ON THE CONCEPT OF NORMAL SHIFT IN NON-METRIC GEOMETRY.

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Abstract. Theory of Newtonian dynamical systems admitting normal shift of hypersurfaces was first developed for the case of Riemannian manifolds. Recently it was generalized for manifolds geometric equipment of which is given by some regular Lagrangian or, equivalently, by some regular Hamiltonian dynamical system. In present paper we consider further generalization of this theory for the case, when geometry of manifold is given by generalized Legendre transformation.

1. What is normal shift ? Brief historical overview.

Phenomenon of normal shift is very simple by its nature. Let's consider it in three-dimensional Euclidean space $\mathbb{R}^3$. Suppose that σ is some smooth orientable surface in $\mathbb{R}^3$. At each point $p \in \sigma$ one can draw unit normal vector $\mathbf{n}$ such that $\mathbf{n} = \mathbf{n}(p)$ would be a smooth vector-valued function on σ. Let's move each point p of σ in the direction of vector $\mathbf{n}(p)$ to the distance t which is the same for all points $p \in \sigma$. Then moved points p_t would form another surface σ_t as shown on Fig. 1.1. Changing parameter t we would obtain one-parametric family of surfaces. This construction is known as Bonnet transformation.

In Bonnet construction initial surface σ is transformed by moving each point of σ. Trajectories of motion in this case are straight lines directed along normal vectors and points of σ move along them with a constant speed $|\mathbf{v}| = 1$. Therefore parameter t, which is the distance of displacement, can also be interpreted as time variable. Bonnet noted that all surfaces σ_t in his construction are perpendicular to the trajectories of moving points. For this reason his construction is also known as **normal displacement** or **normal shift**.

Basic observation by Bonnet, i. e. orthogonality of surfaces σ_t and shift trajectories, gave an impetus for generalization of his construction. This was done by me

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and my student A. Yu. Boldin in preprint [1] (see also [2], [3]). We replaced straight lines in Bonnet construction by trajectories of Newtonian dynamical system:

$$\dot{\mathbf{r}} = \mathbf{v}, \qquad \dot{\mathbf{v}} = \mathbf{F}(\mathbf{r}, \mathbf{v}). \tag{1.1}$$

This dynamical system describes the motion of a particle of unit mass according to Newton's second law. Vector $\mathbf{F}$ in right hand side of (1.1) is a vector of force acting on this particle. In order to initialize a shift of surface σ we applied initial data

$$\mathbf{r}\Bigr|_{t=0} = \mathbf{r}(p), \qquad \mathbf{v}\Bigr|_{t=0} = \nu(p)\cdot\mathbf{n}(p) \tag{1.2}$$

to differential equations (1.1). For classical Bonnet transformation $|\mathbf{v}| = 1$. In our construction modulus of velocity vector is not constant. It's initial value on σ is determined by some scalar function $\nu = \nu(p)$.

Initial data (1.2) with parameter $p \in \sigma$ determine a family of trajectories of Newtonian dynamical system (1.1). Points of σ moving along these trajectories form one-parametric family of surfaces σ_t. Thus we have generalization of Bonnet construction. This is the shift of σ along trajectories of Newtonian dynamical system (1.1). It's clear that this is normal shift for initial instant of time $t = 0$. However, in general, orthogonality of σ_t and shift trajectories gets broken at any other instant of time $t \neq 0$. Only for special Newtonian dynamical systems, i. e. for special force fields $\mathbf{F} = \mathbf{F}(\mathbf{r}, \mathbf{v})$, one can keep orthogonality for $t \neq 0$ at the expense of proper choice of function $\nu = \nu(p)$ in (1.2).

Definition 1.1. Shift of surface $\sigma \subset \mathbb{R}^3$ along trajectories of dynamical system (1.1) determined by initial data (1.2) is called **normal shift** if all displaced surfaces σ_t are perpendicular to shift trajectories.

Definition 1.2. We say that Newtonian dynamical system (1.1) satisfies **normality condition** if for sufficiently small part of any surface $\sigma \subset \mathbb{R}^3$ there is a function $\nu = \nu(p)$ on it determining normal shift of this part of σ.

Words "sufficiently small part" of σ in definition 1.2 mean that for any fixed point $p_0 \in \sigma$ there is some sufficiently small open neighborhood of p_0 in σ, where proper function $\nu = \nu(p)$ does exist. Let $\nu_0 \neq 0$ be some arbitrary nonzero constant. Then we can normalize function $\nu(p)$ at the fixed point p_0:

$$\nu(p_0) = \nu_0. \tag{1.3}$$

Definition 1.3. Newtonian dynamical system (1.1) satisfies **strong normality condition** if for any surface $\sigma \subset \mathbb{R}^3$, for any fixed point $p_0 \in \sigma$, and for any constant $\nu_0 \neq 0$ there is some open neighborhood of p_0 and some smooth function $\nu(p)$ normalized by the condition (1.3) in this neighborhood such that it determines normal shift in the sense of definition 1.1.

Newtonian dynamical systems satisfying strong normality condition form special subclass which appears to be interesting object for study. We call them **systems admitting normal shift** of hypersurfaces. In simpler words, these are systems capable to implement normal displacement of any hypersurface σ in $\mathbb{R}^3$ with any predefined value ν_0 of initial velocity.

It's obvious that definitions 1.1, 1.2, and 1.3 can be formulated for higher dimensional Euclidean spaces and for Riemannian manifolds as well. One should only replace surfaces by hypersurfaces and replace $\mathbb{R}^3$ by $\mathbb{R}^n$ or by arbitrary smooth manifold M with Riemannian metric $\mathbf{g}$. This was done in papers [4–6]. Instead of (1.1) in Riemannian manifolds we write differential equations

$$\dot{x}^i = v^i, \qquad\qquad \nabla_t v^i = F^i(x^1, \ldots, x^n, v^1, \ldots, v^n), \tag{1.4}$$

where $i = 1, \ldots, n$. Here $x^1, \ldots, x^n$ are coordinates of a point p in some local chart of Riemannian manifold M, while $v^1, \ldots, v^n$ are components of velocity vector $\mathbf{v} \in T_p(M)$. Vector $\mathbf{F} \in T_p(M)$ with components $F^1, \ldots, F^n$ determines force field of Newtonian dynamical system (1.4).

In papers [5] and [6] we have shown that strong normality condition applied to Newtonian dynamical system (1.4) leads to a system of partial differential equations for components of force vector $\mathbf{F} = \mathbf{F}(p, \mathbf{v})$. This system subdivides into two parts:

– **weak normality equations** written for $n \geqslant 2$;
– **additional normality equations** written for $n \geqslant 3$.

Note that $n = 2$ is lower limit for the dimension of manifold M. Indeed, for $n = 1$, hypersurfaces are points, concept of normal shift in this case has no meaning. Note also that additional normality equations are written for $n \geqslant 3$. Therefore $n = 2$ is exceptional dimension. Theory of dynamical systems admitting normal shift in two-dimensional case $n = 2$ is rather different from that of multidimensional case $n \geqslant 3$. This fact is reflected in theses [7] and [8].

In two-dimensional case strong normality condition is equivalent to weak normality equations for force field $\mathbf{F}$. Weak normality equations in this case can be reduced to one nonlinear partial differential equation for one scalar function of four variables. This equation cannot be solved explicitly in general. However, one can construct some special explicit solutions of it (see paper [9] and thesis [8], where numerous examples are given).

In multidimensional case $n \geqslant 3$ strong normality condition is equivalent to complete system of weak normality equations and additional normality equations for components force field $\mathbf{F}$. As appeared, in this case one can find an explicit formula for general solution of this rather huge system of PDE's (see papers [10], [11], and Chapter VII of thesis [7]). Here is this formula:

$$F_i = \frac{h(W)}{W_v} \cdot \frac{v_i}{|\mathbf{v}|} - \sum_{k=1}^{n} \frac{\nabla_k W}{W_v} \cdot \frac{2\, v^k\, v_i - |\mathbf{v}|^2\, \delta^k_i}{|\mathbf{v}|}. \tag{1.5}$$

Formula (1.5) contain two arbitrary functions W and h, where $h = h(w)$ is an arbitrary smooth function of one variable, while $W = W(x^1, \ldots, x^n, v)$ is an arbitrary smooth function of $n + 1$ variables with nonzero derivative

$$W_v = \frac{\partial W}{\partial v} \neq 0.$$

By $\nabla_k W$ in formula (1.5) we denote partial derivatives

$$\nabla_k W = \frac{\partial W}{\partial x^k},$$

while v_i and v^k are covariant and contravariant components of velocity vector $\mathbf{v}$:

$$v_i = \sum_{k=1}^{n} g_{ik}\,v^k.$$

When substituted into (1.5), last $(n+1)$-th argument v of W_v and $\nabla_k W$ is replaced by modulus of velocity vector: $v = |\mathbf{v}|$.

Division of complete system of normality equations into two parts is not artificial by its nature. As appeared, weak normality equations have their own geometrical interpretation. In paper [12] normal blow-up of points in Riemannian manifolds was considered. There it was found that force field of Newtonian dynamical systems admitting normal blow-up of points should satisfy weak normality equations only. This means that in multidimensional case $n \geqslant 3$ they could form larger class of dynamical systems than those admitting normal shift of hypersurfaces. Examples given in paper [13] show that they actually do form larger class.

Papers [14] and [15] are devoted to the study of global geometric structures associated with Newtonian dynamical systems admitting normal shift of hypersurfaces in multidimensional case $n \geqslant 3$. This study is based mainly on explicit formula (1.5). In addition, one should note that theory of dynamical systems admitting normal shift of hypersurfaces was generalized to the case of Finslerian manifolds (see Chapter VIII of thesis [7]). Weak and additional normality equations were derived. However, explicit formula like (1.5) for this case is not yet obtained.

During one year after the conference dedicated to Centenary Anniversary of I. G. Petrovsky, May 2001, I was looking for applications of the theory constructed. Results are represented by papers [16] and [17]. The idea is very simple. It is known that in the limit of short waves all wave propagation phenomena can be described in terms of rays and beams (geometrical optics is an example). In this limit amplitude of scalar wave is described by asymptotic expansion

$$u = \sum_{\alpha=0}^{\infty} \frac{\varphi_{(\alpha)}}{(i\,\lambda)^{\alpha}} \cdot e^{i\lambda S}, \quad \lambda \to \infty, \tag{1.6}$$

known as Debye's ansatz (see [18]). Here in (1.6) function $S = S(x^1, \ldots, x^n)$ is a phase of propagating wave. This function satisfies Hamilton-Jacobi equation

$$H(x^1, \ldots, x^n, \nabla_1 S, \ldots, \nabla_n S) = 0. \tag{1.7}$$

Here $\nabla_1 S, \ldots, \nabla_n S$ are components of momentum covector $\mathbf{p} = \nabla S$:

$$p_i = \nabla_i S = \frac{\partial S}{\partial x^i}. \tag{1.8}$$

Hamilton function $H = H(x^1, \ldots, x^n, p_1, \ldots, p_n)$ in (1.7) is determined by wave operator describing physical properties of medium, where wave propagation process occur. Usually it is polynomial with respect to components of momentum covector (1.8). But in more complicated cases it may be non-polynomial as well.

Note that Hamilton-Jacobi equation (1.7) is a first order PDE with respect to phase function $S = S(x^1, \ldots, x^n)$. Its solution is written in terms of characteristic lines. In present case they are given by Hamilton equations

$$\dot{x}^i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial x^i}. \tag{1.9}$$

Physically, characteristic lines or trajectories of Hamiltonian dynamical system (1.9) are interpreted as rays or beams in wave propagation process. **Wave fronts** in this process are hypersurfaces, where phase is constant. In other words, they are level hypersurfaces of phase function $S = S(x^1, \ldots, x^n)$. These hypersurfaces represent the position of wave at various time instants, so we can say that wave propagates moving along trajectories of Hamiltonian dynamical system (1.9). However, time variable t, with respect to which ordinary differential equations (1.9) are written, is not an actual time in wave propagation process. Actual time is proportional to the value of phase itself, therefore we can take $t = S$. As shown in [17], passing to this new time variable, we obtain the following modified Hamilton equations:

$$\dot{x}^i = \frac{1}{\Omega} \cdot \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{1}{\Omega} \cdot \frac{\partial H}{\partial x^i}. \tag{1.10}$$

Denominator Ω, which is often interpreted as kinetic energy, is given by formula

$$\Omega = \sum_{k=1}^{n} \frac{\partial H}{\partial p_k}\, p_k. \tag{1.11}$$

Now suppose that we have Riemannian manifold M. Then we can define modulus of momentum covector $p = |\mathbf{p}|$ and can consider Hamilton function of special form $H = W(x^1, \ldots, x^n, |\mathbf{p}|)$. Modified Hamiltonian dynamical system (1.10) with such special Hamilton function appears to be equivalent to Newtonian dynamical system (1.4) with force field given by the following formula:

$$F_i = -\sum_{k=1}^{n} \frac{\nabla_k W}{W_v} \cdot \frac{2\, v^k\, v_i - |\mathbf{v}|^2\, \delta^k_i}{|\mathbf{v}|}. \tag{1.12}$$

This is the main result of paper [17]. Comparing (1.5) and (1.12) we conclude that part of Newtonian dynamical systems admitting normal shift of hypersurfaces in Riemannian geometry can be interpreted as the equations of wave front dynamics in physics. And conversely, for some wave propagation phenomena we observe something like **conservation law**: wave front dynamics **preserves orthogonality** of wave fronts and rays. However, unlike mathematical theorems, true laws of nature cannot be sharply specific, i. e. applicable to some objects and not applicable to others. One should expect that the above **orthogonality law** can be generalized for all wave propagation phenomena.

Now suppose that modified Hamilton equations (1.10) are written for arbitrary smooth manifold M. In the absence of metric one should revise the concept of orthogonality itself. This was done in paper [19]. Note that modified Hamilton equations (1.10) define dynamics in cotangent bundle T^*M. Therefore at each point of trajectory $p = p(t)$ in M we have momentum covector $\mathbf{p} = \mathbf{p}(t)$. Therefore if trajectory $p = p(t)$ crosses some hypersurface σ and if $\boldsymbol{\tau}$ is a vector tangent to σ, then we can consider scalar product of $\boldsymbol{\tau}$ and momentum covector $\mathbf{p}$:

$$\varphi = \langle \mathbf{p} \,|\, \boldsymbol{\tau} \rangle = \sum_{k=1}^{n} \tau^k \, p_k. \tag{1.13}$$

We see that scalar product (1.13) does not require any metric. If $\varphi = 0$ for all $\boldsymbol{\tau} \in T_p(\sigma)$, then we say that trajectory $p = p(t)$ is perpendicular to hypersurface σ.

Let σ be some arbitrary orientable hypersurface in M. In the absence of metric we cannot define normal vector for σ. However, at each point p of σ there is normal covector $\mathbf{n} = \mathbf{n}(p)$ perpendicular to σ in the sense of scalar product (1.13). It is defined uniquely up to a scalar factor: $\mathbf{n} \to \alpha \cdot \mathbf{n}$. We can choose $\mathbf{n} = \mathbf{n}(p)$ to be smooth covector-valued function on σ. Then we can define the following initial data for ordinary differential equations (1.10):

$$x^i\Big|_{t=0} = x^i(p), \qquad\qquad p_i\Big|_{t=0} = \nu(p) \cdot n_i(p) \tag{1.14}$$

This defines motion of the points of σ along trajectories of dynamical system (1.10).

Definition 1.4. Shift of hypersurface $\sigma \subset M$ along trajectories of a dynamical system in cotangent bundle T^*M determined by initial data (1.14) is called **normal shift** if all displaced hypersurfaces σ_t are perpendicular to shift trajectories in the sense of scalar product determined by formula (1.13).

Definition 1.5. Dynamical system in cotangent bundle T^*M satisfies **strong normality condition** if for any hypersurface $\sigma \subset M$, for any fixed point $p_0 \in \sigma$, and for any constant $\nu_0 \neq 0$ there is some open neighborhood of p_0 and some smooth function $\nu(p)$ normalized by the condition (1.3) in this neighborhood such that it determines normal shift of σ in the sense of definition 1.4.

In paper [19] it was shown that any modified Hamiltonian dynamical system (1.10) with arbitrary Hamilton function H (provided $\Omega \neq 0$) satisfies strong normality condition. This means that dynamical system (1.10) form background for studying strong normality condition. In Riemannian geometry such background is given by geodesic flows (these are Newtonian dynamical systems (1.4) with identically zero force field $\mathbf{F} = 0$). In paper [19] I considered the following class of dynamical systems in cotangent bundle T^*M:

$$\dot{x}^i = \frac{1}{\Omega} \cdot \frac{\partial H}{\partial p_i}, \qquad\qquad \dot{p}_i = -\frac{1}{\Omega} \cdot \frac{\partial H}{\partial x^i} + Q_i. \tag{1.15}$$

Under sufficiently non-restrictive conditions for Hamilton function H (see [19]) differential equations (1.15) can be transformed to the form similar to (1.4):

$$\dot{x}^i = v^i, \qquad\qquad \dot{v}^i = \Phi^i(x^1, \ldots, x^n, v^1, \ldots, v^n). \tag{1.16}$$

Therefore we say that (1.15) is **relative form** of Newtonian dynamical system (1.16) as related to Hamiltonian system (1.10). Covector $\mathbf{Q}$ with components $Q_1, \ldots, Q_n$ in (1.15) plays the same role as force vector $\mathbf{F}$ in (1.4).

Note that Newtonian dynamical system in relative form (1.15) is a dynamical system in cotangent bundle T^*M. Therefore definitions 1.4 and 1.5 can be applied to it. This was actually done in paper [20]. In that paper it was shown that strong normality condition for Newtonian dynamical system (1.15) is equivalent to a system of partial differential equations for components of covector $\mathbf{Q} = \mathbf{Q}(p, \mathbf{p})$. This system of partial differential equations subdivides into two parts:

– **weak normality equations** written for $n \geqslant 2$;
– **additional normality equations** written for $n \geqslant 3$.

Studying these newly derived normality equations is rather interesting problem. However, below we go further and we consider more general situation, when Hamiltonian and/or Lagrangian dynamical system in M is not given.

2. The idea of further generalization.

In the absence of special geometric structures (like Riemannian metric or Hamilton function) the only way of defining Newtonian dynamics in M is given by the equations (1.16). Can we use scalar product (1.13) for to define normal shift in this case. The answer is **yes**, provided we have some way to determine momentum covector $\mathbf{p}$! For instance, momentum covector $\mathbf{p}$ can be given explicitly as a function of dynamic variables p and $\mathbf{v}$. In local chart this looks like

$$\begin{cases} p_1 = p_1(x^1, \ldots, x^n, v^1, \ldots, v^n), \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ p_n = p_n(x^1, \ldots, x^n, v^1, \ldots, v^n). \end{cases} \tag{2.1}$$

If coordinates $x^1, \ldots, x^n$ of the point $p \in M$ are fixed, then functions (2.1) express n variables $p_1, \ldots, p_n$ through another set of n variables $v^1, \ldots, v^n$. Therefore one can treat (2.1) as coordinate representation of a map

$$\lambda\colon TM \to T^*M. \tag{2.2}$$

This map is an analog of well known Legendre transformation (see [21]), for which functions (2.1) are determined by Lagrange function $L = L(p, \mathbf{v})$. Below we shall call it **generalized Legendre transformation** and for the sake of convenience we shall assume this map (2.2) to be diffeomorphism.

Using generalized Legendre transformation, we can transform Newtonian dynamical system (1.16) to new dynamic variables p and $\mathbf{p}$ forming point $q = (p, \mathbf{p})$ of cotangent bundle TM. As a result we get the following differential equations:

$$\begin{cases} \dot{x}^i = V^i(x^1, \ldots, x^n, p_1, \ldots, p_n), \\ \dot{p}_i = \Theta_i(x^1, \ldots, x^n, p_1, \ldots, p_n). \end{cases} \tag{2.3}$$

Here $V^1, \ldots, V^n$ are functions that implement inversion of the map (2.2):

$$\begin{cases} v^1 = V^1(x^1, \ldots, x^n, p_1, \ldots, p_n), \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ v^n = V^n(x^1, \ldots, x^n, p_1, \ldots, p_n). \end{cases} \tag{2.4}$$

They define kinematic part of the equations of Newtonian dynamics (2.3). Their role is similar to the role of Hamilton function in (1.15). Functions $\Theta_1, \ldots, \Theta_n$ define dynamical part of the equations (2.3), they play the same role as functions $Q_1, \ldots, Q_n$ in (1.15) and functions $\Phi^1, \ldots, \Phi^n$ in (1.16).

Note that Newtonian dynamical system written as (2.3) is a dynamical system in cotangent bundle T^*M. Definitions 1.4 and 1.5 are applicable to it. Therefore we can start new theory of dynamical systems admitting normal shift of hypersurfaces for manifolds equipped only with generalized Legendre map. Our main goal here is to derive weak and additional normality equations for this case.

3. Newtonian dynamics in manifolds and associated geometric structures.

Let's consider Newtonian dynamical system in some smooth manifold M. In the absence of metric we cannot use covariant derivative ∇_t in (1.4). Therefore we write it as (1.16). This is the system of first order ordinary differential equations given by Newtonian vector field $\mathbf{\Phi}$ in tangent bundle TM:

$$\mathbf{\Phi} = \sum_{i=1}^{n} v^i \cdot \frac{\partial}{\partial x^i} + \sum_{i=1}^{n} \Phi^i \cdot \frac{\partial}{\partial v^i}. \tag{3.1}$$

Let $q = (p, \mathbf{v})$ be a point of tangent bundle TM. Then canonical projection $\pi : TM \to M$ maps it to the point $p \in M$. Canonical projection induces linear map $\pi_* : T_q(TM) \to T_p(M)$. Applying this map to vector (3.1) at the point $q \in TM$, we get velocity vector $\mathbf{v}$ at the point $p \in M$:

$$\pi_*(\mathbf{\Phi}) = \mathbf{v}. \tag{3.2}$$

The equality (3.2) can be taken for the definition of Newtonian vector field (3.1). Linear map π_* in (3.2) acts from $T_q(TM)$ to $T_p(M)$. However, one can consider a map acting in opposite direction.

Definition 3.1. Suppose that for each point q of tangent bundle TM we have linear map $f\colon T_p(M) \to T_q(TM)$ from tangent space $T_p(M)$ at the point $p = \pi(q)$ to tangent space $T_q(TM)$ at the point q. This construction is called a **lift** of vectors from M to tangent bundle TM.

Definition 3.2. Lift of vectors f from M to TM is called **smooth lift** if it maps each smooth vector field in M to a smooth vector field in TM.

Definition 3.3. Lift of vectors f from M to TM is called **vertical lift** if composition $\pi_* \circ f$ is identically zero: $\pi_* \circ f = 0$.

Definition 3.4. Lift of vectors f from M to TM is called **horizontal lift** if composition $\pi_* \circ f$ is the field of identical operators on M, i. e. $\pi_* \circ f = \mathrm{id}$.

Each smooth manifold M possesses canonical vertical lift of vectors from M to TM. It is defined as follows. Let $\mathbf{X}$ be some vector field in M and let $\mathbf{X}_p$ be its value at the point $p \in M$. Then we can define one-parametric group of diffeomorphisms φ_t in TM that maps point $q = (p, \mathbf{v})$ to the point $\varphi_t(q) = (p, \mathbf{v} + t \cdot \mathbf{X}_p)$. Vector field

$\mathbf{Y}$ in tangent bundle TM associated with one-parametric group of diffeomorphisms φ_t is taken for the result of lifting vector field $\mathbf{X}$:

$$\mathbf{Y} = w(\mathbf{X}). \tag{3.3}$$

Let's write (3.3) in local chart. Suppose that $\mathbf{X}$ is given by its coordinates:

$$\mathbf{X} = X^1 \cdot \frac{\partial}{\partial x^1} + \ldots + X^n \cdot \frac{\partial}{\partial x^n}. \tag{3.4}$$

Then result of applying canonical vertical lift w to $\mathbf{X}$ is given by formula

$$\mathbf{Y} = w(\mathbf{X}) = X^1 \cdot \frac{\partial}{\partial v^1} + \ldots + X^n \cdot \frac{\partial}{\partial v^n}. \tag{3.5}$$

Looking at (3.4) and (3.5), we see that for each point q of tangent bundle w is injective map that maps tangent space $T_p(M)$ onto the vertical subspace

$$V_q(TM) = \operatorname{Ker} \pi_*. \tag{3.6}$$

Let h be some horizontal lift of vectors from M to TM. For each point q of tangent bundle h is also an injective. Let's denote by $H_q(TM)$ its image

$$H_q(TM) = \operatorname{Im} h. \tag{3.7}$$

Subspaces (3.6) and (3.7) are transversal and complementary to each other:

$$T_q(TM) = H_q(TM) \oplus V_q(TM). \tag{3.8}$$

Lemma 3.1. *Defining horizontal lift of vectors from M to TM is equivalent to fixing horizontal subspace $H_q(TM)$ complementary to vertical subspace $V_q(TM)$ in $T_q(TM)$ at each point q of tangent bundle TM.*

Suppose that we have some horizontal lift h. Let's denote by $\mathbf{E}_1, \ldots, \mathbf{E}_n$ base of coordinate vector fields for some local chart of the manifold M:

$$\mathbf{E}_1 = \frac{\partial}{\partial x^1}, \ \ldots, \ \mathbf{E}_n = \frac{\partial}{\partial x^n}. \tag{3.9}$$

Due to (3.8) and due to the equality $\pi_* \circ h = \mathrm{id}$, applying h to $\mathbf{E}_i$, we get:

$$h(\mathbf{E}_i) = \frac{\partial}{\partial x^i} - \sum_{k=1}^{n} \Gamma^k_i \cdot \frac{\partial}{\partial v^k}. \tag{3.10}$$

Quantities $\Gamma^k_i = \Gamma^k_i(x^1, \ldots, x^n, v^1, \ldots, v^n)$ represent horizontal lift h in local chart. If they are smooth functions of their arguments, then h is smooth lift. Under the change of local chart in M these quantities are transformed as follows:

$$\Gamma^k_i = \sum_{m=1}^{n} \sum_{a=1}^{n} S^k_m \, T^a_i \, \tilde{\Gamma}^m_a + \sum_{m=1}^{n} \sum_{r=1}^{n} S^k_m \, \frac{\partial T^m_r}{\partial x^i} \, v^r. \tag{3.11}$$

Here S^k_m, T^a_i, and T^m_r are components of direct and inverse transition matrices (Jacobi matrices) for the change of local coordinates:

$$S^i_j = \frac{\partial x^i}{\partial \tilde{x}^j}, \qquad\qquad T^i_j = \frac{\partial \tilde{x}^i}{\partial x^j}. \tag{3.12}$$

In arbitrary smooth manifold there is no canonical horizontal lift of vectors. However, if Newtonian dynamical system (1.16) is given, then there is horizontal lift h, canonically associated with it (see [22] and references therein). Let $\mathbf{\Phi}$ be Newtonian vector field (3.1) and let $\mathbf{X}$ be some arbitrary vector field in TM. Then we associate with $\mathbf{X}$ the following vector field denoted by $\mathbf{H}(\mathbf{X})$:

$$\mathbf{H}(\mathbf{X}) = \frac{\mathbf{X} + [w\circ\pi_*(\mathbf{X}),\,\mathbf{\Phi}] - w\circ\pi_*([\mathbf{X},\,\mathbf{\Phi}])}{2}. \tag{3.13}$$

Here by square brackets we denote commutator of two vector fields.

Lemma 3.2. *For any smooth function φ and for arbitrary smooth vector field $\mathbf{X}$ in TM the equality $\mathbf{H}(\varphi\cdot\mathbf{X}) = \varphi\cdot\mathbf{H}(\mathbf{X})$ is fulfilled.*

Proof. By direct calculations we find that $[\varphi\cdot\mathbf{X},\,\mathbf{\Phi}] = \varphi\cdot[\mathbf{X},\,\mathbf{\Phi}] - \mathbf{\Phi}\varphi\cdot\mathbf{X}$. Here by $\mathbf{\Phi}\varphi$ we denote derivative of function φ along vector $\mathbf{\Phi}$:

$$\mathbf{\Phi}\varphi = \sum_{i=1}^{n} v^i\,\frac{\partial\varphi}{\partial x^i} + \sum_{i=1}^{n}\Phi^i\,\frac{\partial\varphi}{\partial v^i}.$$

Composition $w\circ\pi_*$ acts as linear operator at each point $q\in TM$. Therefore

$$w\circ\pi_*([\varphi\cdot\mathbf{X},\,\mathbf{\Phi}]) = \varphi\cdot w\circ\pi_*([\mathbf{X},\,\mathbf{\Phi}]) - \mathbf{\Phi}\varphi\cdot w\circ\pi_*(\mathbf{X}), \tag{3.14}$$

$$w\circ\pi_*(\varphi\cdot\mathbf{X}) = \varphi\cdot w\circ\pi_*(\mathbf{X}),$$

$$[w\circ\pi_*(\varphi\cdot\mathbf{X}),\,\mathbf{\Phi}] = \varphi\cdot[w\circ\pi_*(\mathbf{X}),\,\mathbf{\Phi}] - \mathbf{\Phi}\varphi\cdot w\circ\pi_*(\mathbf{X}). \tag{3.15}$$

When subtracting (3.14) from (3.15), terms containing $\mathbf{\Phi}\varphi$ cancel each other. This proves required equality $\mathbf{H}(\varphi\cdot\mathbf{X}) = \varphi\cdot\mathbf{H}(\mathbf{X})$. $\square$

Formula (3.13) defines an operator $\mathbf{H}$ acting on vector field $\mathbf{X}$ and yielding another vector field $\mathbf{H}(\mathbf{X})$. Commutators in (3.13) contain differentiation with respect to $\mathbf{X}$. Therefore one might expect $\mathbf{H}$ to be differential operator. Lemma 3.2 means that $\mathbf{H}$ is non-differential linear operator acting pointwise at each point q of tangent bundle TM.

Lemma 3.3. *Linear operator $\mathbf{H}\colon T_q(TM)\to T_q(TM)$ defined by formula (3.13) satisfies the equalities $\mathbf{H}\circ\mathbf{H} = \mathbf{H}$, $\pi_*\circ\mathbf{H} = \pi_*$, and $\mathbf{H}\circ w = 0$.*

Proof. In order to prove required equalities we shall use direct calculations in local coordinates. Let's take an arbitrary vector field $\mathbf{X}$ in TM represented as

$$\mathbf{X} = \sum_{i=1}^{n} X^i\cdot\frac{\partial}{\partial x^i} + \sum_{i=1}^{n} Y^i\cdot\frac{\partial}{\partial v^i}. \tag{3.16}$$

Newtonian vector field $\mathbf{\Phi}$ is represented by formula (3.1). Hence for $[\mathbf{X},\,\mathbf{\Phi}]$ we have

$$\begin{gathered}[\mathbf{X},\,\mathbf{\Phi}]=\sum_{k=1}^{n}Y^k\cdot\frac{\partial}{\partial x^k}+\sum_{k=1}^{n}\sum_{i=1}^{n}\left(X^i\,\frac{\partial\Phi^k}{\partial x^i}+Y^i\,\frac{\partial\Phi^k}{\partial v^i}\right)\cdot\frac{\partial}{\partial v^k}-\\ -\sum_{i=1}^{n}\sum_{k=1}^{n}\left(v^k\,\frac{\partial X^i}{\partial x^k}+\Phi^k\,\frac{\partial X^i}{\partial v^k}\right)\cdot\frac{\partial}{\partial x^i}-\sum_{i=1}^{n}\sum_{k=1}^{n}\left(v^k\,\frac{\partial Y^i}{\partial x^k}+\Phi^k\,\frac{\partial Y^i}{\partial v^k}\right)\cdot\frac{\partial}{\partial v^i}.\end{gathered}$$

Applying operator $w\circ\pi_*$ to (3.16) and to the above expression, we obtain

$$w\circ\pi_*(\mathbf{X})=\sum_{i=1}^{n}X^i\cdot\frac{\partial}{\partial v^i},\tag{3.17}$$

$$w\circ\pi_*([\mathbf{X},\,\mathbf{\Phi}])=\sum_{k=1}^{n}Y^k\cdot\frac{\partial}{\partial v^k}-\sum_{i=1}^{n}\sum_{k=1}^{n}\left(v^k\,\frac{\partial X^i}{\partial x^k}+\Phi^k\,\frac{\partial X^i}{\partial v^k}\right)\cdot\frac{\partial}{\partial v^i}.\tag{3.18}$$

Now, relying upon formula (3.17), we calculate the commutator $[w\circ\pi_*(\mathbf{X}),\,\mathbf{\Phi}]$:

$$\begin{gathered}[w\circ\pi_*(\mathbf{X}),\,\mathbf{\Phi}]=\sum_{k=1}^{n}X^k\cdot\frac{\partial}{\partial x^k}+\sum_{k=1}^{n}\sum_{i=1}^{n}X^i\,\frac{\partial\Phi^k}{\partial v^i}\cdot\frac{\partial}{\partial v^k}-\\ -\sum_{i=1}^{n}\sum_{k=1}^{n}\left(v^k\,\frac{\partial X^i}{\partial x^k}+\Phi^k\,\frac{\partial X^i}{\partial v^k}\right)\cdot\frac{\partial}{\partial v^i}.\end{gathered}\tag{3.19}$$

Then subtract (3.18) from (3.19). As a result we obtain the following equality:

$$\begin{gathered}[w\circ\pi_*(\mathbf{X}),\,\mathbf{\Phi}]-w\circ\pi_*([\mathbf{X},\,\mathbf{\Phi}])=\sum_{k=1}^{n}X^k\cdot\frac{\partial}{\partial x^k}+\\ +\sum_{k=1}^{n}\sum_{i=1}^{n}X^i\,\frac{\partial\Phi^k}{\partial v^i}\cdot\frac{\partial}{\partial v^k}-\sum_{k=1}^{n}Y^k\cdot\frac{\partial}{\partial v^k}.\end{gathered}$$

And ultimately, for the result of applying operator $\mathbf{H}$ to vector $\mathbf{X}$ given by formula (3.16) we obtain the following expression:

$$\mathbf{H}(\mathbf{X})=\sum_{k=1}^{n}X^k\cdot\frac{\partial}{\partial x^k}+\frac{1}{2}\sum_{s=1}^{n}\sum_{i=1}^{n}X^i\,\frac{\partial\Phi^s}{\partial v^i}\cdot\frac{\partial}{\partial v^s}.\tag{3.20}$$

Looking at the expression in right hand side of (3.20), we see that the required equalities $\mathbf{H}\circ\mathbf{H}=\mathbf{H}$, $\pi_*\circ\mathbf{H}=\pi_*$, and $\mathbf{H}\circ w=0$ for operator $\mathbf{H}$ appear to be obvious. Lemma 3.3 is proved. $\square$

First equality $\mathbf{H}\circ\mathbf{H}=\mathbf{H}$ means that $\mathbf{H}$ is an operator of projection. It projects $T_q(TM)$ onto some subspace $H_q(TM)$, where $H_q(TM)=\operatorname{Im}\mathbf{H}$. Each projection operator breaks the space into direct sum of its kernel and its image:

$$T_q(TM)=\operatorname{Ker}\mathbf{H}\oplus\operatorname{Im}\mathbf{H}.\tag{3.21}$$

Second equality $\pi_*\circ\mathbf{H}=\pi_*$ yields $\operatorname{Ker}\mathbf{H}\subseteq\operatorname{Ker}\pi_*$, while third equality $\mathbf{H}\circ w=0$ means that $\operatorname{Im}w\subseteq\operatorname{Ker}\mathbf{H}$. Taking into account (3.6), we obtain that kernel of operator $\mathbf{H}$ coincides with vertical subspace $V_q(TM)$:

$$\operatorname{Ker}\mathbf{H}=\operatorname{Ker}\pi_*=\operatorname{Im}w=V_q(TM). \tag{3.22}$$

Due to (3.22) the equality (3.21) can be rewritten as (3.8). This means that operator $\mathbf{H}$ defines horizontal subspace $H_q(TM)=\operatorname{Im}\mathbf{H}$ in $T_q(TM)$. Applying lemma 3.1, we derive the following theorem.

Theorem 3.1. *Each Newtonian dynamical system* (1.16) *in smooth manifold M generates horizontal lift of vectors from M to TM canonically associated with it.*

This result is not new (see [22] and references therein). We reproduced it here for the sake of completeness of our consideration.

Let h be horizontal lift canonically associated with Newtonian dynamical system (1.16). Then $\mathbf{H}=h\circ\pi_*$ and $h(\mathbf{E}_i)=h\circ\pi_*(\partial/\partial x^i)=\mathbf{H}(\partial/\partial x^i)$. Applying formula (3.20) to vector field $\mathbf{X}=\mathbf{E}_i$, we now obtain the following equality:

$$h(\mathbf{E}_i)=\mathbf{H}(\partial/\partial x^i)=\frac{\partial}{\partial x^i}+\sum_{k=1}^{n}\frac{1}{2}\,\frac{\partial\Phi^k}{\partial v^i}\cdot\frac{\partial}{\partial v^k}. \tag{3.23}$$

Comparing (3.23) with (3.10), for components of h we derive explicit formula:

$$\Gamma^k_i=-\frac{1}{2}\,\frac{\partial\Phi^k}{\partial v^i}. \tag{3.24}$$

4. Extended tensor fields and covariant differentiations.

Concept of extended tensor fields is closely related to Newtonian dynamical system. Indeed, if we look at right hand side of (1.4), we see that $\mathbf{F}=\mathbf{F}(p,\mathbf{v})$ is a vector-valued function with values in tangent spaces $T_p(M)$. However, it doesn't fit standard definition of vector field in M since it depends not only on a point $p\in M$, but also upon velocity vector $\mathbf{v}$ at this point. Both p and $\mathbf{v}$ form a point $q=(p,\mathbf{v})$ of tangent bundle TM. Function $\mathbf{F}=\mathbf{F}(p,\mathbf{v})$ is an example of extended vector field. Vector-function $\mathbf{V}=\mathbf{V}(p,\mathbf{p})$ in (2.3) with components (2.4) is another example. Its arguments form a point $q=(p,\mathbf{p})$ if cotangent bundle T^*M.

In order to formulate definition of extended tensor fields in general let's consider the following tensor product of tangent spaces and their dual spaced:

$$T^r_s(p,M)=\overbrace{T_p(M)\otimes\ldots\otimes T_p(M)}^{r\text{ times}}\otimes\underbrace{T^*_p(M)\otimes\ldots\otimes T^*_p(M)}_{s\text{ times}}$$

Definition 4.1. Extended tensor field $\mathbf{X}$ of type (r,s) in $\mathbf{v}$-representation is a tensor-valued function with argument $q=(p,\mathbf{v})$ in tangent bundle TM and with value $\mathbf{X}(q)$ in tensor space $T^r_s(p,M)$, where $p=\pi(q)$.

Definition 4.2. Extended tensor field $\mathbf{X}$ of type (r, s) in **p**-representation is a tensor-valued function with argument $q = (p, \mathbf{p})$ in cotangent bundle T^*M and with value $\mathbf{X}(q)$ in tensor space $T^r_s(p, M)$, where $p = \pi(q)$.

At first let's consider extended tensor fields in **v**-representation. Denote by $T^r_s(M)$ the set of smooth extended tensor fields of type (r, s) and take the sum

$$\mathbf{T}(M) = \bigoplus_{r=0}^{\infty} \bigoplus_{s=0}^{\infty} T^r_s(M) \tag{4.1}$$

In $\mathbf{T}(M)$ we have all standard tensorial operations like summation, multiplication by scalars, tensor product, and contraction. Direct sum (4.1) possesses structure of graded algebra over the ring of smooth scalar functions in TM, we denote this ring by $\mathfrak{F}(TM)$. Algebra $\mathbf{T}(M)$ is called **extended algebra of tensor fields**.

Definition 4.3. A map $D\colon \mathbf{T}(M) \to \mathbf{T}(M)$ is called **differentiation** of extended algebra of tensor fields, if the following conditions are fulfilled:

(1) concordance with grading: $D(T^r_s(M)) \subset T^r_s(M)$;
(2) $\mathbb{R}$-linearity: $D(\mathbf{X} + \mathbf{Y}) = D(\mathbf{X}) + D(\mathbf{Y})$ and $D(\lambda\mathbf{X}) = \lambda D(\mathbf{X})$ for $\lambda \in \mathbb{R}$;
(3) commutation with contractions: $D(C(\mathbf{X})) = C(D(\mathbf{X}))$;
(4) Leibniz rule: $D(\mathbf{X} \otimes \mathbf{Y}) = D(\mathbf{X}) \otimes \mathbf{Y} + \mathbf{X} \otimes D(\mathbf{Y})$.

Let's denote by $\mathfrak{D}(M)$ the total set of differentiations of extended algebra of tensor fields $\mathbf{T}(M)$. It is easy to see that it possesses the structure of $\mathfrak{F}(TM)$-module. The set of extended vector fields $T^1_0(M)$ is equipped with the same structure of module over the ring $\mathfrak{F}(TM)$. This coincidence motivates the following definition.

Definition 4.4. Covariant differentiation ∇ in the algebra of extended tensor fields $\mathbf{T}(M)$ is a homomorphism of $\mathfrak{F}(TM)$-modules $\nabla\colon T^1_0(M) \to \mathfrak{D}(M)$. Image of vector field $\mathbf{Y}$ under such homomorphism is called covariant differentiation along vector field $\mathbf{Y}$. It is denoted by $\nabla_{\mathbf{Y}}$.

In Chapter III of thesis [7] it was shown that each covariant differentiation ∇ in extended algebra of tensor fields is associated with some lift of vectors from M to TM. It is called **horizontal covariant differentiation** (or **vertical covariant differentiation**) if corresponding lift of vectors is horizontal (or vertical). In each smooth manifold there is canonical vertical covariant differentiation associated with canonical vertical lift of vectors w. We denote it by $\tilde{\nabla}$. In local chart this covariant differentiation is given by the following formula:

$$\tilde{\nabla}_m X^{i_1 \ldots\, i_r}_{j_1 \ldots\, j_s} = \frac{\partial X^{i_1 \ldots\, i_r}_{j_1 \ldots\, j_s}}{\partial v^m}. \tag{4.2}$$

Due to (4.2) this covariant differentiation is also called **velocity gradient**.

To define horizontal covariant differentiation, apart from horizontal lift of vectors, one need some extended affine connection Γ. Its components are given by formula

$$\nabla_{\mathbf{E}_i} \mathbf{E}_j = \sum_{k=1}^{n} \Gamma^k_{ij}\, \mathbf{E}_k. \tag{4.3}$$

Here $\mathbf{E}_1, \ldots, \mathbf{E}_n$ are coordinate vector fields (3.9) in M. Unlike traditional affine connection, components of extended affine connection Γ^k_{ij} depend on double set of arguments, i. e. on coordinates of point $p \in M$ and on components of vector $\mathbf{v}$:

$$\Gamma^k_{ij} = \Gamma^k_{ij}(x^1, \ldots, x^n, v^1, \ldots, v^n). \tag{4.4}$$

Under the change of local chart quantities (4.4) are transformed as follows:

$$\Gamma^k_{ij} = \sum_{m=1}^{n}\sum_{a=1}^{n}\sum_{c=1}^{n} S^k_m\, T^a_i\, T^c_j\, \tilde{\Gamma}^m_{ac} + \sum_{m=1}^{n} S^k_m\,\frac{\partial T^m_i}{\partial x^j} \tag{4.5}$$

(matrices S and T are defined in (3.12)). If components of horizontal lift h in (3.10) and components of extended affine connection Γ in (4.3) are given, then horizontal covariant differentiation ∇ in local coordinates is given by the following formula:

$$\begin{aligned}\nabla_m X^{i_1\ldots\, i_r}_{j_1\ldots\, j_s} &= \frac{\partial X^{i_1\ldots\, i_r}_{j_1\ldots\, j_s}}{\partial x^m} - \sum_{a=1}^{n}\sum_{b=1}^{n}\Gamma^b_m\,\frac{\partial X^{i_1\ldots\, i_r}_{j_1\ldots\, j_s}}{\partial v^b} + \\ &+ \sum_{k=1}^{r}\sum_{a_k=1}^{n}\Gamma^{i_k}_{m\,a_k}\,X^{i_1\ldots\, a_k\ldots\, i_r}_{j_1\ldots\ldots\ldots\, j_s} - \sum_{k=1}^{s}\sum_{b_k=1}^{n}\Gamma^{b_k}_{m\,j_k}\,X^{i_1\ldots\ldots\ldots\, i_r}_{j_1\ldots\, b_k\ldots\, j_s}.\end{aligned} \tag{4.6}$$

Note that Γ^k_i and Γ^k_{ij} are two independent sets of parameters in (4.6). The only condition is that they should obey transformation rules (3.11) and (4.5) respectively. However, one can prove the following two lemmas.

Lemma 4.1. *If horizontal lift of vectors from M to TM is given and if Γ^k_i are its components, then there is symmetric extended affine connection with components*

$$\Gamma^k_{ij} = \frac{1}{2}\,\frac{\partial \Gamma^k_i}{\partial v^j} + \frac{1}{2}\,\frac{\partial \Gamma^k_j}{\partial v^i}. \tag{4.7}$$

Lemma 4.2. *If extended affine connection with components Γ^k_{ij} is given, then there is horizontal lift of vectors from M to TM with components*

$$\Gamma^k_i = \sum_{j=1}^{n} \Gamma^k_{ij}\, v^j. \tag{4.8}$$

These two lemmas are proved by direct calculations on the base of formulas (3.11) and (4.5). Applying (4.7) to (3.24), we obtain

$$\Gamma^k_{ij} = -\frac{1}{2}\,\frac{\partial^2 \Phi^k}{\partial v^i\,\partial v^j}. \tag{4.9}$$

As a result we have proved the following theorem.

Theorem 4.1. *Each Newtonian dynamical system* (1.16) *in smooth manifold M generates extended affine connection with components* (4.9) *canonically associated with this dynamical system.*

Usually equalities (4.7) and (4.8) cannot be fulfilled simultaneously. In all previous papers we defined Γ^k_i through Γ^k_{ij} by means of formula (4.8). This is equivalent to the equality $\nabla_i v^k = 0$. In present paper we keep this tradition. Therefore formula (4.6) for horizontal covariant derivative looks like

$$\begin{gathered}\nabla_m X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s} = \frac{\partial X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}}{\partial x^m} - \sum_{a=1}^{n}\sum_{b=1}^{n}\sum_{c=1}^{n} v^c\,\Gamma^b_{cm}\,\frac{\partial X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}}{\partial v^b} +\\ + \sum_{k=1}^{r}\sum_{a_k=1}^{n}\Gamma^{i_k}_{m\,a_k}\,X^{i_1\ldots\,a_k\ldots\,i_r}_{j_1\ldots\ldots\ldots\,j_s} - \sum_{k=1}^{s}\sum_{b_k=1}^{n}\Gamma^{b_k}_{m\,j_k}\,X^{i_1\ldots\ldots\ldots\,i_r}_{j_1\ldots\,b_k\ldots\,j_s}.\end{gathered}\tag{4.10}$$

Theory of differentiation for extended tensor fields in $\mathbf{p}$-representation is a little bit different. Here we also can define **extended algebra of tensor fields** by means of direct sum (4.1). It is algebra over the ring of smooth functions in cotangent bundle $\mathfrak{F}(T^*M)$. Definition 4.3 remains unchanged. Definition 4.4 is replaced by the following two definitions.

Definition 4.5. Covariant differentiation ∇ in the algebra of extended vector fields $\mathbf{T}(M)$ is a homomorphism of $\mathfrak{F}(T^*M)$-modules $\nabla\colon T^1_0(M) \to \mathfrak{D}(M)$. Image of vector field $\mathbf{Y}$ under such homomorphism is called covariant differentiation along vector field $\mathbf{Y}$. It is denoted by $\nabla_{\mathbf{Y}}$.

Definition 4.6. Contravariant differentiation ∇ in the algebra of extended vector fields $\mathbf{T}(M)$ is a homomorphism of $\mathfrak{F}(T^*M)$-modules $\nabla\colon T^0_1(M) \to \mathfrak{D}(M)$. Image of covector field $\mathbf{q}$ under such homomorphism is called contravariant differentiation along covector field $\mathbf{q}$. It is denoted by $\nabla_{\mathbf{q}}$.

Instead of velocity gradient (4.2) here in $\mathbf{p}$-representation we have **canonical vertical contravariant differentiation** $\tilde{\nabla}$. It is given by formula

$$\tilde{\nabla}^m X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s} = \frac{\partial X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}}{\partial p_m}.\tag{4.11}$$

Due to (4.11) contravariant differentiation $\tilde{\nabla}$ is called **momentum gradient**. Instead of canonical vertical lift of vectors in $\mathbf{p}$-representation for each point $q = (p, \mathbf{p})$ of cotangent bundle T^*M we have injective map $w\colon T^*_p(p) \to T_q(TM)$ associated with differentiation $\tilde{\nabla}$. Suppose that covector $\mathbf{q}$ is given by its coordinates:

$$\mathbf{q} = q_1 \cdot dx^1 + \ldots + q_n \cdot dx^n.\tag{4.12}$$

Applying w to $\mathbf{q}$, we obtain a vector $\mathbf{Y} = w(\mathbf{q})$ given by the following expression:

$$\mathbf{Y} = w(\mathbf{q}) = q_1 \cdot \frac{\partial}{\partial p_1} + \ldots + q_n \cdot \frac{\partial}{\partial p_n}.\tag{4.13}$$

Formulas (4.12) and (4.13) are similar to formulas (3.4) and (3.5). However, there is invariant way to determine vector $\mathbf{Y}$. It is generated by one-parametric group of diffeomorphisms φ_t in T^*M which is defined as follows. If $q = (p, \mathbf{p})$ is a point of cotangent bundle T^*M, then $\varphi_t(q) = (p, \mathbf{p} + t \cdot \mathbf{q})$.

Injective map $w : T^*_p(p) \to T_q(TM)$ defined just above satisfy the equality $\pi_* \circ w = 0$. Its image coincides with vertical subspace in $T_q(T^*M)$:

$$\operatorname{Im} w = V_q(T^*M) = \operatorname{Ker} \pi_*.$$

Each horizontal covariant derivative in **p**-representation is given by formula

$$\nabla_m X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s} = \frac{\partial X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}}{\partial x^m} + \sum_{a=1}^{n}\sum_{b=1}^{n} \Gamma_{mb}\,\frac{\partial X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}}{\partial p_b} +$$
$$+\sum_{k=1}^{r}\sum_{a_k=1}^{n} \Gamma^{i_k}_{m\,a_k}\,X^{i_1\ldots\,a_k\ldots\,i_r}_{j_1\ldots\ldots\ldots\,j_s} - \sum_{k=1}^{s}\sum_{b_k=1}^{n} \Gamma^{b_k}_{m\,j_k}\,X^{i_1\ldots\ldots\ldots\,i_r}_{j_1\ldots\,b_k\ldots\,j_s}.$$

Here Γ_{ij} and Γ^k_{ij} are components of some horizontal lift of vectors from M to T^*M and some extended affine connection respectively. Quantities Γ^k_{ij} are defined by the equality (4.3). They satisfy the same transformation rule (4.5) as quantities (4.4). However, they differ from (4.4) since they have another set of arguments:

$$\Gamma^k_{ij} = \Gamma^k_{ij}(x^1, \ldots, x^n, p_1, \ldots, p_n). \tag{4.14}$$

Horizontal lift of vectors h from M to T^*M satisfies the equality $\pi_* \circ h = \mathrm{id}$. It's components are determined by the following formula:

$$h(\mathbf{E}_i) = \frac{\partial}{\partial x^i} + \sum_{k=1}^{n} \Gamma_{ik} \cdot \frac{\partial}{\partial p_k}.$$

Under change of local chart quantities Γ_{ij} are transformed as follows:

$$\Gamma_{ij} = \sum_{a=1}^{n}\sum_{c=1}^{n} T^a_i\,T^c_j\,\tilde{\Gamma}_{ac} + \sum_{k=1}^{n}\sum_{m=1}^{n} p_k\,S^k_m\,\frac{\partial T^m_i}{\partial x^j}. \tag{4.15}$$

Comparing formula (4.15) with (4.5), one can prove two lemmas which are similar to lemma 4.1 and lemma 4.2.

Lemma 4.3. *If horizontal lift of vectors from M to T^*M is given and if Γ_{ij} are its components, then there is symmetric extended affine connection with components*

$$\Gamma^k_{ij} = \frac{\partial \Gamma_{ij}}{\partial p_k}. \tag{4.16}$$

Lemma 4.4. *If extended affine connection with components Γ^k_{ij} is given, then there is horizontal lift of vectors from M to T^*M with components*

$$\Gamma_{ij} = \sum_{k=1}^{n} \Gamma^k_{ij}\,p_k. \tag{4.17}$$

Usually equalities (4.16) and (4.17) cannot be fulfilled simultaneously. In previous papers we defined Γ_{ij} by means of formula (4.17). This is equivalent to

$$\nabla_i p_j = 0.$$

In present paper we keep this tradition. Connection components Γ^k_{ij} will be imported to **p**-representation by means of generalized Legendre transformation (2.2) (see below). Then formula for horizontal covariant differentiation looks like

$$
\begin{aligned}
\nabla_m X^{i_1\ldots\, i_r}_{j_1\ldots\, j_s} &= \frac{\partial X^{i_1\ldots\, i_r}_{j_1\ldots\, j_s}}{\partial x^m} - \sum_{a=1}^{n}\sum_{b=1}^{n}\sum_{c=1}^{n} p_c\,\Gamma^c_{mb}\,\frac{\partial X^{i_1\ldots\, i_r}_{j_1\ldots\, j_s}}{\partial p_b} + \\
&+ \sum_{k=1}^{r}\sum_{a_k=1}^{n}\Gamma^{i_k}_{m\,a_k}\,X^{i_1\ldots\, a_k\ldots\, i_r}_{j_1\ldots\ldots\ldots\, j_s} - \sum_{k=1}^{s}\sum_{b_k=1}^{n}\Gamma^{b_k}_{m\,j_k}\,X^{i_1\ldots\ldots\ldots\, i_r}_{j_1\ldots\, b_k\ldots\, j_s}.
\end{aligned}
\tag{4.18}
$$

Formulas (4.10) and (4.18) mean that we do not use horizontal lift with components (3.24) directly, but only for deriving extended connection with components (4.9).

5. Generalized Legendre transformation.

In section 2 we learned that some extended affine connection and some horizontal lift of vectors from M to TM are canonically associated with Newtonian dynamical system (1.16). As for dynamical system (2.3) in T^*M, I don't know similar direct constructions for Γ^k_{ij} and Γ_{ij} in this case. Therefore now we shall import these quantities from $\mathbf{v}$-representation to $\mathbf{p}$-representation by means of generalized Legendre transformation λ. For Γ^k_{ij} in (4.14) we write:

$$
\Gamma^k_{ij} = \Gamma^k_{ij} \circ \lambda^{-1}. \tag{5.1}
$$

This equality (5.1) means that we simply change arguments in (4.4) by substituting functions (2.4) for $v^1, \ldots, v^n$.

Quantities Γ^k_{ij} in $\mathbf{v}$-representation are given by formula (4.9). Using formula (5.1), one can find their counterparts Γ^k_{ij} in $\mathbf{p}$-representation. However, it would be natural to express them through functions $V^1, \ldots, V^n$ and $\Theta_1, \ldots, \Theta_n$ that determine dynamical system (2.3). Differential equations (2.3) describe the same dynamics as differential equations (1.16), but in other variables $x^1, \ldots, x^n, p_1, \ldots, p_n$. Therefore we can calculate time derivative of velocity vector $\mathbf{v}$ as follows:

$$
\dot v^k = \frac{dV^k}{dt} = \sum_{i=1}^{n}\frac{\partial V^k}{\partial x^i}\,V^i + \sum_{i=1}^{n}\frac{\partial V^k}{\partial p_i}\,\Theta_i. \tag{5.2}
$$

Comparing (5.2) with second part of equations (1.16), we obtain the equality

$$
\Phi^k \circ \lambda^{-1} = \sum_{i=1}^{n}\frac{\partial V^k}{\partial x^i}\,V^i + \sum_{i=1}^{n}\frac{\partial V^k}{\partial p_i}\,\Theta_i. \tag{5.3}
$$

Note that partial derivatives $\partial V^k/\partial p_r$ in (5.3) form Jacobi matrix for transformation given by functions (2.4), when $x^1, \ldots, x^n$ are fixed. Let's denote

$$
g^{ir} = \frac{\partial V^i}{\partial p_r} = \tilde\nabla^r V^i. \tag{5.4}
$$

Formula (5.4) determines components of an extended tensor field $\mathbf{g}$. Tensor $\mathbf{g}$ plays the role similar to that of dual metric tensor in Riemannian geometry. It is nondegenerate: $\det \mathbf{g} \neq 0$, but, in general, it is not symmetric: $g^{ij} \neq g^{ji}$. By g_{ij} we denote components of inverse matrix for (5.4), i. e.

$$\sum_{j=1}^{n} g^{ij}\, g_{jk} = \delta^i_k, \qquad\qquad \sum_{j=1}^{n} g_{ij}\, g^{jk} = \delta^k_i. \tag{5.5}$$

They form another extended tensor field, which traditionally is denoted by the same symbol $\mathbf{g}$. Though being non-symmetric extended tensor field, it is direct analog of metric tensor in Riemannian geometry.

Let $f = f(x^1, \ldots, x^n, v^1, \ldots, v^n)$ be some function in $\mathbf{v}$-representation for some local chart of M. This might be component of extended tensor field, component of extended connection, or coordinate representation of some other geometric object. Then $f \circ \lambda^{-1}$ is its $\mathbf{p}$-representation. Using explicit form (2.4) of the map λ^{-1}, we can derive the following transformation rules for partial derivatives:

$$\begin{aligned} &\frac{\partial(f \circ \lambda^{-1})}{\partial p_k} = \sum_{i=1}^{n} \frac{\partial V^i}{\partial p_k} \cdot \left(\frac{\partial f}{\partial v^i} \circ \lambda^{-1}\right),\\ &\frac{\partial(f \circ \lambda^{-1})}{\partial x^k} = \sum_{i=1}^{n} \frac{\partial V^i}{\partial x^k} \cdot \left(\frac{\partial f}{\partial v^i} \circ \lambda^{-1}\right) + \frac{\partial f}{\partial v^i} \circ \lambda^{-1}. \end{aligned} \tag{5.6}$$

Taking into account (5.4) and (5.5), we can rewrite (5.6) as

$$\frac{\partial f}{\partial v^i} \circ \lambda^{-1} = \sum_{k=1}^{n} g_{ki} \cdot \frac{\partial(f \circ \lambda^{-1})}{\partial p_k}. \tag{5.7}$$

Now let's apply (5.7) to the function $f = \Phi^k(x^1, \ldots, x^n, v^1, \ldots, v^n)$ in (5.3):

$$\frac{\partial \Phi^k}{\partial v^i} \circ \lambda^{-1} = \sum_{r=1}^{n} g_{ri} \cdot \frac{\partial}{\partial p_r} \left(\,\sum_{m=1}^{n} \frac{\partial V^k}{\partial x^m}\, V^m + \sum_{m=1}^{n} \frac{\partial V^k}{\partial p_m}\, \Theta_m\right).$$

Applying (5.7) once more and taking into account (4.9) and (5.1), we obtain

$$\Gamma^k_{ij} = \sum_{r=1}^{n} \sum_{s=1}^{n} g_{sj} \cdot \frac{\partial}{\partial p_s} \left(g_{ri} \cdot \frac{\partial}{\partial p_r} \left(\,\sum_{m=1}^{n} \frac{\partial V^k}{\partial x^m}\, V^m + \sum_{m=1}^{n} \frac{\partial V^k}{\partial p_m}\, \Theta_m\right)\right).$$

For the sake of convenience this formula should be slightly transformed:

$$\begin{aligned} \Gamma^k_{ij} = &\sum_{r=1}^{n} \sum_{s=1}^{n} g_{ri}\, g_{sj} \cdot \frac{\partial^2}{\partial p_r\, \partial p_s} \left(\,\sum_{m=1}^{n} \frac{\partial V^k}{\partial x^m}\, V^m + \sum_{m=1}^{n} \frac{\partial V^k}{\partial p_m}\, \Theta_m\right) - \\ -&\sum_{r=1}^{n} \sum_{s=1}^{n} \sum_{\alpha=1}^{n} g_{ri}\, g_{sj}\, \frac{\partial^2 V^\alpha}{\partial p_r\, \partial p_s} \cdot \frac{\partial}{\partial p_\alpha} \left(\,\sum_{m=1}^{n} \frac{\partial V^k}{\partial x^m}\, V^m + \sum_{m=1}^{n} \frac{\partial V^k}{\partial p_m}\, \Theta_m\right). \end{aligned} \tag{5.8}$$

Looking at (5.8), it is obvious that Γ^k_{ij} keep symmetry $\Gamma^k_{ij} = \Gamma^k_{ji}$ in $\mathbf{p}$-representation as well. It is not surprising due to (5.1), but now it is explicitly evident.

Theorem 5.1. *Each Newtonian dynamical system* (2.3) *in cotangent bundle T^*M generates extended affine connection with components* (5.8) *canonically associated with this dynamical system.*

Theorem 5.1 reformulates previous theorem 4.1 with respect to the same Newtonian dynamics, but transferred from tangent bundle to cotangent bundle by means of generalized Legendre transformation (2.2).

6. Regularity condition.

Now we are ready to study Newtonian dynamics given by the equations (2.3) without referring to its $\mathbf{v}$-representation (1.16). If we consider trajectory of dynamical system as a curve $p = p(t)$ in M, then velocity vector on this trajectory is given by the value of extended vector field $\mathbf{V} = \mathbf{V}(p, \mathbf{p})$. Let's define another extended vector field $\mathbf{W} = \mathbf{W}(p, \mathbf{p})$ with components

$$W^s = \sum_{r=1}^{n} \tilde{\nabla}^s V^r\, p_r \tag{6.1}$$

and consider scalar product of this vector field $\mathbf{W}$ and momentum covector $\mathbf{p}$:

$$\Omega = \langle \mathbf{p} \,|\, \mathbf{W} \rangle = \sum_{s=1}^{n} p_s\, W^s(x^1, \ldots, x^n, v^1, \ldots, v^n). \tag{6.2}$$

Extended scalar field $\Omega = \Omega(p, \mathbf{p})$ in (6.2) is somewhat like *kinetic energy* in mechanics (compare (6.2) with (1.11) and (1.13)). In mechanics both momentum and kinetic energy represent the "amount of motion" stored in moving object. This motivates the following definition.

Definition 6.1. Generalized Legendre map (2.2) given by components of extended vector field $\mathbf{V}$ in (2.4) is called **regular** if

(1) it is diffeomorphic;
(2) $V(p, \mathbf{p}) = 0$ is equivalent to $\mathbf{p} = 0$;
(3) $\Omega(p, \mathbf{p}) \neq 0$ for $\mathbf{p} \neq 0$.

Third part of regularity condition means that vector $\mathbf{W} = \mathbf{W}(p, \mathbf{p})$ is transversal to null-space of momentum covector $\mathbf{p}$. Therefore one can consider operator-valued extended tensor field $\mathbf{P}$ with the following components:

$$P^i_j = \delta^i_j - \frac{W^i\, p_j}{\Omega}. \tag{6.3}$$

Operator $\mathbf{P} = \mathbf{P}(p, \mathbf{p})$ with components (6.3) is a projector onto null-space of momentum covector $\mathbf{p}$ along vector $\mathbf{W} = \mathbf{W}(p, \mathbf{p})$. It is defined everywhere in T^*M except for those points $q = (p, \mathbf{p})$, where $\mathbf{p} = 0$.

7. Force covector.

Let's consider Newtonian dynamical system (2.3) in smooth manifold M. Suppose that M is equipped with symmetric extended affine connection Γ. This might be connection (5.8) canonically associated with dynamical system (2.3) or any other symmetric extended affine connection which is not related to (2.3) at all. In both cases one can introduce extended covector field $\mathbf{Q}$ with components

$$Q_i = \Theta_i - \sum_{j=1}^{n}\sum_{k=1}^{n} \Gamma^k_{ij}\, p_k\, V^j. \tag{7.1}$$

Then, using $Q_1, \ldots, Q_n$, one can write differential equations (2.3) as follows:

$$\begin{cases} \dot{x}^i = V^i(x^1, \ldots, x^n, p_1, \ldots, p_n),\\ \nabla_t p_i = Q_i(x^1, \ldots, x^n, p_1, \ldots, p_n). \end{cases} \tag{7.2}$$

Components of covector $\mathbf{Q}$ in (7.2) play the same role as components of vector $\mathbf{F}$ in (1.4). Therefore covector field $\mathbf{Q}$ is called **force covector** or **force field** of Newtonian dynamical system (7.2).

8. Weak normality condition and weak normality equations.

Let's consider one-parametric family of trajectories of Newtonian dynamical system (7.2). This is a family of parametric curves $q = q(t, y)$ in T^*M, where t is time variable and y is additional parameter. In local chart this family of curves is represented by the following set of $2n$ functions

$$\begin{cases} x^1 = x^1(t, y),\\ \dots\dots\dots\dots\\ x^n = x^n(t, y), \end{cases} \qquad\qquad \begin{cases} p_1 = p_1(t, y),\\ \dots\dots\dots\dots\\ p_n = p_n(t, y), \end{cases} \tag{8.1}$$

Differentiating first part of these functions (8.1) with respect to additional parameter y, we obtain vector-function $\boldsymbol{\tau} = \boldsymbol{\tau}(t, y)$ with components

$$\tau^i = \frac{\partial x^i}{\partial y}. \tag{8.2}$$

Functions $p_1, \ldots, p_n$ in (8.1) are components of covector-function. Therefore we should apply covariant derivative to them:

$$\xi_i = \nabla_{\boldsymbol{\tau}} p_i = \frac{\partial p_i}{\partial y} - \sum_{j=1}^{n}\sum_{k=1}^{n} \Gamma^k_{ij}\, p_k\, \tau^j. \tag{8.3}$$

As a result we get covector-function $\boldsymbol{\xi} = \boldsymbol{\xi}(t, y)$ with components (8.3). Vector $\boldsymbol{\tau}$ is called **variation vector** or, more exactly, **vector of variation of trajectory**, while covector $\boldsymbol{\xi}$ is called **covector of variation of momentum**.

Components of both functions $\boldsymbol{\tau}(t, y)$ and $\boldsymbol{\xi}(t, y)$ satisfy a system of ordinary differential equations with respect to time variable. In order to derive these equations let's differentiate equations (7.2) with respect to parameter y. This yields

$$\nabla_t \tau^i = \sum_{k=1}^{n} \nabla_k V^i \cdot \tau^k + \sum_{k=1}^{n} \tilde{\nabla}^k V^i \cdot \xi_k, \tag{8.4}$$

$$\begin{aligned}&\nabla_t\xi_i+\sum_{k=1}^{n}\left(\sum_{j=1}^{n}\sum_{s=1}^{n}R^s_{ijk}\,p_s\,V^j-\sum_{j=1}^{n}\sum_{s=1}^{n}D^{sj}_{ik}\,p_s\,Q_j\right)\cdot\tau^k+\\&+\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{s=1}^{n}D^{sk}_{ij}\,p_s\,V^j\cdot\xi_k=\sum_{k=1}^{n}\nabla_kQ_i\cdot\tau^k+\sum_{k=1}^{n}\tilde\nabla^kQ_i\cdot\xi_k.\end{aligned}\tag{8.5}$$

Here in (8.5) we used components of curvature tensors $\mathbf{R}$ and $\mathbf{D}$. First is an analog of standard curvature tensor of Riemannian geometry, it is given by formula

$$\begin{aligned}R^k_{rij}&=\frac{\partial\Gamma^k_{jr}}{\partial x^i}-\frac{\partial\Gamma^k_{ir}}{\partial x^j}+\sum_{m=1}^{n}\Gamma^k_{im}\,\Gamma^m_{jr}-\sum_{m=1}^{n}\Gamma^k_{jm}\,\Gamma^m_{ir}+\\&+\sum_{m=1}^{n}\sum_{\alpha=1}^{n}p_\alpha\,\Gamma^\alpha_{mi}\,\frac{\partial\Gamma^k_{jr}}{\partial p^m}-\sum_{m=1}^{n}\sum_{\alpha=1}^{n}p_\alpha\,\Gamma^\alpha_{mj}\,\frac{\partial\Gamma^k_{ir}}{\partial p^m}.\end{aligned}\tag{8.6}$$

Second is a tensor of dynamic curvature, it is nonzero only for extended connections, when Γ^k_{ij} do actually depend on components of momentum covector:

$$D^{kr}_{ij}=-\frac{\partial\Gamma^k_{ij}}{\partial p_r}.\tag{8.7}$$

In the next step we consider scalar product of vector $\boldsymbol{\tau}$ and momentum covector $\mathbf{p}$. This scalar product introduced by formula (1.13) determines so called **deviation function** φ. Like $\boldsymbol{\tau}=\boldsymbol{\tau}(t,y)$ and $\boldsymbol{\xi}=\boldsymbol{\xi}(t,y)$, this is a function of time variable t and additional parameter y. In general case deviation function (1.13) satisfies linear homogeneous ODE of the order $2n$ (see theorem 6.1 in [20]). But here we consider special case determined by the following definition.

Definition 8.1. We say that Newtonian dynamical system (2.3) satisfies **weak** normality condition if for each its trajectory $q=q(t)$ there is some second order homogeneous linear ordinary differential equation

$$\ddot\varphi=\mathcal{A}(t)\,\dot\varphi+\mathcal{B}(t)\,\varphi\tag{8.8}$$

such that all deviation functions on the trajectory satisfy this differential equation.

Saying *"all deviation functions"*, in definition 8.1 we imply that each trajectory $q=q(t)$ can be included into one-parametric family of trajectories by various possible ways. Each such inclusion defines some variation vector $\boldsymbol{\tau}=\boldsymbol{\tau}(t)$ and corresponding deviation function $\varphi=\varphi(t)$ on that trajectory. Functions $\mathcal{A}(t)$ and $\mathcal{B}(t)$ in (8.8) depend on the trajectory $q=q(t)$, but they do not depend on how this trajectory is included into one parametric family of trajectories.

Let's calculate time derivatives of deviation function (1.13). For first order time derivative $\dot\varphi$ we obtain the following expression:

$$\dot\varphi=\nabla_t\varphi=\sum_{i=1}^{n}\nabla_t\tau^i\,p_i+\sum_{i=1}^{n}\tau^i\,\nabla_tp_i.\tag{8.9}$$

Then we substitute (8.4) for $\nabla_t\tau^i$ and (7.2) for ∇_tp_i into (8.9). This yields more detailed expression for first order time derivative of deviation function:

$$\dot{\varphi}=\sum_{k=1}^{n}\left(\sum_{i=1}^{n}\nabla_{k}V^{i}\,p_{i}+Q_{k}\right)\cdot\tau^{k}+\sum_{k=1}^{n}\left(\sum_{i=1}^{n}\tilde{\nabla}^{k}V^{i}\,p_{i}\right)\cdot\xi_{k}. \tag{8.10}$$

Applying ∇_t to (8.10), we derive formula for second order time derivative $\ddot{\varphi}$:

$$\begin{aligned}\ddot{\varphi}&=\sum_{k=1}^{n}\left(\sum_{i=1}^{n}\nabla_{k}V^{i}\,p_{i}+Q_{k}\right)\cdot\nabla_t\tau^{k}+\sum_{k=1}^{n}\left(\sum_{i=1}^{n}\tilde{\nabla}^{k}V^{i}\,p_{i}\right)\cdot\nabla_t\xi_{k}+\\&+\sum_{k=1}^{n}\sum_{r=1}^{n}\left(\sum_{i=1}^{n}\nabla_r\nabla_{k}V^{i}\,p_{i}+\nabla_rQ_{k}\right)V^r\cdot\tau^{k}+\sum_{k=1}^{n}\left(\sum_{i=1}^{n}\nabla_{k}V^{i}\,Q_{i}\right)\cdot\tau^{k}+\\&+\sum_{k=1}^{n}\sum_{r=1}^{n}\left(\sum_{i=1}^{n}\tilde{\nabla}^r\nabla_{k}V^{i}\,p_{i}+\tilde{\nabla}^rQ_{k}\right)Q_r\cdot\tau^{k}+\sum_{k=1}^{n}\left(\sum_{i=1}^{n}\tilde{\nabla}^{k}V^{i}\,Q_{i}\right)\cdot\xi_{k}+\\&+\sum_{k=1}^{n}\sum_{r=1}^{n}\left(\sum_{i=1}^{n}\nabla_r\tilde{\nabla}^{k}V^{i}\,p_{i}\right)V^r\cdot\xi_{k}+\sum_{k=1}^{n}\sum_{r=1}^{n}\left(\sum_{i=1}^{n}\tilde{\nabla}^r\tilde{\nabla}^{k}V^{i}\,p_{i}\right)Q_r\cdot\xi_{k}.\end{aligned}$$

Let's substitute (8.4) for $\nabla_t\tau^k$ and (8.5) for $\nabla_t\xi_k$ into the above equality. It's easy to note that resulting expression for $\ddot{\varphi}$ will be of the form

$$\ddot{\varphi}=\sum_{k=1}^{n}\alpha^k\,\xi_k+\sum_{k=1}^{n}\beta_k\,\tau^k. \tag{8.11}$$

Here $\alpha^1, \ldots, \alpha^n$ are components of extended vector field $\boldsymbol{\alpha}$ given by formula

$$\begin{aligned}\alpha^k&=\sum_{i=1}^{n}\tilde{\nabla}^kV^i\,Q_i+\sum_{r=1}^{n}\sum_{i=1}^{n}\nabla_r\tilde{\nabla}^kV^i\,p_i\,V^r+\sum_{r=1}^{n}\sum_{i=1}^{n}\tilde{\nabla}^r\tilde{\nabla}^kV^i\,p_i\,Q_r+\\&+\sum_{r=1}^{n}\sum_{i=1}^{n}\tilde{\nabla}^rV^i\,p_i\,\tilde{\nabla}^kQ_r-\sum_{r=1}^{n}\sum_{s=1}^{n}\sum_{i=1}^{n}\sum_{j=1}^{n}D^{sk}_{rj}\,\tilde{\nabla}^rV^i\,p_i\,p_s\,V^j+\\&+\sum_{r=1}^{n}\sum_{i=1}^{n}\nabla_rV^i\,p_i\tilde{\nabla}^kV^r+\sum_{r=1}^{n}\tilde{\nabla}^kV^r\,Q_r.\end{aligned}\tag{8.12}$$

Similarly, quantities $\beta_1, \ldots, \beta_n$ are components of extended covector field $\boldsymbol{\beta}$:

$$\begin{aligned}\beta_k&=\sum_{r=1}^{n}\sum_{i=1}^{n}\nabla_r\nabla_kV^i\,p_i\,V^r+\sum_{r=1}^{n}V^r\,\nabla_rQ_k+\sum_{i=1}^{n}\nabla_kV^i\,Q_i+\\&+\sum_{r=1}^{n}\sum_{i=1}^{n}\tilde{\nabla}^r\nabla_kV^i\,p_i\,Q_r+\sum_{r=1}^{n}\tilde{\nabla}^rQ_k\,Q_r+\sum_{r=1}^{n}\nabla_kV^r\,Q_r+\\&+\sum_{r=1}^{n}\sum_{i=1}^{n}\nabla_rV^i\,p_i\,\nabla_kV^r+\sum_{r=1}^{n}\sum_{i=1}^{n}\tilde{\nabla}^rV^i\,p_i\,\nabla_kQ_r-\\&-\sum_{r=1}^{n}\sum_{s=1}^{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(R^s_{rjk}\,\tilde{\nabla}^rV^i\,p_i\,p_s\,V^j-D^{sj}_{rk}\,\tilde{\nabla}^rV^i\,p_i\,p_s\,Q_j\right)\end{aligned}\tag{8.13}$$

Let's compare formula (8.10) for time derivative $\dot{\varphi}$ with our previous notations (6.1). It's easy to see that (8.10) can be written as

$$\dot{\varphi}=\sum_{k=1}^{n}\left(\sum_{i=1}^{n}\nabla_k V^i\,p_i+Q_k\right)\cdot\tau^k+\sum_{k=1}^{n}W^k\cdot\xi_k. \tag{8.14}$$

Second term in (8.14) is scalar product of vector $\mathbf{W}$ and covector $\boldsymbol{\xi}$. Suppose that regularity condition is fulfilled (see definition 6.1). Then we can introduce same scalar product into the formula (8.11) for second order partial derivative $\ddot{\varphi}$:

$$\ddot{\varphi}=\sum_{k=1}^{n}\frac{\langle\mathbf{p}\,|\,\boldsymbol{\alpha}\rangle}{\Omega}\,W^k\cdot\xi_k+\sum_{k=1}^{n}\sum_{r=1}^{n}\alpha^r\,P^k_r\cdot\xi_k+\sum_{k=1}^{n}\beta_k\,\tau^k. \tag{8.15}$$

Here we used formula (6.3) written in the following form:

$$\delta^k_r=P^k_r+\frac{W^k\,p_r}{\Omega}. \tag{8.16}$$

Combining formulas (8.14) and (8.15) for time derivatives $\dot{\varphi}$ and $\ddot{\varphi}$, we obtain the equality with right hand side free of scalar product $\langle\boldsymbol{\xi}\,|\,\mathbf{W}\rangle$:

$$\begin{aligned}\ddot{\varphi}-\frac{\langle\mathbf{p}\,|\,\boldsymbol{\alpha}\rangle}{\Omega}\,\dot{\varphi}&=\sum_{k=1}^{n}\sum_{r=1}^{n}\alpha^r\,P^k_r\cdot\xi_k+\sum_{k=1}^{n}\beta_k\cdot\tau^k-\\-\sum_{k=1}^{n}&\left(\sum_{i=1}^{n}\sum_{s=1}^{n}\frac{\nabla_kV^i\,p_i\,\alpha^s\,p_s}{\Omega}+\sum_{s=1}^{n}\frac{Q_k\,\alpha^s\,p_s}{\Omega}\right)\cdot\tau^k.\end{aligned} \tag{8.17}$$

For the sake of brevity in further calculations we introduce the following notations:

$$\eta_k=\beta_k-\sum_{i=1}^{n}\sum_{s=1}^{n}\frac{\nabla_kV^i\,p_i\,\alpha^s\,p_s}{\Omega}-\sum_{s=1}^{n}\frac{Q_k\,\alpha^s\,p_s}{\Omega}. \tag{8.18}$$

Quantities $\eta_1,\,\ldots\,,\,\eta_n$ in (8.18) are components of covector field $\boldsymbol{\eta}$. Using notations (8.18), we can simplify the above equality (8.17):

$$\ddot{\varphi}-\frac{\langle\mathbf{p}\,|\,\boldsymbol{\alpha}\rangle}{\Omega}\,\dot{\varphi}=\sum_{k=1}^{n}\sum_{r=1}^{n}\alpha^r\,P^k_r\cdot\xi_k+\sum_{k=1}^{n}\eta_k\cdot\tau^k. \tag{8.19}$$

Deviation function φ is scalar product of vector $\boldsymbol{\tau}$ and covector $\mathbf{p}$. Using (8.16), we can introduce such scalar product into right hand side of (8.19). Then we get

$$\ddot{\varphi}-\frac{\langle\mathbf{p}\,|\,\boldsymbol{\alpha}\rangle}{\Omega}\,\dot{\varphi}-\frac{\langle\boldsymbol{\eta}\,|\,\mathbf{W}\rangle}{\Omega}\,\varphi=\sum_{k=1}^{n}\sum_{r=1}^{n}\alpha^r\,P^k_r\cdot\xi_k+\sum_{k=1}^{n}\sum_{r=1}^{n}\eta_r\,P^r_k\cdot\tau^k. \tag{8.20}$$

Now let's recall the definition 8.1. Suppose that the following equalities are fulfilled:

$$\sum_{r=1}^{n}\alpha^r\,P^k_r=0,\qquad\qquad\sum_{r=1}^{n}\eta_r\,P^r_k=0. \tag{8.21}$$

Then deviation function φ satisfies second order ordinary differential equation (8.8) with coefficients $\mathcal{A}$ and $\mathcal{B}$ determined by vector field $\boldsymbol{\alpha}$ and covector field $\boldsymbol{\eta}$:

$$\mathcal{A} = \frac{\langle \mathbf{p} \,|\, \boldsymbol{\alpha} \rangle}{\Omega}, \qquad \mathcal{B} = \frac{\langle \boldsymbol{\eta} \,|\, \mathbf{W} \rangle}{\Omega}. \tag{8.22}$$

This means that equations (8.21) are sufficient for weak normality condition formulated in definition 8.1 to be fulfilled.

Conversely, suppose that weak normality condition for Newtonian dynamical system (7.2) is fulfilled. Combining (8.8) with (8.20), we derive

$$\tilde{\mathcal{A}}\,\dot{\varphi} + \tilde{\mathcal{B}}\,\varphi = \sum_{k=1}^{n}\sum_{r=1}^{n} \alpha^r\, P^k_r \cdot \xi_k + \sum_{k=1}^{n}\sum_{r=1}^{n} \eta_r\, P^r_k \cdot \tau^k, \tag{8.23}$$

where

$$\tilde{\mathcal{A}} = \mathcal{A} - \frac{\langle \mathbf{p} \,|\, \boldsymbol{\alpha} \rangle}{\Omega}, \qquad \tilde{\mathcal{B}} = \mathcal{B} - \frac{\langle \boldsymbol{\eta} \,|\, \mathbf{W} \rangle}{\Omega}. \tag{8.24}$$

Substituting (8.14) and (1.13) for $\dot{\varphi}$ and φ into (8.23), we obtain

$$\tilde{\mathcal{A}}\ \langle \boldsymbol{\xi} \,|\, \mathbf{W} \rangle + \tilde{\mathcal{C}}\ \langle \mathbf{p} \,|\, \boldsymbol{\tau} \rangle = \sum_{k=1}^{n}\sum_{r=1}^{n} \alpha^r\, P^k_r \cdot \xi_k + \sum_{k=1}^{n}\sum_{r=1}^{n} \tilde{\eta}_r\, P^r_k \cdot \tau^k. \tag{8.25}$$

Here for more convenience we introduced the following auxiliary notations:

$$\tilde{\mathcal{C}} = \tilde{\mathcal{B}} + \tilde{\mathcal{A}} \left(\sum_{r=1}^{n}\sum_{i=1}^{n} \frac{\nabla_r V^i\, p_i\, W^r}{\Omega} + \sum_{r=1}^{n} \frac{W^r\, Q_r}{\Omega} \right), \tag{8.26}$$

$$\tilde{\eta}_r = \eta_r - \tilde{\mathcal{A}} \left(\sum_{i=1}^{n} \nabla_r V^i\, p_i + Q_r \right). \tag{8.27}$$

Now conceptual point! Looking at formula (8.25), let's remember that for a fixed trajectory $p = p(t)$ of Newtonian dynamical system (7.2) functions $\tau^1(t), \ldots, \tau^n(t)$ and $\xi_1(t), \ldots, \xi_n(t)$ satisfy system of first order linear homogeneous ordinary differential equations (see (8.4) and (8.5)). Solutions of these differential equations form $2n$-dimensional linear space $\mathfrak{T}$, while components of vector-function $\boldsymbol{\tau}(t)$ and covector-function $\boldsymbol{\xi}(t)$ for any fixed time instant $t = t_0$ can be treated as coordinates in $\mathfrak{T}$. In other words, linear $\mathfrak{T}$ is isomorphic to direct sum

$$\mathfrak{T} \cong T_p(M) \oplus T^*_p(M), \tag{8.28}$$

where $p = p(t_0)$. In regular case (see definition 6.1) denominator $\Omega = \Omega(p, \mathbf{p})$ defined by formula (6.2) is nonzero for $\mathbf{p} \neq 0$. Hence vector $\mathbf{W} = \mathbf{W}(p, \mathbf{p})$ is transversal to null-space of momentum covector $\mathbf{p}$:

$$\mathfrak{P} = \operatorname{Im} \mathbf{P} = \{ \mathbf{X} \in T_p(M) :\ \langle \mathbf{p} \,|\, \mathbf{X} \rangle = 0 \}. \tag{8.29}$$

Therefore tangent space $T_p(P)$ in (8.28) breaks into direct sum

$$T_p(M) = \langle \mathbf{W} \rangle \oplus \mathfrak{P}, \tag{8.30}$$

where $\langle \mathbf{W} \rangle$ is linear span of vector $\mathbf{W}$. We can write similar expansion for $T^*_p(M)$:

$$T^*_p(M) = \langle \mathbf{p} \rangle \oplus \mathfrak{W}. \tag{8.31}$$

Here $\langle \mathbf{p} \rangle$ is linear span of covector $\mathbf{p}$ and $\mathfrak{W}$ is null-space of $\mathbf{W}$:

$$\mathfrak{W} = \operatorname{Im} \mathbf{P}^* = \left\{ \mathbf{y} \in T^*_p(M) : \ \langle \mathbf{y} \,|\, \mathbf{W} \rangle = 0 \right\}. \tag{8.32}$$

Substituting (8.30) and (8.31) into (8.28) we get the following expansion:

$$\mathfrak{T} \cong \underbrace{\langle \mathbf{W} \rangle \oplus \langle \mathbf{p} \rangle} \oplus \underbrace{\mathfrak{P} \oplus \mathfrak{W}} = \mathfrak{T}_1 \oplus \mathfrak{T}_2. \tag{8.33}$$

Expansion (8.33) of the space $\mathfrak{T}$ generates conjugate expansion of dual space $\mathfrak{T}^*$:

$$\mathfrak{T}^* \cong \mathfrak{T}^*_1 \oplus \mathfrak{T}^*_2. \tag{8.34}$$

The above equality (8.25) is an equality of functions. However, if we treat pair of $\boldsymbol{\tau}$ and $\boldsymbol{\xi}$ as a point of $\mathfrak{T}$, then for any fixed $t = t_0$ both sides of the equality (8.25) can be treated as linear functionals in $\mathfrak{T}$, i. e. they are elements of $\mathfrak{T}^*$. Note that left hand side of (8.25) is in $\mathfrak{T}^*_1$, while right hand side is an element of $\mathfrak{T}^*_2$. Subspaces $\mathfrak{T}^*_1$ and $\mathfrak{T}^*_2$ in direct sum (8.34) have zero intersection. Therefore both sides of (8.25) are zero (as elements of $\mathfrak{T}^*$ and as functions of t as well, since $t = t_0$ as an arbitrary fixed instant of time). Thus we have the equality

$$\tilde{\mathcal{A}} \ \langle \boldsymbol{\xi} \,|\, \mathbf{W} \rangle + \tilde{\mathcal{C}} \ \langle \mathbf{p} \,|\, \boldsymbol{\tau} \rangle = 0. \tag{8.35}$$

Due to expansion $\mathfrak{T}^*_1 \cong \langle \mathbf{W} \rangle^* \oplus \langle \mathbf{p} \rangle^*$ from the equality (8.35) we derive

$$\tilde{\mathcal{A}} = 0, \qquad\qquad \tilde{\mathcal{C}} = 0. \tag{8.36}$$

Applying (8.26), (8.27), and (8.24) to (8.36), we get $\tilde{\mathcal{B}} = 0$, $\tilde{\eta}_k = \eta_k$ and come back to (8.22). Substituting $\tilde{\mathcal{A}} = 0$ and $\tilde{\mathcal{B}} = 0$ into (8.23), we obtain the equality

$$\sum_{k=1}^{n} \sum_{r=1}^{n} \alpha^r \, P^k_r \cdot \xi_k + \sum_{k=1}^{n} \sum_{r=1}^{n} \eta_r \, P^r_k \cdot \tau^k = 0,$$

which breaks into two separate parts equivalent to (8.21). Thus, assuming that weak normality condition is fulfilled, we have derived again the equalities (8.21). When written explicitly, they form a system of partial differential equations relating extended vector field $\mathbf{V}$, extended covector field $\mathbf{Q}$, and extended affine connection Γ. These equations are called **weak normality equations**. Just above we have proved the following theorem for them.

Theorem 8.1. *Weak normality condition for Newtonian dynamical system* (7.2) *is equivalent to weak normality equations* (8.21) *that should be fulfilled at all points* $q = (p, \mathbf{p})$ *of cotangent bundle* T^*M*, where* $\mathbf{p} \neq 0$.

Using (8.12), (8.13), (8.18), one can easily write weak normality equations (8.21) explicitly. However, we shall not do it, since they are rather huge.

9. Additional normality condition.

Let's proceed with studying normal shift phenomenon assuming that weak normality condition for Newtonian dynamical system (7.2) is fulfilled. For this purpose let's consider some hypersurface σ and apply initial data (1.14) to differential equations (7.2). As a result we get $(n-1)$-parametric family of trajectories of Newtonian dynamical system (7.2). Let's fix some point $p_0 \in \sigma$. If $y^1, \ldots, y^{n-1}$ are local coordinates on σ in some neighborhood of the point p_0 and if $x^1, \ldots, x^n$ are local coordinates in M in n-dimensional neighborhood of this point, then our $(n-1)$-parametric family of trajectories is represented by functions

$$\begin{cases} x^1 = x^1(t, y^1, \ldots, y^{n-1}), \\ \dots\dots\dots\dots\dots\dots \\ x^n = x^n(t, y^1, \ldots, y^{n-1}), \end{cases} \qquad \begin{cases} p_1 = p_1(t, y^1, \ldots, y^{n-1}), \\ \dots\dots\dots\dots\dots\dots \\ p_n = p_n(t, y^1, \ldots, y^{n-1}). \end{cases} \tag{9.1}$$

Comparing (9.1) with (8.1) and looking at (8.2) and (8.3), we see that now we can define vector-functions $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_{n-1}$ and covector-functions $\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_{n-1}$. Their components in local chart are given by the following derivatives:

$$\tau^s_i = \frac{\partial x^s}{\partial y^i}, \qquad \xi_{si} = \nabla_{\boldsymbol{\tau}_i} p_s. \tag{9.2}$$

Vector-functions $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_{n-1}$ with components (9.2) define $n-1$ deviation functions $\varphi_1, \ldots, \varphi_{n-1}$ according to the formula (1.13):

$$\varphi_i = \varphi_i(t, y^1, \ldots, y^{n-1}) = \langle \mathbf{p} \,|\, \boldsymbol{\tau}_i \rangle = \sum_{s=1}^{n} \tau^s_i \, p_s. \tag{9.3}$$

We assumed that weak normality condition is fulfilled (see definition 8.1). Therefore each function φ_i in (9.3) satisfies differential equation of the form (8.8):

$$\ddot{\varphi}_i = \mathcal{A}(t, y^1, \ldots, y^{n-1})\,\dot{\varphi}_i + \mathcal{B}(t, y^1, \ldots, y^{n-1})\,\varphi_i. \tag{9.4}$$

According to definition 1.4, in order to have normal shift we should provide vanishing of all deviation functions $\varphi_i, \ldots, \varphi_{n-1}$ in (9.3). Due to the equation (9.4) it is sufficient to provide the following initial data for them:

$$\varphi_i\Big|_{t=0} = 0, \qquad \dot{\varphi}_i\Big|_{t=0} = 0. \tag{9.5}$$

First part of initial conditions (9.5) is fulfilled due to initial data (1.14). Second part of these initial conditions should be fulfilled at the expense of proper choice of scalar function $\nu = \nu(p) = \nu(y^1, \ldots, y^n)$ in (1.14). In order to calculate time derivative $\dot{\varphi}_i$ we can use formula (8.14). Here it is written as follows:

$$\dot{\varphi}_i = \sum_{s=1}^{n} \left(\sum_{r=1}^{n} \nabla_s V^r \, p_r + Q_s \right) \cdot \tau^s_i + \sum_{s=1}^{n} W^s \cdot \xi_{si}. \tag{9.6}$$

Then we should substitute $t = 0$ into (9.6). Vectors $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_{n-1}$ at initial instant of time $t = 0$ form base of coordinate vectors in tangent space to initial hypersurface

σ. Initial value of momentum covector $\mathbf{p}$ is given by (1.14). Thus we have to calculate initial values for covectors $\boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_{n-1}$ in formula (9.6):

$$\xi_{si}\Bigr|_{t=0} = \nabla_{\boldsymbol{\tau}_i} p_s\Bigr|_{t=0} = \frac{\partial\nu}{\partial y^i}\cdot n_s + \nu\cdot\nabla_{\boldsymbol{\tau}_i} n_s. \tag{9.7}$$

Substituting (9.7) into (9.6) and then substituting (9.6) into (9.5), we get

$$\frac{\partial\nu}{\partial y^i} = -\frac{(\nu)^2}{\Omega}\sum_{s=1}^{n} W^s\cdot\nabla_{\boldsymbol{\tau}_i} n_s - \frac{\nu}{\Omega}\sum_{s=1}^{n}\left(\sum_{r=1}^{n}\nabla_s V^r\, p_r + Q_s\right)\cdot\tau^s_i. \tag{9.8}$$

In two-dimensional case $n = \dim M = 2$ we have only one variable $y = y^1$. Then (9.8) turns to ordinary differential equation with respect to function $\nu = \nu(y)$. Normalizing condition (1.3) yields initial value problem for this ordinary differential equation, which is always solvable (at least, locally).

In multidimensional case $n \geqslant 3$ we have several variables $y^1, \ldots, y^{n-1}$. In this case partial differential equations (9.8) form complete system of Pfaff equations with respect to function $\nu = \nu(y^1, \ldots, y^{n-1})$. Initial value problem

$$\nu(p_0) = \nu_0 \tag{9.9}$$

at some fixed point $p_0 \in \sigma$ with local coordinates $y^1_0, \ldots, y^{n-1}_0$ is typical for Pfaff equations (9.8). However, now it is not unconditionally solvable (even locally).

Definition 9.1. Complete system of Pfaff equations (9.8) is called *compatible* if initial value problem (9.9) for it is locally solvable for all $p_0 \in M$ and for all $\nu_0 \neq 0$.

Let's write Pfaff equations (9.8) formally, denoting by ψ_i their right hand sides:

$$\frac{\partial\nu}{\partial y^i} = \psi_i(\nu, y^1, \ldots, y^{n-1}) \tag{9.10}$$

Due to (9.10) we can calculate mixed partial derivatives of ν in two different ways

$$\frac{\partial^2\nu}{\partial y^i\,\partial y^j} = \frac{\partial\psi_i}{\partial y^j} + \frac{\partial\psi_i}{\partial\nu}\,\psi_j = \vartheta_{ij}(\nu, y^1, \ldots, y^{n-1}), \tag{9.11}$$

$$\frac{\partial^2\nu}{\partial y^j\,\partial y^i} = \frac{\partial\psi_j}{\partial y^i} + \frac{\partial\psi_j}{\partial\nu}\,\psi_i = \vartheta_{ji}(\nu, y^1, \ldots, y^{n-1}). \tag{9.12}$$

Equating (9.11) and (9.12), we get compatibility condition for (9.10):

$$\vartheta_{ij}(\nu, y^1, \ldots, y^{n-1}) = \vartheta_{ji}(\nu, y^1, \ldots, y^{n-1}). \tag{9.13}$$

Lemma 9.1. *Pfaff equations* (9.10) *are compatible in the sense of definition 9.1 if and only if for $\nu \neq 0$ left and right hands sides of* (9.13) *are equal to each other identically as functions of n independent variables $y^1, \ldots, y^{n-1}$, and ν.*

Lemma 9.1 is standard result in the theory of Pfaff equations. Proof of this lemma can be found in thesis [7].

Definition 9.2. We say that Newtonian dynamical system satisfies **additional** normality condition if Pfaff equations for the function $\nu(p)$ in (1.14) derived from initial conditions (9.5) are compatible for any hypersurface σ in M.

For the sake of convenience we introduce extended covector field $\mathbf{U}$ with components

$$U_s=\sum_{r=1}^{n}\nabla_s V^r\,p_r+Q_s. \tag{9.14}$$

Then Pfaff equations (9.8) are written as follows:

$$\frac{\partial\nu}{\partial y^i}=-\frac{(\nu)^2}{\Omega}\sum_{s=1}^{n}W^s\cdot\nabla_{\boldsymbol{\tau}_i}n_s-\frac{\nu}{\Omega}\sum_{s=1}^{n}U_s\cdot\tau^s_i. \tag{9.15}$$

Now, relying upon definition 9.2, we shall derive explicit form of compatibility equation (9.13). For this purpose let's calculate partial derivatives (9.11) and (9.12) in explicit form. For partial derivatives (9.11) we obtain

$$\begin{aligned}\frac{\partial^2\nu}{\partial y^i\,\partial y^j}&=\nabla_{\boldsymbol{\tau}_j}\psi_i=\frac{2\,(\nu)^3}{\Omega^2}\sum_{s=1}^{n}\sum_{r=1}^{n}W^s\,W^r\,\nabla_{\boldsymbol{\tau}_i}n_s\,\nabla_{\boldsymbol{\tau}_j}n_r+\\&+\frac{2\,(\nu)^2}{\Omega^2}\sum_{s=1}^{n}\sum_{r=1}^{n}W^s\,U_r\,\nabla_{\boldsymbol{\tau}_i}n_s\,\tau^r_j+\frac{(\nu)^2}{\Omega^2}\sum_{s=1}^{n}\sum_{r=1}^{n}U_s\,W^r\,\tau^s_i\,\nabla_{\boldsymbol{\tau}_j}n_r+\\&+\frac{\nu}{\Omega^2}\sum_{s=1}^{n}\sum_{r=1}^{n}U_s\,U_r\,\tau^s_i\,\tau^r_j-(\nu)^2\sum_{s=1}^{n}\nabla_{\boldsymbol{\tau}_j}\left(\frac{W^s}{\Omega}\right)\cdot\nabla_{\boldsymbol{\tau}_i}n_s-\\&-\nu\sum_{s=1}^{n}\nabla_{\boldsymbol{\tau}_j}\left(\frac{U_s}{\Omega}\right)\,\tau^s_i-\frac{(\nu)^2}{\Omega}\sum_{s=1}^{n}W^s\,\nabla_{\boldsymbol{\tau}_j}\nabla_{\boldsymbol{\tau}_i}n_s-\frac{\nu}{\Omega}\sum_{s=1}^{n}U_s\,\nabla_{\boldsymbol{\tau}_j}\tau^s_i.\end{aligned} \tag{9.16}$$

In order to transform (9.16) we need to bring about some preliminary calculations. For covariant derivative $\nabla_{\boldsymbol{\tau}_j}\tau^s_i$ in (9.16) we have

$$\nabla_{\boldsymbol{\tau}_j}\tau^s_i=\frac{\partial\tau^s_i}{\partial y^j}+\sum_{r=1}^{n}\sum_{q=1}^{n}\Gamma^s_{rq}\,\tau^r_i\,\tau^q_j=\frac{\partial^2x^s}{\partial y^i\,\partial y^j}+\sum_{r=1}^{n}\sum_{q=1}^{n}\Gamma^s_{rq}\,\tau^r_i\,\tau^q_j.$$

Taking into account symmetry of connection components $\Gamma^s_{rq}=\Gamma^s_{qr}$, we derive

$$\nabla_{\boldsymbol{\tau}_i}\tau^s_j-\nabla_{\boldsymbol{\tau}_j}\tau^s_i=0. \tag{9.17}$$

In a similar way by direct calculations we derive the following identities:

$$\begin{aligned}\nabla_{\boldsymbol{\tau}_j}\left(\frac{W^s}{\Omega}\right)&=\sum_{r=1}^{n}\nabla_r\left(\frac{W^s}{\Omega}\right)\cdot\tau^r_j+\sum_{r=1}^{n}\tilde{\nabla}^r\left(\frac{W^s}{\Omega}\right)\cdot\xi_{rj},\\\nabla_{\boldsymbol{\tau}_j}\left(\frac{U_s}{\Omega}\right)&=\sum_{r=1}^{n}\nabla_r\left(\frac{U_s}{\Omega}\right)\cdot\tau^r_j+\sum_{r=1}^{n}\tilde{\nabla}^r\left(\frac{U_s}{\Omega}\right)\cdot\xi_{rj}.\end{aligned} \tag{9.18}$$

Formulas (9.18) are special cases of general formula applicable to arbitrary extended tensor field $\mathbf{X}$. If $X^{i_1\dots\,i_r}_{j_1\dots\,j_s}$ are components of $\mathbf{X}$ in local chart, then we have

$$\nabla_{\boldsymbol{\tau}_q} X^{i_1\dots\,i_r}_{j_1\dots\,j_s} = \sum_{r=1}^{n} \nabla_r X^{i_1\dots\,i_r}_{j_1\dots\,j_s}\cdot\tau^r_q + \sum_{r=1}^{n} \tilde\nabla^r X^{i_1\dots\,i_r}_{j_1\dots\,j_s}\cdot\xi_{rq}$$

And finally, there is an identity for commutator of two covariant derivatives:

$$\begin{gathered}
[\nabla_{\boldsymbol{\tau}_i},\,\nabla_{\boldsymbol{\tau}_j}]n_s = -\sum_{q=1}^{n}\sum_{\alpha=1}^{n}\sum_{\gamma=1}^{n}\frac{p_q}{\nu}\,R^q_{s\alpha\gamma}\,\tau^\alpha_i\,\tau^\gamma_j\,+\\
+\sum_{r=1}^{n}\sum_{\alpha=1}^{n}\sum_{\gamma=1}^{n}\frac{p_\alpha}{\nu}\,D^{\alpha\gamma}_{sr}\,\tau^r_j\,\xi_{\gamma i}-\sum_{r=1}^{n}\sum_{\alpha=1}^{n}\sum_{\gamma=1}^{n}\frac{p_\alpha}{\nu}\,D^{\alpha\gamma}_{sr}\,\tau^r_i\,\xi_{\gamma j}.
\end{gathered}\tag{9.19}$$

Combining (9.15) and (9.7), for the quantities ξ_{rj} in (9.18) we derive

$$\xi_{rj} = \sum_{s=1}^{n}\nu\,P^s_r\,\nabla_{\boldsymbol{\tau}_j}n_s - \sum_{s=1}^{n}\frac{U_s\,p_r}{\Omega}\,\tau^s_j.\tag{9.20}$$

Here P^s_r are components of projection operator $\mathbf{P}$ introduced in (6.3). Further

$$\tilde\nabla^r\!\left(\frac{W^s}{\Omega}\right) = \frac{1}{\Omega^2}\left(\Omega\,\tilde\nabla^r W^s - W^s\,W^r - \sum_{q=1}^{n} W^s\,p_q\,\tilde\nabla^r W^q\right).$$

Its is easy to see that right hand side of this formula simplifies when we introduce components of projector operator $\mathbf{P}$. Indeed, we have

$$\tilde\nabla^r\!\left(\frac{W^s}{\Omega}\right) = -\frac{W^s\,W^r}{\Omega^2} + \sum_{q=1}^{n}\frac{\tilde\nabla^r W^q}{\Omega}\,P^s_q.\tag{9.21}$$

Combining (9.20) and (9.21), we obtain the following equality:

$$\begin{gathered}
\sum_{r=1}^{n}\tilde\nabla^r\!\left(\frac{W^s}{\Omega}\right)\,\xi_{rj} = \sum_{k=1}^{n}\sum_{r=1}^{n}\sum_{q=1}^{n}\nu\,P^k_r\,\frac{\tilde\nabla^r W^q}{\Omega}\,P^s_q\,\nabla_{\boldsymbol{\tau}_j}n_k\,+\\
+\sum_{k=1}^{n}\frac{W^s}{\Omega^2}\,U_k\,\tau^k_j - \sum_{k=1}^{n}\sum_{r=1}^{n}\sum_{q=1}^{n}\frac{p_r\,\tilde\nabla^r W^q}{\Omega^2}\,P^s_q\,U_k\,\tau^k_j.
\end{gathered}\tag{9.22}$$

In similar way for the first term in right hand side of first equality (9.18) we derive

$$\sum_{r=1}^{n}\nabla_r\!\left(\frac{W^s}{\Omega}\right)\cdot\tau^r_j = \sum_{r=1}^{n}\sum_{q=1}^{n}\frac{\nabla_r W^q}{\Omega}\,P^s_q\,\tau^r_j.\tag{9.23}$$

Now we are able to proceed with transforming the equality (9.16). Note that terms symmetric in indices i and j make no contribution to ultimate compatibility equation (9.13). Therefore further we shall omit them replacing by dots. Taking into account (9.17), (9.18), (9.19), (9.22), and (9.23), for (9.13) we derive

$$\begin{aligned}
&\theta_{ij}-\theta_{ji}=(\nu)^3\sum_{k=1}^{n}\sum_{q=1}^{n}\sum_{r=1}^{n}\sum_{s=1}^{n}\frac{\tilde\nabla^r W^s-\tilde\nabla^s W^r}{\Omega}\,(P^q_r\,\nabla_{\boldsymbol\tau_i}n_q)\,(P^k_s\,\nabla_{\boldsymbol\tau_j}n_k)\,+\\
&\quad+(\nu)^2\sum_{q=1}^{n}\sum_{r=1}^{n}\sum_{s=1}^{n}\left(\frac{\tilde\nabla^r U_s}{\Omega}+\sum_{m=1}^{n}\frac{\tilde\nabla^m W^r-\tilde\nabla^r W^m}{\Omega^2}\,U_s\,p_m-\frac{\nabla_s W^r}{\Omega}\,+\right.\\
&\quad\left.+\sum_{m=1}^{n}\sum_{k=1}^{n}\frac{W^k\,p_m\,D^{mr}_{ks}}{\Omega}\right)(P^q_r\,\nabla_{\boldsymbol\tau_i}n_q)\,\tau^s_j\,-(\nu)^2\sum_{q=1}^{n}\sum_{r=1}^{n}\sum_{s=1}^{n}\left(\frac{\tilde\nabla^r U_s}{\Omega}\,+\right.\\
&\quad\left.+\sum_{m=1}^{n}\frac{\tilde\nabla^m W^r-\tilde\nabla^r W^m}{\Omega^2}\,U_s\,p_m-\frac{\nabla_s W^r}{\Omega}+\sum_{m=1}^{n}\sum_{k=1}^{n}\frac{W^k\,p_m\,D^{mr}_{ks}}{\Omega}\right)\times\\
&\quad\times(P^q_r\,\nabla_{\boldsymbol\tau_j}n_q)\,\tau^s_i+\nu\sum_{r=1}^{n}\sum_{s=1}^{n}\left(\sum_{m=1}^{n}\frac{p_m\,\tilde\nabla^m U_r}{\Omega^2}\,U_s-\sum_{m=1}^{n}\frac{p_m\,\tilde\nabla^m U_s}{\Omega^2}\,U_r\,+\right.\\
&\quad\left.+\frac{\nabla_r U_s}{\Omega}-\frac{\nabla_s U_r}{\Omega}+\sum_{m=1}^{n}\frac{p_m\,\nabla_s W^m\,U_r}{\Omega^2}-\sum_{m=1}^{n}\frac{p_m\,\nabla_r W^m\,U_s}{\Omega^2}\right)\tau^r_i\,\tau^s_j\,+\\
&\quad+\nu\sum_{r=1}^{n}\sum_{s=1}^{n}\sum_{k=1}^{n}\sum_{q=1}^{n}W^k\,p_q\left(\sum_{m=1}^{n}\frac{D^{mq}_{kr}\,U_s-D^{mq}_{ks}\,U_r}{\Omega^2}\,p_m-\frac{R^q_{krs}}{\Omega}\right)\tau^r_i\,\tau^s_j.
\end{aligned}$$

In deriving the above equality we also used the following quite obvious formulas:

$$\begin{aligned}
&\tilde\nabla^r\left(\frac{U_s}{\Omega}\right)=\frac{\tilde\nabla^r U_s}{\Omega}-\frac{W^r\,U_s}{\Omega^2}-\sum_{m=1}^{n}\frac{p_m\,\tilde\nabla^r W^m\,U_s}{\Omega^2},\\
&\nabla_r\left(\frac{U_s}{\Omega}\right)=\frac{\nabla_r U_s}{\Omega}-\sum_{m=1}^{n}\frac{p_m\,\nabla_r W^m\,U_s}{\Omega^2}.
\end{aligned}\tag{9.24}$$

Using (9.24), for two summands in right hand side of second equality (9.18) we get

$$\begin{aligned}
\sum_{r=1}^{n}\tilde\nabla^r\left(\frac{U_s}{\Omega}\right)\cdot\xi_{rj}&=-\nu\sum_{r=1}^{n}\sum_{k=1}^{n}\sum_{m=1}^{n}\frac{p_m\,\tilde\nabla^r W^m\,U_s}{\Omega^2}\,P^k_r\,\nabla_{\boldsymbol\tau_j}n_k\,+\\
&+\nu\sum_{r=1}^{n}\sum_{k=1}^{n}\frac{\tilde\nabla^r U_s}{\Omega}\,P^k_r\,\nabla_{\boldsymbol\tau_j}n_k-\sum_{k=1}^{n}\sum_{m=1}^{n}\frac{p_m\,\tilde\nabla^m U_s}{\Omega^2}\,U_k\,\tau^k_j\,+\\
&+\sum_{k=1}^{n}\frac{1}{\Omega^2}\,U_s\,U_k\,\tau^k_j+\sum_{k=1}^{n}\sum_{q=1}^{n}\sum_{m=1}^{n}\frac{p_q\,p_m\,\tilde\nabla^q W^m}{\Omega^3}\,U_s\,U_k\,\tau^k_j,
\end{aligned}\tag{9.25}$$

$$\sum_{r=1}^{n}\nabla_r\left(\frac{U_s}{\Omega}\right)\cdot\tau^r_j=\sum_{r=1}^{n}\frac{\nabla_r U_s}{\Omega}\,\tau^r_j-\sum_{r=1}^{n}\sum_{m=1}^{n}\frac{p_m\,\nabla_r W^m\,U_s}{\Omega^2}\,\tau^r_j.\tag{9.26}$$

Right hand sides of the equalities (9.25) and (9.26) are reflected in the above formula for $\theta_{ij}-\theta_{ji}$. In deriving this formula for $\theta_{ij}-\theta_{ji}$ we have made the following transformations for second term in right hand side of commutator identity (9.19):

$$\begin{gathered}\sum_{r=1}^{n}\sum_{\alpha=1}^{n}\sum_{\gamma=1}^{n}\frac{p_\alpha}{\nu}\,D^{\alpha\gamma}_{sr}\,\tau^r_j\,\xi_{\gamma i}=\sum_{k=1}^{n}\sum_{m=1}^{n}\sum_{r=1}^{n}\frac{p_m}{\nu}\,D^{mr}_{sk}\,\tau^k_j\,\xi_{ri}=\\=\sum_{k=1}^{n}\sum_{q=1}^{n}\sum_{m=1}^{n}\sum_{r=1}^{n}\left(p_m\,D^{mr}_{sk}\,(P^q_r\,\nabla_{\boldsymbol\tau_i}n_q)\,\tau^k_j-\frac{p_m\,p_q}{\nu\,\Omega}\,D^{mq}_{sk}\,U_r\,\tau^r_i\,\tau^k_j\right).\end{gathered}\tag{9.27}$$

Terms from right hand side of (9.27) are also reflected in the above formula for $\theta_{ij}-\theta_{ji}$. Looking at this formula, we see that compatibility equation (9.13) providing compatibility of Pfaff equations (9.8) has the following structure:

$$\begin{gathered}(\nu)^3\sum_{k=1}^{n}\sum_{q=1}^{n}\sum_{r=1}^{n}\sum_{s=1}^{n}\frac{A^{rs}-A^{sr}}{\Omega}\,(P^q_r\,\nabla_{\boldsymbol\tau_i}n_q)\,(P^k_s\,\nabla_{\boldsymbol\tau_j}n_k)\,+\\+\,(\nu)^2\sum_{q=1}^{n}\sum_{r=1}^{n}\sum_{s=1}^{n}\frac{B^r_s}{\Omega}\,(P^q_r\,\nabla_{\boldsymbol\tau_i}n_q)\,\tau^s_j\,-(\nu)^2\sum_{q=1}^{n}\sum_{r=1}^{n}\sum_{s=1}^{n}\frac{B^r_s}{\Omega}\,\times\\\times\,(P^q_r\,\nabla_{\boldsymbol\tau_j}n_q)\,\tau^s_i+\nu\sum_{r=1}^{n}\sum_{s=1}^{n}\frac{C_{rs}-C_{sr}}{\Omega}\,\tau^r_i\,\tau^s_j=0.\end{gathered}\tag{9.28}$$

Here A^{rs}, B^r_s, and C_{rs} are components of three extended tensor fields $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ respectively. They are given by explicit formulas

$$A^{rs}=\tilde\nabla^r W^s,\tag{9.29}$$

$$\begin{aligned}B^r_s&=\tilde\nabla^r U_s+\sum_{m=1}^{n}\sum_{k=1}^{n}W^k\,p_m\,D^{mr}_{ks}\,-\\&-\,\nabla_s W^r+\sum_{m=1}^{n}\frac{\tilde\nabla^m W^r-\tilde\nabla^r W^m}{\Omega}\,U_s\,p_m,\end{aligned}\tag{9.30}$$

$$\begin{aligned}C_{rs}&=\nabla_r U_s-\sum_{m=1}^{n}\frac{U_r\,\tilde\nabla^m U_s+U_s\,\nabla_r W^m}{\Omega}\,p_m\,-\\&-\sum_{k=1}^{n}\sum_{q=1}^{n}\left(\sum_{m=1}^{n}\frac{D^{mq}_{ks}\,U_r}{\Omega}\,p_m+\frac{R^q_{krs}}{2}\right)W^k\,p_q.\end{aligned}\tag{9.31}$$

Further study of compatibility equations (9.28) with coefficients (9.29), (9.30), (9.31) require some information concerning geometry of hypersurfaces in non-metric geometry of manifolds equipped with symmetric extended connection Γ and generalized Legendre transformation (2.2) given by extended vector field $\mathbf{V}$ in $\mathbf{p}$-representation (see (2.4)).

10. Geometry of hypersurfaces.

Let's fix some arbitrary point $q_0=(p_0,\mathbf{p})$ of cotangent bundle T^*M. This means that we fix some point $p_0\in M$ and some covector $\mathbf{p}\in T^*_{p_0}(M)$. Assume that $\mathbf{p}\neq 0$. Then null-space of covector $\mathbf{p}$ is a hyperplane in tangent space $T_{p_0}(M)$ (see (8.29)). Let σ be some smooth hypersurface passing through the point p_0 and tangent to null-space of fixed momentum covector $\mathbf{p}$ at that point. If $\mathbf{n} = \mathbf{n}(p)$ is smooth normal covector field on σ, then for $p = p_0$ we have

$$\mathbf{p} = \nu_0 \cdot \mathbf{n}(p_0), \text{ where } \nu_0 \neq 0. \tag{10.1}$$

Taking constant $\nu_0 \neq 0$ from (10.1), we can expand it up to a smooth nonzero function $\nu = \nu(p)$ on hypersurface σ (or at least in some neighborhood of marked point p_0 on σ). Function $\nu(p)$ satisfies the equality

$$\nu(p_0) = \nu_0, \tag{10.2}$$

which is just the same as normalizing condition in (9.9). So, we can substitute $\nu(p)$ into (1.14) and use it for defining shift of σ along trajectories of Newtonian dynamical system (7.2). Now we assume that dynamical system (7.2) satisfies additional normality condition (see definition 9.2). This means that for any hypersurface σ passing through our marked point $p_0 \in M$ and for any nonzero constant ν_0 in (10.2) Pfaff equations (9.8) are compatible. Hence compatibility equations (9.28) are fulfilled. Note that in (9.28) we have explicit entries of $\nu = \nu(p)$ and implicit entries of ν through $\mathbf{p} = \nu \cdot \mathbf{n}$ in arguments of extended tensor fields $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$. Moreover, we have implicit entries of ν in covariant derivatives

$$\nabla_{\boldsymbol{\tau}_i} n_s = \frac{\partial n_s}{\partial y^i} - \sum_{k=1}^{n} \sum_{r=1}^{n} \Gamma^k_{sr}\, n_k\, \tau^r_i \tag{10.3}$$

due to connection components Γ^k_{sr} that depend on momentum covector $\mathbf{p} = \nu \cdot \mathbf{n}$ (see (4.14)). But in (9.28) we have no derivatives of function $\nu = \nu(p)$. Therefore, if we write (9.28) only at our fixed point $p = p_0$, we can replace all entries of ν by normalizing constant ν_0 from (10.2). Using (10.1), we can express $\mathbf{n} = \mathbf{n}(p_0)$ through our fixed momentum covector $\mathbf{p}$ at the point $p = p_0$:

$$\mathbf{n} = \mathbf{n}(p_0) = \frac{\mathbf{p}}{\nu_0}. \tag{10.4}$$

In this form $\mathbf{n}(p_0)$ is not too specific property of hypersurface σ. It determines only tangent hyperplane to σ at fixed point $p = p_0$. The only parameters in (9.28) that depend on fine structure of hypersurface σ at the point p_0 are covariant derivatives (10.3). Using them, in [20] we have defined a map $f\colon T_p(\sigma) \to T^*_p(M)$. Indeed, if τ is some arbitrary vector tangent to σ, then $\boldsymbol{\tau} = \alpha^1 \cdot \boldsymbol{\tau}_1 + \ldots + \alpha^{n-1} \cdot \boldsymbol{\tau}_{n-1}$. Let

$$f(\boldsymbol{\tau}) = \nabla_{\boldsymbol{\tau}} \mathbf{n} = \sum_{r=1}^{n} \sum_{j=1}^{n-1} \bigl(\alpha^j\, \nabla_{\boldsymbol{\tau}_j} n_r\bigr) \cdot dx^r. \tag{10.5}$$

It is easy to see that (10.5) defines linear map from tangent hyperplane $T_p(\sigma)$ at the point $p \in \sigma$ to cotangent space $T^*_p(M)$. We consider composite map

$$\mathbf{b} = -\mathbf{P}^* \circ f \circ \mathbf{P}. \tag{10.6}$$

Projection operator $\mathbf{P}^*$ in (10.6) is a conjugate operator for projector $\mathbf{P}$ with components (6.3). Remember that $\mathbf{P}$ projects onto the subspace $T_p(\sigma) \in T_p(M)$. Therefore linear map $\mathbf{b}\colon T_p(M) \to T^*_p(M)$ is correctly defined by formula (10.6).

Linear map $\mathbf{b}$ defined by formula (10.6) is associated with second fundamental form of hypersurface σ. Indeed, let's define bilinear form

$$b(\mathbf{X},\mathbf{Y})=\langle\mathbf{b}(\mathbf{Y})\,|\,\mathbf{X}\rangle\,. \tag{10.7}$$

Due to the presence of projection operators $\mathbf{P}$ and $\mathbf{P}^*$ in (10.6) we have

$$b(\mathbf{X},\mathbf{Y})=b(\mathbf{P}(\mathbf{X}),\mathbf{Y})=b(\mathbf{X},\mathbf{P}(\mathbf{Y})). \tag{10.8}$$

Theorem 10.1. *Bilinear form* (10.7) *defined by linear map* (10.6) *is symmetric.*

When restricted to tangent space $T_p(\sigma)$ of hypersurface σ bilinear form (10.7) yields second fundamental form of σ. Its components

$$\beta_{ij}=b(\boldsymbol{\tau}_i,\boldsymbol{\tau}_j) \tag{10.9}$$

define tensor field in inner geometry of hypersurface σ. From (10.8) we derive the equality $b(\mathbf{X},\mathbf{Y})=b(\mathbf{P}(\mathbf{X}),\mathbf{P}(\mathbf{Y}))$. It means that bilinear form (10.7) and linear map (10.6) in outer space are completely determined by components of second fundamental form (10.9). Further we need the following theorem.

Theorem 10.2. *Let $q_0=(p_0,\mathbf{p})$ be some fixed point of cotangent bundle T^*M with $\mathbf{p}\neq 0$ and let projector $\mathbf{P}$ be the value of projector-valued extended tensor field* (6.3) *at this point. Then any symmetric quadratic form b in $T_{p_0}(M)$ satisfying the equality* (10.8) *can be determined by some hypersurface σ passing through the point p_0 and tangent to null-space of covector $\mathbf{p}$ at this point.*

Theorems 10.1 and 10.2 are proved in paper [20]. Though these theorems are very important for further study of compatibility equations (9.28), we shall not repeat their proofs in present paper.

11. Additional normality equations.

As in previous section, let's fix some point $p_0\in M$ and some covector $\mathbf{p}\neq 0$ at this point. This means that we fix some point $q_0=(p_0,\mathbf{p})$ of cotangent bundle T^*M. Let's fix some arbitrary nonzero constant $\nu_0\neq 0$ and then use formula (10.4) for to define another nonzero covector $\mathbf{n}\neq 0$ at our fixed point p_0. Further, let's consider various hypersurfaces passing through the point p_0 tangent to null-space of covector $\mathbf{n}$. For each such hypersurface σ covector $\mathbf{n}$ is normal covector at the point p_0. It can be expanded up to a smooth normal covector field $\mathbf{n}=\mathbf{n}(p)$ (at least in some neighborhood of marked point p_0). Therefore we can build σ into a framework of shift construction defined by of Newtonian dynamical (7.2). If this dynamical system satisfies additional normality condition (see definition 9.2), then we can choose smooth function $\nu=\nu(p)$ normalized by the condition (10.2) and such that compatibility equations (9.28) are fulfilled.

Let $\boldsymbol{\tau}_1,\,\ldots\,,\boldsymbol{\tau}_{n-1}$ be basic tangent vectors of σ. Applying linear map (10.6) to them, we get a set of $n-1$ covectors $\boldsymbol{\theta}_1,\,\ldots\,,\boldsymbol{\theta}_{n-1}$. In other words, we denote

$$\boldsymbol{\theta}_i=\mathbf{b}(\boldsymbol{\tau}_i). \tag{11.1}$$

Let's calculate components of covectors (11.1) in local chart. Using formulas (10.5) and (10.6) defining linear map $\mathbf{b}\colon T_p(M)\to T^*_p(M)$, we derive

$$\theta_{ri}=-\sum_{q=1}^{n}P^q_r\,\nabla_{\boldsymbol{\tau}_i}n_q=\sum_{s=1}^{n}b_{rs}\,\tau^s_i. \tag{11.2}$$

Here b_{rs} are components of bilinear form (10.7) in local chart. Comparing (9.28) and (11.2), we see that compatibility equations (9.28) can be written as follows:

$$\begin{gathered}(\nu_0)^3\sum_{k=1}^{n}\sum_{q=1}^{n}\sum_{r=1}^{n}\sum_{s=1}^{n}\frac{A^{rs}-A^{sr}}{\Omega}\,(b_{rq}\,\tau^q_i)\,(b_{sk}\,\tau^k_j)\,-\\ -(\nu_0)^2\sum_{q=1}^{n}\sum_{r=1}^{n}\sum_{s=1}^{n}\frac{B^r_s}{\Omega}\,(b_{rq}\,\tau^q_i)\,\tau^s_j\,+(\nu_0)^2\sum_{q=1}^{n}\sum_{r=1}^{n}\sum_{s=1}^{n}\frac{B^r_s}{\Omega}\,\times\\ \times(b_{rq}\,\tau^q_j)\,\tau^s_i+\nu_0\sum_{r=1}^{n}\sum_{s=1}^{n}\frac{C_{rs}-C_{sr}}{\Omega}\,\tau^r_i\,\tau^s_j=0.\end{gathered} \tag{11.3}$$

Varying hypersurface σ, we can vary components of bilinear form b in (11.3). In particular, we can 1) change the sign of b; 2) get zero quadratic form for b. Due to these facts (they follow from theorem 10.2) compatibility equations (11.3) split into three separate parts. Now they are written as follows:

$$\sum_{r=1}^{n}\sum_{s=1}^{n}(A^{rs}-A^{sr})\,\theta_{ri}\,\theta_{sj}=0, \tag{11.4}$$

$$\sum_{r=1}^{n}\sum_{s=1}^{n}\sum_{q=1}^{n}(B^q_s\,b_{qr}-B^q_r\,b_{qs})\,\tau^r_i\,\tau^s_j=0, \tag{11.5}$$

$$\sum_{r=1}^{n}\sum_{s=1}^{n}(C_{rs}-C_{sr})\,\tau^r_i\,\tau^s_j=0. \tag{11.6}$$

Here in (11.4) we used (11.2) again. Vectors $\boldsymbol{\tau}_1,\,\ldots\,,\,\boldsymbol{\tau}_{n-1}$ depend on the choice of local chart on hypersurface σ. For a fixed point $p=p_0$ they can be treated as arbitrary $n-1$ vectors forming base in tangent hyperplane $T_p(\sigma)$. Similarly, due to theorem 10.2 covectors $\boldsymbol{\theta}_1,\,\ldots\,,\,\boldsymbol{\theta}_{n-1}$ can be treated as arbitrary $n-1$ covectors in null-space of vector $\mathbf{W}$ (see (8.32)). Therefore (11.4) and (11.6) reduce to

$$\sum_{r=1}^{n}\sum_{s=1}^{n}(A^{rs}-A^{sr})\,P^i_r\,P^j_s=0, \tag{11.7}$$

$$\sum_{r=1}^{n}\sum_{s=1}^{n}(C_{rs}-C_{sr})\,P^r_i\,P^s_j=0. \tag{11.8}$$

In this form equations (11.7) and (11.8) do not depend on any particular hypersurface we used to derive them.

As for (11.5), we should bring it to similar form independent on σ. For this purpose remember that we can treat $\boldsymbol{\tau}_1,\,\ldots\,,\,\boldsymbol{\tau}_{n-1}$ as arbitrary $n-1$ vectors in tangent hyperplane $T_p(\sigma)$. Therefore equation (11.5) reduces to the following form:

$$\sum_{r=1}^{n}\sum_{s=1}^{n}\sum_{q=1}^{n}(B^q_s\,b_{qr}-B^q_r\,b_{qs})\,P^r_i\,P^s_j=0. \tag{11.9}$$

In the next step we rewrite (11.9) in coordinate-free form. It looks like

$$b(\mathbf{P}\circ\mathbf{B}\circ\mathbf{P}(\mathbf{X}),\mathbf{Y})=b(\mathbf{X},\mathbf{P}\circ\mathbf{B}\circ\mathbf{P}(\mathbf{Y})) \tag{11.10}$$

Here $\mathbf{B}$ is linear operator in $T_{p_0}(M)$ determined by components of extended tensor field (9.30). Due to theorem 10.2 the above equality (11.10) means that composite operator $\mathbf{P}\circ\mathbf{B}\circ\mathbf{P}$ is symmetric with respect to any symmetric bilinear form b in $T_{p_0}(M)$ for which (10.8) is fulfilled. Hence composite operator $\mathbf{P}\circ\mathbf{B}\circ\mathbf{P}$ can differ from projector $\mathbf{P}$ only by some scalar factor λ:

$$\mathbf{P}\circ\mathbf{B}\circ\mathbf{P}=\lambda\cdot\mathbf{P}. \tag{11.11}$$

This fact is proved in paper [20]. Scalar factor λ is given by trace formula

$$\lambda=\frac{\operatorname{tr}(\mathbf{P}\circ\mathbf{B}\circ\mathbf{P})}{n-1}=\frac{\operatorname{tr}(\mathbf{B}\circ\mathbf{P})}{n-1}=\sum_{r=1}^{n}\sum_{s=1}^{n}\frac{B^r_s\,P^s_r}{n-1}. \tag{11.12}$$

Formulas (11.11) and (11.12) written in local chart yield required equations

$$\sum_{r=1}^{n}\sum_{s=1}^{n}P^i_r\,B^r_s\,P^s_j=\sum_{r=1}^{n}\sum_{s=1}^{n}\frac{B^r_s\,P^s_r}{n-1}\,P^i_j. \tag{11.13}$$

In form (11.13) equations (11.5) do not depend on any particular hypersurface used in deriving these equations.

Equations (11.7), (11.8), and (11.13) taken together form a system of *additional normality equations*. Due to the above notations (6.1), (6.2), (6.3), (9.14), (9.29), (9.30), and (9.31) they are partial differential equations for components of extended vector field $\mathbf{V}$ and extended covector field $\mathbf{Q}$ that determine Newtonian dynamical system (7.2). We have derived them assuming that Newtonian dynamical system (7.2) satisfies additional normality condition formulated in definition 9.2. Conversely, if additional normality equations are fulfilled, then compatibility equations (9.28) turn to identities. Therefore Pfaff equations (9.8) appear to be compatible. Thus, we have proved the following theorem analogous to theorem 8.1.

Theorem 11.1. *Additional normality condition for Newtonian dynamical system* (7.2) *is equivalent to the system of additional normality equations* (11.7), (11.8), *and* (11.13) *that should be fulfilled at all points* $q=(p,\mathbf{p})$ *of cotangent bundle* T^*M, *where* $\mathbf{p}\neq 0$.

Now suppose that both weak and additional normality conditions are fulfilled. In this case all deviation functions (9.3) satisfy second order ordinary differential equation (8.8) and we can provide initial data (9.5) for them by proper choice of function $\nu=\nu(p)$ on any predefined hypersurface σ in M. Then all deviation functions do vanish, and we have normal shift of hypersurface σ. So, we see that weak and additional normality conditions are complementary to each other, and if both are fulfilled, we can arrange normal shift of any predefined hypersurface along trajectories of Newtonian dynamical system (7.2).

Definition 11.1. We say that Newtonian dynamical system satisfies **complete** normality condition if both **weak** and **additional** normality conditions for this system are fulfilled.

According to theorems 8.1 and 11.1, weak and additional normality conditions are equivalent to weak and additional normality equations for parameters $\mathbf{V}$ and $\mathbf{Q}$ determining Newtonian dynamical system (7.2). Therefore complete normality condition is equivalent to complete system of normality equations including (8.21), (11.7), (11.8), and (11.13). As we noted just above, complete normality condition is sufficient for strong normality condition to be fulfilled (see definition 1.5). We shall strengthen this result in the next section.

12. Equivalence of strong and complete normality conditions.

Part of the statement declared in the title of this section is already proved. Indeed, we know that complete normality condition implies strong normality condition. Let's prove converse implication. Assuming that Newtonian dynamical system (7.2) satisfies strong normality condition, we should prove that it satisfies weak and additional normality conditions.

In the first step let's prove that additional normality condition is fulfilled. For this purpose let's take some arbitrary hypersurface σ with marked point $p = p_0$ and smooth normal covector field $\mathbf{n} = \mathbf{n}(p)$ in some neighborhood of marked point. Then let's take some nonzero constant $\nu_0 \neq 0$ and let's apply strong normality condition, which is fulfilled by assumption (see definition 1.5). As a result we get smooth function $\nu = \nu(p)$ normalized by condition (10.2) and such that it provide initial data (1.14) for normal shift of σ. Due to normality of shift all deviation functions (9.3) are identically zero. Hence initial conditions (9.5) for them are fulfilled. Writing (9.5) explicitly, we find that our function $\nu = \nu(p)$ is a solution for Pfaff equations (9.8). Now, varying constant $\nu_0 \neq 0$ in (10.2), we prove that Pfaff equations (9.8) are compatible (see definition 9.1). Thus, additional normality condition is proved (see definition 9.2).

In the second step we shall derive weak normality condition assuming that strong normality condition is fulfilled. This is a little bit more complicated. For this purpose we fix some point $q_0 = (p_0, \mathbf{p})$ of cotangent bundle T^*M with $\mathbf{p} \neq 0$. Initial point p_0 and momentum covector $\mathbf{p}$ at this point form initial data for Newtonian dynamical system (7.2). They define a trajectory $p = p(t)$ passing through initial point p_0. Null-space of initial covector $\mathbf{p}$ is a hyperplane in tangent space $T_{p_0}(M)$. Let's consider various hypersurfaces passing through initial point p_0 tangent to this hyperplane and denote by σ one of them. If $\mathbf{n} = \mathbf{n}(p)$ is normal covector of this hypersurface σ, then at the point $p = p_0$ we have the equality (10.1) that determine normalizing constant $\nu_0 \neq 0$ for (10.2). Applying strong normality condition (see definition 1.5), we find smooth function $\nu = \nu(p)$ on σ in some neighborhood of initial point p_0 that provide initial data (1.14) for normal shift of σ. Thus, our fixed trajectory $p = p(t)$ passing through initial point p_0 appears to be shift trajectory among many others. Due to normality of shift all corresponding deviation functions

(9.3) on this trajectory are identically zero. Therefore we can write

$$\ddot{\varphi}_i\Big|_{t=0} = 0. \tag{12.1}$$

Note that initial conditions (9.5) are also fulfilled. Therefore we can use formula (8.20) for $\ddot{\varphi}$ in left hand side of (12.1). Here it is written as follows:

$$\ddot{\varphi}_i\Big|_{t=0} = \sum_{k=1}^{n}\sum_{r=1}^{n}\alpha^r\,P^k_r\cdot\xi_{ki}\Big|_{t=0} + \sum_{k=1}^{n}\sum_{r=1}^{n}\eta_r\,P^r_k\cdot\tau^k_i\Big|_{t=0}. \tag{12.2}$$

Vectors $\boldsymbol{\tau}_1,\,\ldots\,,\,\boldsymbol{\tau}_{n-1}$ at initial instant of time $t = 0$ form base in tangent hyperplane to initial hypersurface σ. As we noted in section 11, for fixed point $p = p_0$ they can be treated as arbitrary $n-1$ vectors in null-space of momentum covector $\mathbf{p} = \nu_0\cdot\mathbf{n}(p_0)$. For components of covectors $\boldsymbol{\xi}_1,\,\ldots\,,\,\boldsymbol{\xi}_{n-1}$ at initial instant of time $t = 0$ we can use formula (9.20). Applying (9.20) and (11.2), we get

$$\sum_{k=1}^{n} P^k_r\cdot\xi_{ki}\Big|_{t=0} = \sum_{s=1}^{n}\nu_0\,P^s_r\,\nabla_{\boldsymbol{\tau}_i}n_s = -\sum_{s=1}^{n}\nu_0\,b_{rs}\,\tau^s_i. \tag{12.3}$$

Combining (12.1), (12.2), and (12.3), we derive the equality

$$-\sum_{k=1}^{n}\sum_{r=1}^{n}\sum_{s=1}^{n}\nu_0\,\alpha^r\,P^k_r b_{ks}\,\tau^s_i + \sum_{k=1}^{n}\sum_{r=1}^{n}\eta_r\,P^r_k\cdot\tau^k_i = 0. \tag{12.4}$$

Just like in (11.3), by varying hypersurface σ and by applying theorem 10.2 we can break (12.4) into two separate equalities

$$\sum_{k=1}^{n}\sum_{r=1}^{n}\alpha^r\,P^k_r\cdot\theta_{ki} = 0, \qquad\qquad \sum_{k=1}^{n}\sum_{r=1}^{n}\eta_r\,P^r_k\cdot\tau^k_i = 0,$$

which are equivalent to weak normality equations (8.21). Applying theorem 8.1, we find that weak normality condition is fulfilled, i. e. strong normality condition implies weak normality condition. Ultimately, we have proved the following theorem.

Theorem 12.1. *Strong and complete normality conditions for Newtonian dynamical system* (7.2) *are equivalent to each other.*

13. Connection invariance.

Theorem 12.1 is a basic result in the theory of dynamical Newtonian dynamical systems admitting normal shift, while definition 1.5 is basic definition of this theory. Comparing them we see that strong normality condition formulated in definition 1.5 is applicable either to general Newtonian dynamical system of the form (2.3), and to special one given by the equations (7.2). Theorem 12.1 is formulated only for Newtonian dynamical system (7.2), which implies presence of some symmetric extended connection Γ in M. Theorem 5.1 gives one way to avoid this discrepancy. We can use symmetric affine connection (5.8) canonically associated with dynamical system (2.3) and by means of this connection we can rewrite (2.3) in form of (7.2) (see formula (7.1)). However, there is another way. Below we shall prove that the whole theory constructed in sections 6–13 is invariant under gauge transformations changing one connection for another:

$$\Gamma^k_{ij} \to \Gamma^k_{ij} + T^k_{ij}. \tag{13.1}$$

Here T^k_{ij} are components of some symmetric extended tensor field $\mathbf{T}$ of type $(1,2)$. Applying gauge transformation (13.1) to dynamical system (7.2), we change covariant derivatives $\nabla_t p_i$ in left hand side. In order to keep corresponding connection-free equations (2.3) unchanged we should change components of covector $\mathbf{Q}$ as follows:

$$Q_i \to Q_i - \sum_{k=1}^{n}\sum_{s=1}^{n} T^k_{is}\,p_k\,V^s. \tag{13.2}$$

Having fixed gauge transformations by formulas (13.1) and (13.2), now we shall apply them to all normality equations (8.21), (11.7), (11.8), and (11.13) for to prove their invariance under these transformations. First of all note that vector field $\mathbf{V}$ in (7.2), vector field $\mathbf{W}$ introduced by formula (6.1), scalar field Ω given by formula (6.2), and projector field $\mathbf{P}$ with components (6.3) are invariant under gauge transformations defined by (13.1), (13.2):

$$V^s \to V^s, \qquad\qquad W^s \to W^s, \tag{13.3}$$

$$\Omega \to \Omega, \qquad\qquad P^i_j \to P^i_j. \tag{13.4}$$

As for covector field $\mathbf{U}$ in (9.14), here we have the following transformation rule:

$$U_s \to U_s + \sum_{q=1}^{n}\sum_{m=1}^{n} p_m\,T^m_{sq}\,W^q. \tag{13.5}$$

Applying (13.1) to curvature tensors (8.6) and (8.7), we derive

$$\begin{aligned}
R^k_{rij} \to R^k_{rij} &+ \nabla_i T^k_{jr} - \nabla_j T^k_{ir} - \sum_{s=1}^{n}\sum_{m=1}^{n} p_s\,D^{km}_{jr}\,T^s_{mi}\,+\\
&+ \sum_{s=1}^{n}\sum_{m=1}^{n} p_s\,D^{km}_{ir}\,T^s_{mj} + \sum_{m=1}^{n}\left(T^k_{im}\,T^m_{jr} - T^k_{jm}\,T^m_{ir}\right) +\\
&+ \sum_{s=1}^{n}\sum_{m=1}^{n} p_s\,T^s_{mi}\,\tilde\nabla^m T^k_{jr} - \sum_{s=1}^{n}\sum_{m=1}^{n} p_s\,T^s_{mj}\,\tilde\nabla^m T^k_{ir},
\end{aligned} \tag{13.6}$$

$$D^{kr}_{ij} \to D^{kr}_{ij} - \tilde\nabla^r T^k_{ij}. \tag{13.7}$$

Now we can apply (13.1), (13.2), (13.3), (13.4), and (13.7) to vector field $\boldsymbol{\alpha}$ with components (8.12) used in weak normality equations (8.21):

$$\alpha^k \to \alpha^k - \sum_{i=1}^{n}\sum_{r=1}^{n}\sum_{s=1}^{n} \tilde\nabla^k V^i\,T^r_{is}\,p_r\,V^s + \Biggl(\sum_{r=1}^{n}\sum_{s=1}^{n}\sum_{i=1}^{n} T^k_{rs}\,\tilde\nabla^s V^i\,p_i\,V^r\,+$$

$$
\begin{gathered}
+\sum_{r=1}^{n}\sum_{s=1}^{n}\sum_{i=1}^{n} T^{i}_{rs}\,\tilde\nabla^{k}V^{s}\,p_{i}\,V^{r}+\sum_{r=1}^{n}\sum_{s=1}^{n}\sum_{i=1}^{n}\sum_{m=1}^{n} p_{m}\,T^{m}_{rs}\,\tilde\nabla^{s}\tilde\nabla^{k}V^{i}\,p_{i}\,V^{r}\Bigg)-\\
-\sum_{r=1}^{n}\sum_{s=1}^{n}\sum_{i=1}^{n}\sum_{m=1}^{n}\tilde\nabla^{r}\tilde\nabla^{k}V^{i}\,p_{i}\,T^{m}_{rs}\,p_{m}\,V^{s}-\sum_{r=1}^{n}\sum_{i=1}^{n}\Bigg(\sum_{s=1}^{n}\tilde\nabla^{r}V^{i}\,p_{i}\,T^{k}_{rs}\,V^{s}-\\
-\sum_{m=1}^{n}\sum_{s=1}^{n}\tilde\nabla^{r}V^{i}\,p_{i}\,\tilde\nabla^{k}T^{m}_{rs}\,p_{m}\,V^{s}-\sum_{m=1}^{n}\sum_{s=1}^{n}\tilde\nabla^{r}V^{i}\,p_{i}\,T^{m}_{rs}\,p_{m}\,\tilde\nabla^{k}V^{s}\Bigg)+\\
+\sum_{r=1}^{n}\sum_{s=1}^{n}\sum_{i=1}^{n}\sum_{m=1}^{n}\tilde\nabla^{k}T^{s}_{rm}\,\tilde\nabla^{r}V^{i}\,p_{i}\,p_{s}\,V^{m}+\sum_{r=1}^{n}\sum_{i=1}^{n}\Bigg(\sum_{s=1}^{n}T^{i}_{rs}\,V^{s}\,p_{i}\tilde\nabla^{k}V^{r}+\\
+\sum_{s=1}^{n}\sum_{m=1}^{n}p_{m}\,T^{m}_{rs}\,\tilde\nabla^{s}V^{i}\,p_{i}\tilde\nabla^{k}V^{r}\Bigg)-\sum_{r=1}^{n}\sum_{s=1}^{n}\sum_{m=1}^{n}\tilde\nabla^{k}V^{r}\,T^{m}_{rs}\,p_{m}\,V^{s}.
\end{gathered}
$$

Looking attentively at the above formula, we see that almost all terms in right hand side do cancel each other. As a result we get the following transformation rule:

$$\alpha^{k}\to\alpha^{k}. \tag{13.8}$$

Formula (8.13) for components of extended covector field $\boldsymbol{\beta}$ is more complicated than formula (8.12). Let's simplify it using notations (9.14):

$$
\begin{gathered}
\beta_{k}=\sum_{r=1}^{n}\nabla_{r}U_{k}\,V^{r}+\sum_{r=1}^{n}\tilde\nabla^{r}U_{k}\,Q_{r}+\sum_{r=1}^{n}\nabla_{k}V^{r}\,U_{r}+\\
+\sum_{r=1}^{n}\nabla_{k}Q_{r}\,W^{r}-\sum_{r=1}^{n}\sum_{s=1}^{n}\sum_{m=1}^{n}(R^{s}_{rmk}\,V^{m}-D^{sm}_{rk}\,Q_{m})\,W^{r}\,p_{s}.
\end{gathered}\tag{13.9}
$$

Formula (13.9) for β^{k} is still rather complicated. Therefore we perform some preliminary calculations. Using formula (13.5), we derive

$$
\begin{gathered}
\nabla_{r}U_{s}\to\nabla_{r}U_{s}+\sum_{q=1}^{n}\sum_{m=1}^{n}p_{m}\,\nabla_{r}T^{m}_{sq}\;W^{q}+\sum_{q=1}^{n}\sum_{m=1}^{n}p_{m}\,T^{m}_{sq}\,\nabla_{r}W^{q}-\\
-\sum_{k=1}^{n}T^{k}_{rs}\,U_{k}-\sum_{k=1}^{n}\sum_{q=1}^{n}\sum_{m=1}^{n}p_{m}\,T^{k}_{rs}\;T^{m}_{kq}\;W^{q}\;+\sum_{k=1}^{n}\sum_{u=1}^{n}p_{u}\,T^{u}_{rk}\times\\
\times\tilde\nabla^{k}U_{s}+\sum_{k=1}^{n}\sum_{u=1}^{n}\sum_{q=1}^{n}p_{u}\,T^{u}_{rk}\,T^{k}_{sq}\,W^{q}+\sum_{k=1}^{n}\sum_{u=1}^{n}\sum_{q=1}^{n}\sum_{m=1}^{n}p_{u}\,p_{m}\times\\
\times T^{u}_{rk}\,\tilde\nabla^{k}T^{m}_{sq}\,W^{q}+\sum_{k=1}^{n}\sum_{u=1}^{n}\sum_{q=1}^{n}\sum_{m=1}^{n}p_{u}\,p_{m}\,T^{u}_{rk}\,T^{m}_{sq}\,\tilde\nabla^{k}W^{q},
\end{gathered}\tag{13.10}
$$

$$
\begin{gathered}
\tilde\nabla^{r}U_{s}\to\tilde\nabla^{r}U_{s}+\sum_{q=1}^{n}T^{r}_{sq}\,W^{q}+\sum_{q=1}^{n}\sum_{m=1}^{n}p_{m}\,\tilde\nabla^{r}T^{m}_{sq}\,W^{q}+\\
+\sum_{q=1}^{n}\sum_{m=1}^{n}p_{m}\,T^{m}_{sq}\,\tilde\nabla^{r}W^{q}.
\end{gathered}\tag{13.11}
$$

For covariant derivative $\nabla_k V^i$ we have the following transformation rule:

$$\nabla_k V^i \to \nabla_k V^i + \sum_{s=1}^{n} T^i_{ks}\, V^s + \sum_{s=1}^{n}\sum_{m=1}^{n} p_m\, T^m_{ks}\, \tilde{\nabla}^s V^i. \tag{13.12}$$

Combining formulas (13.11) and (13.2), we derive

$$\begin{gathered}
\sum_{r=1}^{n} \tilde{\nabla}^r U_k\, Q_r \to \sum_{r=1}^{n} \tilde{\nabla}^r U_k\, Q_r + \sum_{r=1}^{n}\sum_{q=1}^{n} T^r_{kq}\, W^q\, Q_r + \sum_{r=1}^{n}\sum_{q=1}^{n}\sum_{m=1}^{n} p_m\, \times\\
\times \tilde{\nabla}^r T^m_{kq}\, W^q\, Q_r + \sum_{r=1}^{n}\sum_{q=1}^{n}\sum_{m=1}^{n} p_m\, T^m_{kq}\, \tilde{\nabla}^r W^q\, Q_r - \sum_{r=1}^{n}\sum_{s=1}^{n}\sum_{u=1}^{n} \tilde{\nabla}^r U_k\, \times\\
\times T^u_{rs}\, p_u\, V^s - \sum_{r=1}^{n}\sum_{s=1}^{n}\sum_{q=1}^{n}\sum_{u=1}^{n} T^r_{kq}\, W^q\, T^u_{rs}\, p_u\, V^s - \sum_{r=1}^{n}\sum_{s=1}^{n}\sum_{q=1}^{n}\sum_{u=1}^{n}\sum_{m=1}^{n} p_u\, \times\\
\times p_m\, \tilde{\nabla}^r T^m_{kq}\, W^q\, T^u_{rs}\, V^s - \sum_{r=1}^{n}\sum_{s=1}^{n}\sum_{q=1}^{n}\sum_{u=1}^{n}\sum_{m=1}^{n} p_u\, p_m\, T^m_{kq}\, T^u_{rs}\, \tilde{\nabla}^r W^q\, V^s.
\end{gathered}$$

In a similar way, combining formulas (13.12) and (13.5), we derive

$$\begin{gathered}
\sum_{r=1}^{n} \nabla_k V^r\, U_r \to \sum_{r=1}^{n} \nabla_k V^r\, U_r + \sum_{r=1}^{n}\sum_{s=1}^{n} T^r_{ks}\, V^s\, U_r + \sum_{r=1}^{n}\sum_{s=1}^{n}\sum_{m=1}^{n} p_m\, \times\\
\times T^m_{ks}\, \tilde{\nabla}^s V^r\, U_r + \sum_{r=1}^{n}\sum_{q=1}^{n}\sum_{m=1}^{n} p_m\, T^m_{rq}\, \nabla_k V^r\, W^q + \sum_{r=1}^{n}\sum_{s=1}^{n}\sum_{q=1}^{n}\sum_{m=1}^{n} p_m\, \times\\
\times T^r_{ks}\, T^m_{rq}\, W^q\, V^s + \sum_{r=1}^{n}\sum_{s=1}^{n}\sum_{q=1}^{n}\sum_{u=1}^{n}\sum_{m=1}^{n} p_u\, p_m\, T^u_{rq}\, T^m_{ks}\, \tilde{\nabla}^s V^r\, W^q.
\end{gathered}$$

Transformation rule for $\nabla_k Q_r$ is similar to (13.10):

$$\begin{gathered}
\nabla_k Q_r \to \nabla_k Q_r - \sum_{s=1}^{n}\sum_{u=1}^{n} p_u\, \nabla_k T^u_{rs}\, V^s - \sum_{s=1}^{n}\sum_{u=1}^{n} p_u\, T^u_{rs}\, \nabla_k V^s -\\
- \sum_{q=1}^{n} T^q_{kr}\, Q_q + \sum_{s=1}^{n}\sum_{u=1}^{n}\sum_{q=1}^{n} p_u\, T^q_{kr}\, T^u_{rs}\, V^s + \sum_{m=1}^{n}\sum_{q=1}^{n} p_m\, T^m_{kq}\, \times\\
\times \tilde{\nabla}^q Q_r - \sum_{s=1}^{n}\sum_{q=1}^{n}\sum_{m=1}^{n} p_m\, T^m_{kq}\, T^q_{rs}\, V^s - \sum_{s=1}^{n}\sum_{q=1}^{n}\sum_{m=1}^{n}\sum_{u=1}^{n} p_u\, p_m\, \times\\
\times T^m_{kq}\, \tilde{\nabla}^q T^u_{rs}\, V^s - \sum_{s=1}^{n}\sum_{q=1}^{n}\sum_{m=1}^{n}\sum_{u=1}^{n} p_u\, p_m\, T^m_{kq}\, T^u_{rs}\, \tilde{\nabla}^q V^s.
\end{gathered} \tag{13.13}$$

For term with curvature tensor D^{sm}_{rk} in (13.9) we use transformation rule (13.7):

$$\sum_{r=1}^{n}\sum_{s=1}^{n}\sum_{m=1}^{n} p_m\, D^{ms}_{rk}\, Q_s\, W^r \to \sum_{r=1}^{n}\sum_{s=1}^{n}\sum_{m=1}^{n} p_m\, D^{ms}_{rk}\, Q_s\, W^r -$$

$$-\sum_{r=1}^{n}\sum_{s=1}^{n}\sum_{m=1}^{n} p_m\,\tilde{\nabla}^s T^m_{rk}\,Q_s\,W^r - \sum_{r=1}^{n}\sum_{s=1}^{n}\sum_{q=1}^{n}\sum_{u=1}^{n}\sum_{m=1}^{n} p_m\,D^{ms}_{rk}\,T^u_{sq}\,p_u\,V^q\,W^r +$$
$$+\sum_{r=1}^{n}\sum_{s=1}^{n}\sum_{q=1}^{n}\sum_{u=1}^{n}\sum_{m=1}^{n} p_m\,\tilde{\nabla}^s T^m_{rk}\,T^u_{sq}\,p_u\,V^q\,W^r.$$

And finally, for term with another curvature tensor R^s_{rmk} in formula (13.9) we use transformation rule (13.6), which is more complicated:

$$\sum_{r=1}^{n}\sum_{s=1}^{n}\sum_{m=1}^{n} p_m\,R^m_{rsk}\,V^s\,W^r \to \sum_{r=1}^{n}\sum_{s=1}^{n}\sum_{m=1}^{n} p_m\,R^m_{rsk}\,V^s\,W^r +$$
$$+\sum_{r=1}^{n}\sum_{s=1}^{n}\sum_{m=1}^{n} p_m\,\nabla_s T^m_{kr}\,V^s\,W^r - \sum_{r=1}^{n}\sum_{s=1}^{n}\sum_{m=1}^{n} p_m\,\nabla_k T^m_{sr}\,V^s\,W^r -$$
$$-\sum_{r=1}^{n}\sum_{s=1}^{n}\sum_{m=1}^{n}\sum_{u=1}^{n}\sum_{q=1}^{n} p_m\,p_u\left(D^{mq}_{kr}\,T^u_{qs}\,V^s\,W^r - D^{mq}_{sr}\,T^u_{qk}\,V^s\,W^r\right) +$$
$$+\sum_{r=1}^{n}\sum_{s=1}^{n}\sum_{m=1}^{n}\sum_{q=1}^{n}\left(p_m\,T^m_{sq}\,T^q_{kr}\,W^r\,V^s - p_m\,T^m_{kq}\,T^q_{sr}\,W^r\,V^s\right) +$$
$$+\sum_{r=1}^{n}\sum_{s=1}^{n}\sum_{m=1}^{n}\sum_{q=1}^{n}\sum_{u=1}^{n} p_u\,p_m\left(T^u_{qs}\,\tilde{\nabla}^q T^m_{kr} - T^u_{qk}\,\tilde{\nabla}^q T^m_{sr}\right) W^r\,V^s.$$

Combining (13.10), (13.11), (13.12), (13.13) and other above formulas, for components of covector field $\boldsymbol{\beta}$ we derive the following very simple transformation rule:

$$\beta_k \to \beta_k + \sum_{m=1}^{n}\sum_{q=1}^{n} T^m_{kq}\,p_m\,\alpha^q. \tag{13.14}$$

In deriving (13.14) we used simplified version of formula (8.12) for vector field $\boldsymbol{\alpha}$:

$$\alpha^k = \sum_{r=1}^{n}\tilde{\nabla}^k V^r\,U_r + \sum_{r=1}^{n}\nabla_r W^k\,V^r + \sum_{r=1}^{n}\tilde{\nabla}^r W^k\,Q_r +$$
$$+\sum_{r=1}^{n} W^r\,\tilde{\nabla}^k Q_r - \sum_{r=1}^{n}\sum_{s=1}^{n}\sum_{q=1}^{n} p_s\,D^{sk}_{rq}\,W^r\,V^q.$$

In a similar way, using notations (9.14), we can simplify formula (8.18):

$$\eta_k = \beta_k - \sum_{s=1}^{n}\frac{U_k\,\alpha^s\,p_s}{\Omega}. \tag{13.15}$$

Substituting (13.14) into (13.15) and using (13.4), (13.5), and (13.8), we obtain

$$\eta_k \to \eta_k + \sum_{s=1}^{n}\sum_{m=1}^{n}\sum_{q=1}^{n} p_m\,T^m_{kq}\,P^q_s\,\alpha^s. \tag{13.16}$$

Formula (13.16) means that in general extended covector field $\boldsymbol{\eta}$ is not invariant under gauge transformations (13.1), (13.2). However, if first equation in (8.21) is fulfilled, then formula (13.16) simplifies. It turns to

$$\eta_k \to \eta_k.$$

Theorem 13.1. *Weak normality equations* (8.21) *for Newtonian dynamical system* (7.2) *are invariant under gauge transformations* (13.1), (13.2).

Now let's proceed with our calculations for additional normality equations (11.7), (11.8), (11.13). Scalar field $\mathbf{A}$ given by formula (9.29) does not depend on $\mathbf{Q}$ and on connection components Γ^k_{ij}. Therefore we have

$$A_{rs} \to A_{rs}. \tag{13.17}$$

For B^r_s, using (13.5), (13.7), and (13.11), from (9.30) we derive

$$B^r_s \to B^r_s - \sum_{m=1}^{n}\sum_{q=1}^{n}\sum_{k=1}^{n} p_k\,T^k_{sq}\left(\tilde{\nabla}^m W^r - \tilde{\nabla}^r W^m\right) P^q_m. \tag{13.18}$$

Now let's substitute (13.18) into (11.13) and use (9.29). As a result we get

$$\begin{gathered}\sum_{r=1}^{n}\sum_{s=1}^{n} P^i_r\,B^r_s\,P^s_j \to \sum_{r=1}^{n}\sum_{s=1}^{n} P^i_r\,B^r_s\,P^s_j\,+\\ +\sum_{r=1}^{n}\sum_{s=1}^{n}\sum_{m=1}^{n}\sum_{q=1}^{n}\sum_{k=1}^{n} p_k\,T^k_{sq}\,P^s_j\,(A^{rm}-A^{mr})\,P^i_r\,P^q_m.\end{gathered} \tag{13.19}$$

Looking at (13.19), we see that in general left hand side of the equation (11.13) is not invariant under gauge transformations (13.1), (13.2). However, if equations (11.7) are fulfilled, then formula (13.19) reduces to

$$\sum_{r=1}^{n}\sum_{s=1}^{n} P^i_r\,B^r_s\,P^s_j \to \sum_{r=1}^{n}\sum_{s=1}^{n} P^i_r\,B^r_s\,P^s_j. \tag{13.20}$$

For the expression in right hand side of (11.13) under the same assumption we have

$$\sum_{r=1}^{n}\sum_{s=1}^{n}\frac{B^r_s\,P^s_r}{n-1}\,P^i_j \to \sum_{r=1}^{n}\sum_{s=1}^{n}\frac{B^r_s\,P^s_r}{n-1}\,P^i_j. \tag{13.21}$$

This follows from trace formula (11.12) for scalar factor λ in (11.11).

Now let's apply gauge transformation (13.1), (13.2) to tensor field $\mathbf{C}$ with components (9.31). By rather huge, but direct calculations we find:

$$\begin{gathered}C_{rs} \to C_{rs} + \ldots + \sum_{k=1}^{n}\sum_{q=1}^{n}\sum_{u=1}^{n} p_u\,T^u_{rk}\,P^k_q\left(\tilde{\nabla}^q U_s - \nabla_s W^q\,+\right.\\ \left.+\sum_{m=1}^{n}\sum_{v=1}^{n} D^{vq}_{ms}\,W^m\,p_v\right) + \sum_{k=1}^{n}\sum_{u=1}^{n}\sum_{m=1}^{n} p_u\,T^u_{rk}\,\frac{\tilde{\nabla}^m W^k - \tilde{\nabla}^k W^m}{\Omega}\,U_s\,p_m\,+\end{gathered}$$

$$+\sum_{k=1}^{n}\sum_{q=1}^{n}\sum_{u=1}^{n}\sum_{m=1}^{n}\sum_{v=1}^{n} p_u\,T^u_{rk}\left(P^k_q\,\tilde\nabla^q W^m\,p_v\,T^v_{sm}-\tilde\nabla^k W^m\,\frac{W^q\,p_m}{\Omega}\,p_v\,T^v_{sq}\right).$$

By dots we denoted terms symmetric with respect to indices r and s. They do not affect ultimate equation (11.8), therefore we need not keep them in explicit form. In right hand side of the above formula we see three distinct terms with sums. For the beginning let's transform second term using formula (8.16):

$$\begin{gathered}\sum_{k=1}^{n}\sum_{u=1}^{n}\sum_{m=1}^{n} p_u\,T^u_{rk}\,\frac{\tilde\nabla^m W^k-\tilde\nabla^k W^m}{\Omega}\,U_s\,p_m=\sum_{k=1}^{n}\sum_{u=1}^{n}\sum_{m=1}^{n}\sum_{q=1}^{n} p_u\,T^u_{rk}\,P^k_q\,\times\\ \times\frac{\tilde\nabla^m W^q-\tilde\nabla^q W^m}{\Omega}\,U_s\,p_m+\sum_{u=1}^{n}\sum_{k=1}^{n}\left(\sum_{q=1}^{n}\sum_{m=1}^{n} p_q\,\frac{\tilde\nabla^m W^q-\tilde\nabla^q W^m}{\Omega}\,p_m\right)\times\\ p_u\,T^u_{rq}\,\frac{W^q}{\Omega}\,U_s=\sum_{k=1}^{n}\sum_{u=1}^{n}\sum_{m=1}^{n}\sum_{q=1}^{n} p_u\,T^u_{rk}\,P^k_q\,\frac{\tilde\nabla^m W^q-\tilde\nabla^q W^m}{\Omega}\,U_s\,p_m.\end{gathered}$$

Substituting this result into formula for C_{rs} and taking into account (9.30), we get

$$\begin{gathered}C_{rs}\to C_{rs}+\ldots+\sum_{k=1}^{n}\sum_{q=1}^{n}\sum_{u=1}^{n} p_u\,T^u_{rk}\,P^k_q\,B^q_s+\sum_{k=1}^{n}\sum_{q=1}^{n}\sum_{u=1}^{n}\sum_{m=1}^{n}\sum_{v=1}^{n} p_u\,T^u_{rk}\,\times\\ \times P^k_q\,\tilde\nabla^q W^m\,p_v\,T^v_{sm}-\sum_{k=1}^{n}\sum_{q=1}^{n}\sum_{u=1}^{n}\sum_{m=1}^{n}\sum_{v=1}^{n} p_u\,T^u_{rk}\,\tilde\nabla^k W^m\,\frac{W^q\,p_m}{\Omega}\,p_v\,T^v_{sq}.\end{gathered}$$

Now we apply the same trick with formula (8.16) to last term in the above formula:

$$\begin{gathered}\sum_{k=1}^{n}\sum_{q=1}^{n}\sum_{u=1}^{n}\sum_{m=1}^{n}\sum_{v=1}^{n} p_u\,T^u_{rk}\,\tilde\nabla^k W^m\,\frac{W^q\,p_m}{\Omega}\,p_v\,T^v_{sq}=\\ =\sum_{k=1}^{n}\sum_{q=1}^{n}\sum_{u=1}^{n}\sum_{m=1}^{n}\sum_{v=1}^{n}\sum_{a=1}^{n} p_u\,T^u_{rk}\,P^k_a\,\tilde\nabla^a W^m\,\frac{W^q\,p_m}{\Omega}\,p_v\,T^v_{sq}+\\ +\sum_{k=1}^{n}\sum_{q=1}^{n}\sum_{u=1}^{n}\sum_{v=1}^{n}\frac{p_u\,T^u_{rk}\,W^k}{\Omega}\left(\sum_{a=1}^{n}\sum_{m=1}^{n} p_a\,\tilde\nabla^a W^m\,p_m\right)\frac{p_v\,T^v_{sq}\,W^q}{\Omega}.\end{gathered}\tag{13.22}$$

Last term in (13.22) is symmetric with respect to indices r and s. Therefore we can omit it adding to those denoted by dots in formula for C_{rs}:

$$\begin{gathered}C_{rs}\to C_{rs}+\ldots+\sum_{k=1}^{n}\sum_{q=1}^{n}\sum_{u=1}^{n} p_u\,T^u_{rk}\,P^k_q\,B^q_s+\\ +\sum_{k=1}^{n}\sum_{q=1}^{n}\sum_{u=1}^{n}\sum_{m=1}^{n}\sum_{v=1}^{n}\sum_{a=1}^{n} p_u\,T^u_{rk}\,P^k_q\,\tilde\nabla^q W^m\,P^a_m\,p_v\,T^v_{sa}.\end{gathered}\tag{13.23}$$

Now we can find transformation rule for left hand side of the equation (11.8). Using formula (13.23) and taking into account equations (11.13) and (11.7), we derive

$$\sum_{r=1}^{n}\sum_{s=1}^{n}(C_{rs}-C_{sr})\,P^r_i\,P^s_j\to\sum_{r=1}^{n}\sum_{s=1}^{n}(C_{rs}-C_{sr})\,P^r_i\,P^s_j. \tag{13.24}$$

Now, summarizing formulas (13.17), (13.20), (13.21), and (13.24), we see that the following theorem is proved.

Theorem 13.2. *Additional normality equations* (11.7), (11.8), (11.13) *for Newtonian dynamical system* (7.2) *are invariant under gauge transformations given by formulas* (13.1), (13.2).

Theorems 13.1 and 13.2 mean that we can apply all normality equations (8.21), (11.7), (11.8), (11.13) to Newtonian dynamical system written in connection-free form (2.3). For this purpose one should set $Q_i=\Theta_i$ in them, and one should choose identically zero connection components $\Gamma^k_{ij}=0$, thus replacing covariant derivatives ∇ and $\tilde{\nabla}$ by corresponding partial derivatives:

$$\nabla_i\to\frac{\partial}{\partial x^i},\qquad\qquad\tilde{\nabla}^i\to\frac{\partial}{\partial p_i}.$$

However, in this form normality equations are not obviously coordinate covariant. Each term in them loose transparent tensorial interpretation provided by covariant derivatives. Generally speaking, we have a problem of constructing present theory in coordinate-free and connection-free form. This is separate problem, it will be studied in future papers.

14. Basic example.

Let $H=H(x^1,\ldots,x^n,\,p_1,\ldots,p_n)$ be Hamilton function for some Hamiltonian dynamical system. It is given by the following well-known differential equations:

$$\dot{x}^i=\frac{\partial H}{\partial p_i},\qquad\qquad\dot{p}_i=-\frac{\partial H}{\partial x^i}. \tag{14.1}$$

Using some symmetric extended affine connection Γ, we can write Hamilton equations (14.1) in terms of covariant derivatives ∇ and $\tilde{\nabla}$:

$$\dot{x}^i=\tilde{\nabla}^iH,\qquad\qquad\nabla_tp_i=-\nabla_iH. \tag{14.2}$$

Using the same Hamilton function $H=H(x^1,\ldots,x^n,\,p_1,\ldots,p_n)$ as in (14.2), we define so called **modified Hamiltonian dynamical system**:

$$\dot{x}^i=\frac{\tilde{\nabla}^iH}{\displaystyle\sum_{s=1}^{n}p_s\,\tilde{\nabla}^sH},\qquad\qquad\nabla_tp_i=-\frac{\nabla_iH}{\displaystyle\sum_{s=1}^{n}p_s\,\tilde{\nabla}^sH}. \tag{14.3}$$

Surely we should choose H such that denominator in (14.3) is non-zero. Dynamical system (14.3) is an example (is special case) of Newtonian dynamical system (7.2) with vector field $\mathbf{V}$ and covector field $\mathbf{Q}$ given by formulas

$$V^i=\frac{\tilde{\nabla}^i H}{\sum\limits_{s=1}^{n} p_s\,\tilde{\nabla}^s H},\qquad\qquad Q_i=-\frac{\nabla_i H}{\sum\limits_{s=1}^{n} p_s\,\tilde{\nabla}^s H}. \tag{14.4}$$

Substituting (14.4) into (6.1) and (6.2), we find

$$W^i=-V^i,\qquad\qquad \Omega=-1. \tag{14.5}$$

Theorem 14.1. *Modified Hamiltonian dynamical system* (14.3) *is an example of Newtonian dynamical system* (7.2) *admitting normal shift of hypersurfaces in the sense of definition 1.5.*

We can write modified dynamical system (14.3) in connection-free form:

$$\dot{x}^i=\frac{\dfrac{\partial H}{\partial p_i}}{\sum\limits_{s=1}^{n} p_s\,\dfrac{\partial H}{\partial p_s}},\qquad\qquad \dot{p}_i=-\frac{\dfrac{\partial H}{\partial x^i}}{\sum\limits_{s=1}^{n} p_s\,\dfrac{\partial H}{\partial p_s}}. \tag{14.6}$$

Theorem 14.2. *Modified Hamiltonian dynamical system* (14.6) *is an example of Newtonian dynamical system* (2.3) *admitting normal shift of hypersurfaces in the sense of definition 1.5.*

Differential equations (14.3) and (14.6) are different representations of the same dynamical system. Therefore theorems 14.1 and 14.2 formulate the same result. This result is not new. It was obtained in paper [19]. In present paper it is built into framework of more general theory and forms basic example for this theory. It proves that class of dynamical systems considered in this theory is not empty, and, moreover, this class is sufficiently large.

15. Dedication.

This paper is dedicated to my mother F. M. Sharipova, who taught Mathematics for many years in School no. 18 of Karakul, Bukhara region, Uzbekistan. She was best in recognizing various constellations on the sky and knew great many of them. In dark south nights in Summer, when I was 10 or even younger, she often told me about stars, comets, planets, and other thing. Possibly this was why later on I have chosen Natural Sciences and Mathematics for my profession. Bright image of my mother is ever kept in my memory.

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