} + \mathbf{L}\wedge\mathbf{E}. \tag{6.5}$$

Now it is clear that further steps require special notations and study of recurrent procedure underlying all above formulas (5.9), (6.2), (6.3), (6.4), (6.5). Let's denote

$$\begin{aligned} &\mathbf{L} = \mathbf{A}_0, && \mathbf{A} = \mathbf{A}_1, && \mathbf{B} = \mathbf{A}_2,\\ &\mathbf{C} = \mathbf{A}_3, && \mathbf{D} = \mathbf{A}_4, && \mathbf{E} = \mathbf{A}_5. \end{aligned} \tag{6.6}$$

In terms of notations (6.6) introduced just above we can rewrite our equations as

$$\begin{aligned} &d\,\mathbf{A}_0 = \mathbf{A}_0\wedge\mathbf{A}_1, && d\,\mathbf{A}_1 = \mathbf{A}_0\wedge\mathbf{A}_2,\\ &d\,\mathbf{A}_2 = \mathbf{A}_0\wedge\mathbf{A}_3 + \mathbf{A}_1\wedge\mathbf{A}_2, && d\,\mathbf{A}_3 = \mathbf{A}_0\wedge\mathbf{A}_4 + 2\,\mathbf{A}_1\wedge\mathbf{A}_3. \end{aligned} \tag{6.7}$$

Equations (6.5) are a little bit more complicated. They are written as follows:

$$d\,\mathbf{A}_4 = \mathbf{A}_0\wedge\mathbf{A}_5 + 3\,\mathbf{A}_1\wedge\mathbf{A}_4 + 2\,\mathbf{A}_2\wedge\mathbf{A}_3. \tag{6.8}$$

Looking at (6.7) and (6.8), one can formulate a conjecture concerning general structure of all such equations, for those, which are already written, and for all others.

Conjecture 6.1. *Differential equations* (5.9) *lead to infinite series of compatibility conditions that in terms of notations* (6.6) *can be written as*

$$d\,\mathbf{A}_k=\sum_{i=0}^{\left[\frac{k}{2}\right]}C^i_{k+1}\,\mathbf{A}_i\wedge\mathbf{A}_{k+1-i},\quad \textit{where}\quad C^1_{k+1}=1. \tag{6.9}$$

Here C^i_{k+1} are some constants similar to binomial coefficients, but not coinciding with them. They should be calculated recurrently. By square brackets in upper limit of sum in (6.9) *we denote entire part of fraction $k/2$.*

First of all let's derive recurrent relationships for coefficients C^i_{k+1} in (6.9). Applying external differentiation d to both sides of (6.9), we get

$$\begin{gathered}0=d\,(d\,\mathbf{A}_k)=\sum_{i=0}^{\left[\frac{k}{2}\right]}C^i_{k+1}\,(d\,\mathbf{A}_i\wedge\mathbf{A}_{k+1-i}-\mathbf{A}_i\wedge d\,\mathbf{A}_{k+1-i})=\\ =\mathbf{A}_0\wedge\mathbf{A}_1\wedge\mathbf{A}_{k+1}-\mathbf{A}_0\wedge d\,\mathbf{A}_{k+1}+\sum_{i=1}^{\left[\frac{k}{2}\right]}C^i_{k+1}\,\bigl(C^0_{i+1}\,\mathbf{A}_0\wedge\mathbf{A}_{i+1}+\\ +\,\ldots\bigr)\wedge\mathbf{A}_{k+1-i}-\sum_{i=1}^{\left[\frac{k}{2}\right]}C^i_{k+1}\,\mathbf{A}_i\wedge\bigl(C^0_{k+2-i}\,\mathbf{A}_0\wedge\mathbf{A}_{k+2-i}+\ldots\bigr).\end{gathered} \tag{6.10}$$

Terms denoted by dots in the above equality have no entry of $\mathbf{A}_0$. Below we shall prove that they do cancel each other. Now from (6.10) we derive

$$\begin{gathered}A_0\wedge\left(-d\,\mathbf{A}_{k+1}+\sum_{i=1}^{\left[\frac{k}{2}\right]}C^i_{k+1}\,\mathbf{A}_{i+1}\wedge\mathbf{A}_{k+1-i}\,+\right.\\ \left.+\,\mathbf{A}_1\wedge\mathbf{A}_{k+1}+\sum_{i=1}^{\left[\frac{k}{2}\right]}C^i_{k+1}\,\mathbf{A}_i\wedge\mathbf{A}_{k+2-i}\right)=0.\end{gathered} \tag{6.11}$$

Applying lemma 6.1 to (6.11), we derive the following equality for $d\,\mathbf{A}_{k+1}$:

$$\begin{gathered}d\,\mathbf{A}_{k+1}=\mathbf{A}_0\wedge\mathbf{A}_{k+2}+\mathbf{A}_1\wedge\mathbf{A}_{k+1}+\\ +\sum_{i=1}^{\left[\frac{k}{2}\right]}C^i_{k+1}\,\mathbf{A}_i\wedge\mathbf{A}_{k+2-i}+\sum_{i=2}^{\left[\frac{k+2}{2}\right]}C^{i-1}_{k+1}\,\mathbf{A}_i\wedge\mathbf{A}_{k+2-i}.\end{gathered} \tag{6.12}$$

Comparing (6.12) and (6.9), we can write the following recurrent formula for C^i_{k+1}:

$$C^i_{k+2}=\begin{cases}1 & \text{for}\ \ i=0;\\ C^{i-1}_{k+1}+C^i_{k+1} & \text{for}\ \ 0<2i<k+1;\\ C^{i-1}_{k+1} & \text{for}\ \ 2i=k+1.\end{cases} \tag{6.13}$$

Though formula (6.13) is quite similar to corresponding recurrent formula for binomial coefficients, it doesn't coincide with that formula.

Now let's study terms denoted by dots in formula (6.10). Total sum of all these terms is given by the following explicit formula:

$$\begin{aligned} S = &\sum_{i=1}^{\left[\frac{k}{2}\right]}\sum_{s=1}^{\left[\frac{i}{2}\right]} C_{k+1}^{i}\,C_{i+1}^{s}\,\mathbf{A}_s\wedge\mathbf{A}_{i+1-s}\wedge\mathbf{A}_{k+1-i}\,-\\ -&\sum_{r=1}^{\left[\frac{k}{2}\right]}\sum_{e=1}^{\left[\frac{k+1-r}{2}\right]} C_{k+1}^{r}\,C_{k+2-r}^{e}\,\mathbf{A}_r\wedge\mathbf{A}_e\wedge\mathbf{A}_{k+2-r-e}.\end{aligned}\tag{6.14}$$

Indices in external product in first sum of formula (6.14) satisfy inequalities

$$1\leqslant i<k+1-i,\qquad\qquad 1\leqslant s<i+1-s.\tag{6.15}$$

Inequalities (6.15) mean that indices in external product $\mathbf{A}_s\wedge\mathbf{A}_{i+1-s}\wedge\mathbf{A}_{k+1-i}$ are properly arranged, i. e. they are in growing order:

$$s<i+1-s<k+1-i.$$

Here are inequalities for indices in external product $\mathbf{A}_r\wedge\mathbf{A}_e\wedge\mathbf{A}_{k+2-r-e}$:

$$1\leqslant r<k+1-r,\qquad\qquad 1\leqslant e<k+2-r-e.\tag{6.16}$$

Inequalities (6.16) cannot provide proper ordering of indices r, e, $k+2-r-e$. Therefore we consider three possible subranges for index r:

$$\text{Subrange 1:}\qquad r<e;\tag{6.17}$$

$$\text{Subrange 2:}\qquad e<r<k+2-r-e;\tag{6.18}$$

$$\text{Subrange 3:}\qquad k+2-r-e<r.\tag{6.19}$$

Inequalities (6.16) define polygon $ABCD$ on re-plane (see Fig. 6.1 below), sides AB and AD are closed, sides BC and CD are open. Subranges (6.17), (6.18), and (6.19) break this polygon into three triangular domains ABE, ADE, and CDE. Segments AE and DE are in open parts of their boundaries.

Subrange 1. In this subrange indices in external product $\mathbf{A}_r\wedge\mathbf{A}_e\wedge\mathbf{A}_{k+2-r-e}$ are properly ordered. Therefore we can match them with indices of another external product $\mathbf{A}_s\wedge\mathbf{A}_{i+1-s}\wedge\mathbf{A}_{k+1-i}$, i. e. we can write

$$r=s,\qquad e=i+1-s,\qquad k+2-r-e=k+1-i.\tag{6.20}$$

Third equality in (6.20) follows from first two ones. Therefore we can treat first two equalities as a map from is-plane to re-plane. This is linear invertible map taking integer points to integer point. So is inverse map:

$$\begin{cases} r=s,\\ e=i+1-s,\end{cases}\qquad\qquad \begin{cases} i=r+e-1,\\ s=r.\end{cases}\tag{6.21}$$

Due to maps (6.21) triangle ABE is associated with triangle FGH (see Fig. 6.2 below). Indeed, we have the following correspondence of sides and inequalities:

$$\begin{array}{lcl} AB\ (r\geqslant 1) & \longrightarrow & FG\ (s\geqslant 1);\\ BE\ (e<k+2-r-e) & \longrightarrow & GH\ (s>2\,i-k);\\ EA\ (e>r) & \longrightarrow & HF\ (s<i+1-s).\end{array}\tag{6.22}$$

Due to inequalities in right column of (6.22) we see that side FG of subrange 1 mapped to is-plane is closed. Other two sides GH and HF are open.

Subrange 2. In this subrange indices in external product $\mathbf{A}_r\wedge\mathbf{A}_e\wedge\mathbf{A}_{k+2-r-e}$ are not properly ordered. We need to transpose first two terms in it. As a result we get external product $\mathbf{A}_e\wedge\mathbf{A}_r\wedge\mathbf{A}_{k+2-r-e}$ that can be matched with external product $\mathbf{A}_s\wedge\mathbf{A}_{i+1-s}\wedge\mathbf{A}_{k+1-i}$. This yields another pair of mutually inverse maps linking re-plane with is-plane. These maps are given by formulas

$$\begin{cases} r=i+1-s,\\ e=s,\end{cases}\qquad\qquad \begin{cases} i=r+e-1,\\ s=e.\end{cases}\tag{6.23}$$

Applying (6.23) to inequalities defining sides of triangle AED, we get

$$\begin{array}{lcl} AE\ (e<r) & \longrightarrow & FH\ (s<i+1-s);\\ ED\ (r<k+2-r-e) & \longrightarrow & HG\ (s>2\,i-k);\\ DA\ (e\geqslant 1) & \longrightarrow & GF\ (s\geqslant 1).\end{array}\tag{6.24}$$

Its important that subrange 2 is mapped onto the same triangle in is-plane as subrange 1, and again side GF is closed, while other two sides FH and HG of triangle FHG are open.

Subrange 3. In this subrange indices in external product $\mathbf{A}_r\wedge\mathbf{A}_e\wedge\mathbf{A}_{k+2-r-e}$ also are not properly ordered. We need to move $\mathbf{A}_r$ to third position. Then we get external product $\mathbf{A}_e \wedge \mathbf{A}_{k+2-r-e} \wedge \mathbf{A}_r$ that can be matched with external product $\mathbf{A}_s \wedge \mathbf{A}_{i+1-s} \wedge \mathbf{A}_{k+1-i}$. This matching yields two maps inverse to each other:

$$\begin{cases} r = k+1-i, \\ e = s, \end{cases} \qquad\qquad \begin{cases} i = k+1-r, \\ s = e. \end{cases} \tag{6.25}$$

Applying (6.25) to inequalities defining sides of triangle DEC, we get

$$\begin{aligned} &DE\ (k+2-r-e<r) && \longrightarrow && GH\ (s>2\,i-k); \\ &EC\ (e<k+2-r-e) && \longrightarrow && HK\ (s<i+1-s); \\ &CD\ (r<k+1-r) && \longrightarrow && KG\ (2\,i>k+1). \end{aligned} \tag{6.26}$$

Formulas (6.26) mean that subrange 3 is mapped onto the smaller triangle GHK (see Fig. 6.2). All three sides of this triangle are open.

Thus, due to (6.22), (6.24), and (6.26) we see that under the action of maps (6.21), (6.23), and (6.25) two parts of tetragone $ABCD$ covers triangle FGH twice, while third part of this tetragone covers smaller triangle GHK. All maps (6.21), (6.23), and (6.25) are given by linear functions with entire coefficients. Hence they map grid of entire points in re-plane onto the grid of entire points in is-plane and vice versa. Note also that inequalities (6.15) define triangle FGK complementary to triangle GHK within triangle FGH. This means that each entire point of triangle FGH with closed side FG is associated with three terms in sum (6.14), except for those on segment GK. And we have two terms in sum (6.14) associated with each inner entire point of segment GK. Therefore in order to prove that $S=0$ in (6.14) we should prove series of identities for coefficients C^i_k. First identity

$$C^i_{k+1}\,C^s_{i+1} - C^s_{k+1}\,C^{i+1-s}_{k+2-s} + C^{i+1-s}_{k+1}\,C^s_{k+1-i+s} = 0 \tag{6.27}$$

should be fulfilled within open triangle FGK. The same identity (6.27) should be fulfilled on its side FG, except for ending points F and G. Next identity

$$C^{i+1-s}_{k+1}\,C^s_{k+1-i+s} - C^s_{k+1}\,C^{i+1-s}_{k+2-s} - C^{k+1-i}_{k+1}\,C^s_{i+1} = 0 \tag{6.28}$$

should be fulfilled within open triangle GHK. For exceptional points, i. e. for entire points within open segment GK, we should prove the identity

$$C^{i+1-s}_{k+1}\,C^s_{k+1-i+s} - C^s_{k+1}\,C^{i+1-s}_{k+2-s} = 0. \tag{6.29}$$

Note that open segment GK has entire points if and only if k is odd number not less than 7, i. e. we should set $k = 2\,m+7$, where m is arbitrary non-negative number. In this case $i = m+4$, while $s = p+2$, where p is arbitrary non-negative number such that $2\,p < m+1$. Under these conditions identity (6.29) reduces to

$$C^{m+3-p}_{2\,m+8}\,C^{p+2}_{m+6+p} - C^{p+2}_{2\,m+8}\,C^{m+3-p}_{2\,m+7-p} = 0. \tag{6.30}$$

In order to prove all these identities we should state formal definition of coefficients C^i_k, other than formula (6.9), which is only a conjecture yet.

Definition 6.1. Normality coefficients C^i_k are determined for all integer $k \geqslant 1$ and all integer i such that $0 \leqslant 2\,i < k$ by recurrent formula

$$C^i_{k+1} = \begin{cases} 1 & \text{for } \ i = 0, \\ C^{i-1}_k + C^i_k & \text{for } \ 0 < 2i < k, \\ C^{i-1}_k & \text{for } \ 2i = k \end{cases} \tag{6.31}$$

and by value of initial coefficient $C^0_1 = 1$ in the series.

It is easy to see that definition 6.1 is correct and self-consistent. Formula (6.31) is actually the same formula as (6.13). Now let's calculate few initial coefficients in the series and let's arrange them as a table. Applying formula (6.31), we get

$$\begin{array}{llllll}
C^0_1 = 1, & & & & & \\
C^0_2 = 1, & & & & & \\
C^0_3 = 1, & C^1_3 = 1, & & & & \\
C^0_4 = 1, & C^1_4 = 2, & & & & \\
C^0_5 = 1, & C^1_5 = 3, & C^2_5 = 2, & & & \\
C^0_6 = 1, & C^1_6 = 4, & C^2_6 = 5, & & & \\
C^0_7 = 1, & C^1_7 = 5, & C^2_7 = 9, & C^3_7 = 5, & & \\
C^0_8 = 1, & C^1_8 = 6, & C^2_8 = 14, & C^3_8 = 14, & & \\
C^0_9 = 1, & C^1_9 = 7, & C^2_9 = 20, & C^3_9 = 28, & C^4_9 = 14, & \\
C^0_{10} = 1, & C^1_{10} = 8, & C^2_{10} = 27, & C^3_{10} = 48, & C^4_{10} = 42, & \\
C^0_{11} = 1, & C^1_{11} = 9, & C^2_{11} = 35, & C^3_{11} = 75, & C^4_{11} = 90, & C^5_{11} = 42, \\
C^0_{12} = 1, & C^1_{12} = 10, & C^2_{12} = 44, & C^3_{12} = 110, & C^4_{12} = 165, & C^5_{12} = 132.
\end{array}$$

One can easily write general formula for elements in first two columns of this table:

$$C^0_k = 1, \qquad\qquad C^1_k = k - 2. \tag{6.32}$$

General formula for elements of third column is less obvious:

$$C^2_k = \frac{(k-2)(k-3)}{2} - 1. \tag{6.33}$$

However, one can go further and write general formula for all elements of the table:

$$C^i_k = \prod^{i}_{s=1} \frac{k-1-s}{s} - \prod^{k-i}_{s=1} \frac{k-1-s}{s}. \tag{6.34}$$

Formula (6.34) generalizes (6.32) and (6.33). In order to prove this general formula it is sufficient to make sure that it is correct for initial part of the above table and then test recursion (6.31) for it. When this is done, proof of the identities (6.27), (6.28), and (6.30) is nothing, but pure calculations.

Thus, we have proved that $S = 0$ in (6.14), and hence we have proved conjecture 6.1. Now we can state it as a theorem.

Theorem 6.1. *Differential equations* (5.9) *with* $\mathbf{L} = \mathbf{A}_0$ *and* $\mathbf{A} = \mathbf{A}_1$ *lead to infinite series of compatibility conditions* (6.9)*, where coefficients* C^i_k *in* (6.9) *are determined by formula* (6.34)*.*

7. An example of solution of normality equations.

Theorem 6.1 and formula (6.9) give a way for constructing special solutions of normality equations (5.9). Let's write first two equations given by formula (6.9) and let's loop them assuming that $\mathbf{A}_2 = \mathbf{A}_0$. Then we have

$$d\,\mathbf{A}_0 = \mathbf{A}_0 \wedge \mathbf{A}_1, \qquad\qquad d\,\mathbf{A}_1 = \mathbf{A}_0 \wedge \mathbf{A}_0 = 0. \tag{7.1}$$

Second equation (7.1) means that $\mathbf{A}_1$ is closed 1-form within separate fibers of tangent bundle. Locally it is represented as $\mathbf{A}_1 = d\varphi$ for some scalar function in TM. First equation (7.1) for $\mathbf{A}_0 = \mathbf{L}$ then is written as

$$d\,\mathbf{L} = \mathbf{L} \wedge d\varphi. \tag{7.2}$$

Let's define another 1-form $\mathbf{M} = e^{\varphi}\,\mathbf{L}$. For this form from (7.2) we derive:

$$d\mathbf{M} = e^{\varphi}\,d\varphi \wedge \mathbf{L} + e^{\varphi}\,d\mathbf{L} = e^{\varphi}\,(d\varphi \wedge \mathbf{L} + \mathbf{L} \wedge d\varphi) = 0.$$

Thus, $\mathbf{M}$ appears to be closed form. Like $\mathbf{A}_1$ above, it is determined by some scalar function: $\mathbf{M} = dL$. For differential form $\mathbf{L}$ this yields

$$\mathbf{L} = e^{-\varphi}\,dL, \tag{7.3}$$

where $\varphi = \varphi(x^1, \ldots, x^n, v^1, \ldots, v^n)$ and $L = L(x^1, \ldots, x^n, v^1, \ldots, v^n)$. Remember that components of differential form $\mathbf{L}$ determine generalized Legendre transformation λ, see (5.8) and functions (1.3). For these functions from (7.3) we derive

$$L_i = e^{-\varphi}\,\frac{\partial L}{\partial v^i}. \tag{7.4}$$

Example. Let's consider three dimensional case $n = 3$ and let's choose functions

$$\varphi = -v^1, \qquad\qquad L = v^1 + \frac{1}{2}\left((v^2)^2 + (v^3)^2\right). \tag{7.5}$$

Applying formula (7.4) to functions (7.5), in this case we get

$$L_1 = e^{v^1}, \qquad\qquad L_2 = v^2\,e^{v^1}, \qquad\qquad L_2 = v^3\,e^{v^1}. \tag{7.6}$$

These three functions define regular fiber-preserving map from TM to T^*M. Its

Jacobi matrix can be calculated explicitly. Indeed, applying (2.2) to (7.6), we get

$$g_{ij}=e^{v^1}\cdot\begin{Vmatrix}1 & 0 & 0\\ v^2 & 1 & 0\\ v^3 & 0 & 1\end{Vmatrix}. \tag{7.7}$$

We also can explicitly calculate inverse matrix for lower triangular matrix (7.7):

$$g^{ij}=e^{-v^1}\cdot\begin{Vmatrix}1 & 0 & 0\\ -v^2 & 1 & 0\\ -v^3 & 0 & 1\end{Vmatrix}. \tag{7.8}$$

Now, using matrix (7.8), we apply formula (2.4) to components of covector $\mathbf{L}$. As a result we get vector $\mathbf{L}$ with the following components:

$$L^1=1-(v^2)^2-(v^3)^2,\qquad\qquad L^2=v^2,\qquad\qquad L^2=v^3. \tag{7.9}$$

Modulus of vector $\mathbf{L}$ calculated in non-symmetric metric (7.7) is given by formula

$$|\mathbf{L}|^2=\sum_{i=1}^{n}L^i\,L_i=e^{v^1}. \tag{7.10}$$

Now we are able to calculate matrix of projection operator $\mathbf{P}$:

$$P^i_j=\begin{Vmatrix}1-L^1 & -L^1\,v^2 & -L^1\,v^3\\ \\ -v^2 & 1-(v^2)^3 & -v^2\,v^3\\ \\ -v^3 & -v^2\,v^3 & 1-(v^3)^2\end{Vmatrix}. \tag{7.11}$$

We used formula (2.3) for P^i_j and formula (7.10) for $|\mathbf{L}|^2$. We keep L^1 in (7.11) as notation for the sake of brevity in order to have formula looking pretty well. Its value is given by formula (7.9).

Next step is to calculate components of tensor field $\mathbf{A}$ given by formula (2.7). Upon alternating matrix A^{rs} we get the following one:

$$A^{rs}-A^{sr}=e^{-v^1}\cdot\begin{Vmatrix}0 & v^2 & v^3\\ -v^2 & 0 & 0\\ -v^3 & 0 & 0\end{Vmatrix}. \tag{7.12}$$

Substituting (7.12) and (7.11) into (2.6), we easily find that normality equations (2.6) are fulfilled. Thus, we have constructed an example of generalized Legendre transformation λ satisfying normality equations. It is given by functions (7.6). This is not classical Legendre transformation. However, it differs from classical one (1.2) only by scalar factor e^{φ} (see formula (7.4)). Therefore we say that (7.6) is trivial example of non-classical Legendre transformation satisfying normality equations.

In order to construct non-trivial solution of normality equations (2.6) one should choose another way of looping for the chain of differential equations (6.9). For example we can set $\mathbf{A}_3=\mathbf{A}_0$. This leads to more complicated calculations than we carried out above. Therefore this example will be studied in separate paper.

8. Acknowledgements.

This work is supported by grant from Russian Fund for Basic Research (project 01-01-00996-a, coordinator of project Ya. T. Sultanaev), and by grant from Academy of Sciences of the Republic Bashkortostan (coordinator N. M. Asadullin). I am grateful to these organizations for financial support.

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[1]Electronic Archive that was initially residing at Los Alamos National Laboratory of USA (LANL). As it is known to me, currently primary server of Archive is at Cornell University. Archive is accessible through Internet **http://arXiv.org**, it has mirror site **http://ru.arXiv.org** at the Institute for Theoretical and Experimental Physics (ITEP, Moscow) and many other mirrors around the world.