DYNAMICAL SYSTEMS ADMITTING NORMAL SHIFT AND WAVE EQUATIONS.

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Abstract. High frequency limit for most of wave phenomena is known as quasi-classical limit or ray optics limit. Propagation of waves in this limit is described in terms of wave fronts and rays. Wave front is a surface of constant phase whose points are moving along rays. As it appears, their motion can be described by Hamilton equations being special case for Newton's equations. In simplest cases (e. g. light propagation in isotropic non-homogeneous refracting media) the dynamics of points preserves orthogonality of wave front and rays. This property was generalized in the theory of dynamical systems admitting normal shift of hypersurfaces. In present paper inverse problem is studied, i. e. the problem of reconstructing wave equation corresponding wave front dynamics for which is described by Newtonian dynamical system admitting normal shift of hypersurfaces in Riemannian manifold.

1. High frequency asymptotics for wave equations.

Let M be a Riemannian manifold of the dimension n. Denote by ∇ covariant differentiation with respect to metric connection in M. Then the expression

$$\mathbf{a}_r \cdot D^r = \sum_{k_1=1}^{n} \dots \sum_{k_r=1}^{n} \left(\frac{1}{i}\right)^r a^{k_1\dots\,k_r}(x^1,\dots,x^n)\cdot\nabla_{k_1}\cdot\ldots\cdot\nabla_{k_r} \tag{1.1}$$

is an elementary differential operator of r-th order. Here $a^{k_1\dots\,k_r}(x^1,\dots,x^n)$ are components of some smooth symmetric tensor field $\mathbf{a}$ of the type $(r,0)$ in local coordinates $x^1,\dots,x^n$ and $i=\sqrt{-1}$. Operators (1.1) are united into polynomial

$$H(p,D) = \sum_{r=0}^{m} \mathbf{a}_r \cdot D^r. \tag{1.2}$$

This is m-th order scalar differential operator in M. Here p is a point of M and D is a formal symbol for differentiation. Operator (1.2) can be applied either to scalar field or tensorial field in M. We shall apply it to scalar field u, but first we introduce large parameter λ in $H(p,D)$. Let's denote

$$H(p,\lambda^{-1}D) = \sum_{r=0}^{m} \lambda^{-r}\,\mathbf{a}_r\cdot D^r. \tag{1.3}$$

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Then consider scalar m-th order wave equation with large parameter λ:

$$H(p,\lambda^{-1}D)u=0. \tag{1.4}$$

In particular cases (1.4) can reduce to Helmholtz equation, to Schredinger equation, and to wave equation (see [1]). The following ansatz for formal asymptotic solution of the equation (1.4) as $\lambda\to\infty$ was suggested by P. Debye:

$$u=\sum_{\alpha=0}^{\infty}\frac{\varphi_{(\alpha)}}{(i\,\lambda)^{\alpha}}\cdot e^{i\lambda S}. \tag{1.5}$$

Here $\varphi_{(\alpha)}$ and S are some smooth scalar fields in M. Scalar field S is interpreted as a phase of wave.

Now let's substitute (1.5) into (1.4). This is done in several steps. First of all let's apply differential operator $\nabla_{k_1}\ldots\nabla_{k_r}$ to $e^{i\lambda S}$. This yields

$$\nabla_{k_1}\ldots\nabla_{k_r}e^{i\lambda S}=\left(\sum_{q=0}^{r}\beta^{(q)}_{k_1\ldots\,k_r}\cdot(i\,\lambda)^{r-q}\right)\cdot e^{i\lambda S}. \tag{1.6}$$

Leading coefficient in this expression can be written explicitly

$$\beta^{(0)}_{k_1\ldots\,k_r}=\nabla_{k_1}S\cdot\ldots\cdot\nabla_{k_r}S. \tag{1.7}$$

Other coefficients $\beta^{(q)}_{k_1\ldots\,k_r}$ can also be calculated for each particular q. It might be difficult to write general formula for all of them, but we do not need this formula.

In the second step we apply operator (1.3) to power series (1.5). Using formula (1.6), for $H(p,\lambda^{-1}D)u$ we obtain the following expression:

$$Hu=\sum_{\alpha=0}^{\infty}\sum_{r=0}^{m}\sum_{s=0}^{r}\sum_{q=s}^{r}\sum_{k_1=1}^{n}\ldots\sum_{k_r=1}^{n}\frac{r!\cdot a^{k_1\ldots\,k_r}\cdot\beta^{(q-s)}_{k_{s+1}\ldots\,k_r}}{s!\,(r-s)!\cdot(i\,\lambda)^{q+\alpha}}\cdot e^{i\lambda S}\cdot\nabla_{k_1}\ldots\nabla_{k_s}\varphi_{(\alpha)}.$$

If we collect all terms with fixed value of α and $\theta=q+\alpha$, we find that the number of such terms in the above expression is finite. Simplest way to prove this fact is to make the following transformations of sums:

$$\sum_{\alpha=0}^{\infty}\sum_{r=0}^{m}\sum_{s=0}^{r}\sum_{q=s}^{r}\to\sum_{\alpha=0}^{\infty}\sum_{r=0}^{m}\sum_{q=0}^{r}\sum_{s=0}^{q}\to\sum_{\alpha=0}^{\infty}\sum_{q=0}^{m}\sum_{r=q}^{m}\sum_{s=0}^{q},$$

$$\sum_{\alpha=0}^{\infty}\sum_{q=0}^{m}\sum_{r=q}^{m}\sum_{s=0}^{q}\to\sum_{\theta=0}^{\infty}\ \sum_{\alpha=\alpha_{\theta,m}}^{\theta}\ \sum_{r=\theta-\alpha}^{m}\ \sum_{s=0}^{\theta-\alpha}.$$

Here $\alpha_{\theta,m}=\max(0,\theta-m)$. Now we introduce differential operator R_ω:

$$R_\omega=\sum_{r=\omega}^{m}\ \sum_{s=0}^{\omega}\ \sum_{k_1=1}^{n}\ldots\sum_{k_r=1}^{n}\frac{r!\cdot a^{k_1\ldots\,k_r}\cdot\beta^{(\omega-s)}_{k_{s+1}\ldots\,k_r}}{s!\,(r-s)!}\cdot\nabla_{k_1}\ldots\nabla_{k_s}. \tag{1.8}$$

Let's rewrite left hand side of the equation (1.4) in terms of operator (1.8):

$$Hu=\sum_{\theta=0}^{\infty}\sum_{\alpha=\alpha_{\theta,m}}^{\theta}\frac{R_{\theta-\alpha}\varphi_{(\alpha)}}{(i\,\lambda)^{\theta}}\cdot e^{i\lambda S}.$$

This is third step in substituting formal power series (1.5) into the equation (1.4). As a result wave equation (1.4) breaks into infinite series of equations for $\varphi_{(\alpha)}$:

$$\sum_{\alpha=\alpha_{\theta,m}}^{\theta}R_{\theta-\alpha}\varphi_{(\alpha)}=0\ \text{ for }\ \theta=0,\,1,\,\dots\,,\,\infty. \tag{1.9}$$

In order to study the equations (1.9) we calculate explicit form of operator (1.8) for $\omega=0$. Using formula (1.7), we derive

$$R_0=\sum_{r=0}^{m}\sum_{k_1=1}^{n}\dots\sum_{k_r=1}^{n}a^{k_1\dots\,k_r}(x^1,\dots,x^n)\cdot\nabla_{k_1}S\cdot\ldots\cdot\nabla_{k_r}S. \tag{1.10}$$

Note that R_0 is a scalar operator, i. e. it is the operator of multiplication by the function $R_0=R_0(x^1,\dots,x^n)$. Therefore first equation in the series (1.9) looks like $R_0\cdot\varphi_{(0)}=0$. Leading term in power series (1.5) for u is nonzero. Therefore the equation $R_0\cdot\varphi_{(0)}=0$ reduces to the following one:

$$R_0=0. \tag{1.11}$$

Taking into account (1.11), we can rewrite (1.9) as

$$\sum_{\alpha=\alpha_{\theta,m}}^{\theta-1}R_{\theta-\alpha}\varphi_{(\alpha)}=0,\ \text{ where }\ \theta=1,\,2,\,\dots\,,\,\infty. \tag{1.12}$$

First equation in the series (1.12) is also very simple. It is a partial differential equation of the first order for the function $\varphi_{(0)}$:

$$R_1\varphi_{(0)}=0. \tag{1.13}$$

Other equations in the series (1.12) can be written in recursive form:

$$R_1\varphi_{(\theta-1)}=-\sum_{\alpha=\alpha_{\theta,m}}^{\theta-2}R_{\theta-\alpha}\varphi_{(\alpha)}\ \text{ for }\ \theta=2,\,3,\,\dots\,,\,\infty. \tag{1.14}$$

Below we calculate explicit expression for operator R_1, but we do not consider the problem of solvability of equations (1.13) and (1.14). This should be done for each particular equation (1.4) taking into account initial-value and/or boundary-value problem stated for this equation.

In order to find explicit expression for operator R_1 we should first find explicit expression for coefficient $\beta^{(1)}_{k_1\dots\,k_r}$ in (1.6). This is done by direct calculations. Applying operator $\nabla_{k_1}\ldots\nabla_{k_r}$ to $e^{i\lambda S}$ and collecting all terms with $(i\,\lambda)^{r-1}$, we get

$$\beta^{(1)}_{k_1}=0\quad\text{for}\quad r=1,\tag{1.15}$$

$$\beta^{(1)}_{k_1\ldots\,k_r}=\sum_{1\leqslant i<j\leqslant r}\nabla_{k_i}\nabla_{k_j}S\cdot\prod_{\substack{1\leqslant q\leqslant r\\ q\neq i,q\neq j}}\nabla_{k_q}S\quad\text{for}\quad r\geqslant 2.\tag{1.16}$$

Now let's substitute the expressions (1.15) and (1.16) into the formula (1.8) and take into account symmetry of coefficients $a^{k_1\ldots\,k_r}$. This yields

$$\begin{aligned}R_1&=\sum^{m}_{r=1}\sum^{n}_{k_1=1}\ldots\sum^{n}_{k_r=1}r\cdot a^{k_1\ldots\,k_r}\cdot\prod^{r}_{\rho=2}\nabla_{k_\rho}S\cdot\nabla_{k_1}+\\&+\sum^{m}_{r=2}\sum^{n}_{k_1=1}\ldots\sum^{n}_{k_r=1}\frac{r(r-1)}{2}\cdot\nabla_{k_1}\nabla_{k_2}S\cdot\prod^{r}_{\rho=3}\nabla_{k_\rho}S.\end{aligned}\tag{1.17}$$

Further we assume that the equations (1.13) and (1.14) with operator (1.17) in left hand side are solvable in some sense, and we assume that that $\varphi_{(0)}\neq 0$. Under these assumptions we concentrate our efforts in studying the equation (1.11).

2. Hamilton-Jacobi equation.

Note that right hand side of the equality (1.10) is similar to that of (1.1). This is not casual circumstance. In order to reveal the nature of this analogy let's denote ∇S by $\mathbf{p}$. Then $\mathbf{p}$ is a covector field with components

$$p_k=\nabla_k S,\ \text{ where }\ k=1,\,\ldots\,,\,n.\tag{2.1}$$

Using notations (2.1), we can rewrite (1.10) as $R_0=H(p,\mathbf{p})$, where

$$H(p,\mathbf{p})=\sum^{m}_{r=0}\sum^{n}_{k_1=1}\ldots\sum^{n}_{k_r=1}a^{k_1\ldots\,k_r}(x^1,\ldots\,,x^n)\cdot p_{k_1}\cdot\ldots\cdot p_{k_r}.\tag{2.2}$$

Right hand side of (2.2) is a polynomial in components of covector $\mathbf{p}$. If $p_1,\,\ldots\,,\,p_n$ are treated as independent variables representing components of some arbitrary covector $\mathbf{p}$ at the point $p\in M$, then polynomial $H(p,\mathbf{p})$ is called the **symbol** of differential operator (1.2). Once $H(p,\mathbf{p})$ is given, the operator (1.2) itself can be obtained by substituting $p_k=-i\,\nabla_k$ into its symbol. This trick is well-known in quantum mechanics, where momentum vector $\mathbf{p}$ is replaced by differential operator. The analogy with quantum mechanics becomes more transparent (see [2]) if we take

$$p_k=-\frac{i}{\lambda}\,\nabla_k\tag{2.3}$$

with $\lambda=1/\hbar$ treated as large parameter. If we substitute (2.3) into the polynomial (2.2), then we obtain the operator (1.3) used in the equation (1.4).

Now let's return back to the equation (1.11). It is first equation in the series (1.9). Due to (1.10) and (2.2) we can write it as follows:

$$H(p,\nabla S)=0.\tag{2.4}$$

The equation (2.4) is first order nonlinear partial differential equation with respect to scalar field S. It is often called **Hamilton-Jacobi equation**. Traditional way of solving such equations is based on the concept of **characteristic lines**. For the equation (2.4) these are parametric curves determined by **Hamilton equations**:

$$\dot{x}^i = \frac{\partial H}{\partial p_i}, \qquad\qquad \dot{p}_i = -\frac{\partial H}{\partial x^i}. \tag{2.5}$$

In paper [3] the following form of Hamilton equations (2.5) was suggested:

$$\dot{x}^i = \tilde{\nabla}^i H, \qquad\qquad \nabla_t p_i = -\nabla_i H. \tag{2.6}$$

It was especially designed for the case of Hamiltonian dynamical systems in Riemannian manifolds. Here ∇_t is standard notation for covariant differentiation of vector-valued and tensor-valued functions with respect to parameter t along the curve. In our particular case covariant derivative $\nabla_t p_i$ is written as

$$\nabla_t p_i = \dot{p}_i - \sum_{k=1}^{n}\sum_{j=1}^{n} \Gamma^k_{ij}\, p_k\, \dot{x}^j. \tag{2.7}$$

Covariant derivatives ∇_i and $\tilde{\nabla}^i$ in (2.6) are less standard objects. The matter is that, writing Hamilton equations (2.6), we treat H as **extended scalar field** in the sense of the following definition.

Definition 2.1. *Extended* tensor field $\mathbf{X}$ of type (r,s) in M is a tensor-valued function that maps each point $q \in G \subseteq T^*M$ to a tensor of the space $T^r_s(p,M)$, where $p = \pi(q)$. Subset G of T^*M is a domain of extended tensor field $\mathbf{X}$. If $G = T^*M$, then $\mathbf{X}$ is called *global* extended tensor field in M.

In local coordinates extended tensor fields are represented by their components:

$$X^{i_1\ldots\, i_r}_{j_1\ldots\, j_s} = X^{i_1\ldots\, i_r}_{j_1\ldots\, j_s}(x^1,\ldots,x^n,p_1,\ldots,p_n). \tag{2.8}$$

In contrast to traditional tensor fields, components of extended tensor field are functions of double set of arguments. First n arguments of $X^{i_1\ldots\, i_r}_{j_1\ldots\, j_s}$ in (2.8) are local coordinates of a point $p \in M$. Others are components of covector $\mathbf{p} \in T^*_p(M)$. Both p and $\mathbf{p}$ form a point $q = (p, \mathbf{p})$ of cotangent bundle T^*M. Further we shall not come deep into the theory of extended tensor fields, referring reader to paper [3] and thesis [4] for more details. But we give explicit formulas for covariant differentiations ∇ and $\tilde{\nabla}$ in local coordinates $x^1, \ldots, x^n, p_1, \ldots, p_n$:

$$\tilde{\nabla}^q X^{i_1\ldots\, i_r}_{j_1\ldots\, j_s} = \frac{\partial X^{i_1\ldots\, i_r}_{j_1\ldots\, j_s}}{\partial p_q}. \tag{2.9}$$

$$\begin{aligned}\nabla_q X^{i_1\ldots\, i_r}_{j_1\ldots\, j_s} &= \frac{\partial X^{i_1\ldots\, i_r}_{j_1\ldots\, j_s}}{\partial x^q} + \sum_{a=1}^{n}\sum_{b=1}^{n} p_a\, \Gamma^a_{qb}\, \frac{\partial X^{i_1\ldots\, i_r}_{j_1\ldots\, j_s}}{\partial p_b} + \\ &+ \sum_{k=1}^{r}\sum_{a_k=1}^{n} \Gamma^{i_k}_{q\,a_k}\, X^{i_1\ldots\, a_k \ldots\, i_r}_{j_1\ldots\ldots\ldots\, j_s} - \sum_{k=1}^{s}\sum_{b_k=1}^{n} \Gamma^{b_k}_{q\,j_k}\, X^{i_1\ldots\ldots\ldots\, i_r}_{j_1\ldots\, b_k \ldots\, j_s}.\end{aligned} \tag{2.10}$$

Covariant differentiation $\tilde{\nabla}$ defined by formula (2.9) is called **momentum gradient**. Second covariant differentiation ∇ defined by formula (2.10) is called **spatial gradient**. Note that for traditional tensor fields formula (2.10) reduces to standard formula for covariant derivatives. Therefore use of symbol ∇ in section 1 and in formula (2.4) does not contradict to formula (2.10).

Let's take some smooth orientable hypersurface σ in M. By $\mathbf{n} = \mathbf{n}(p)$ we denote smooth field of unit normal vectors on σ. Using Riemannian metric $\mathbf{g}$ in the manifold M, we can transform it to unit covector field:

$$n_i = \sum_{j=1}^{n} g_{ij}\, n^j. \tag{2.11}$$

We denote it by the same symbol $\mathbf{n}$. Using covector $\mathbf{n} = \mathbf{n}(p)$, we set up the following boundary-value problem for Hamilton-Jacobi equation (2.4):

$$S\,\Big|_{p\in\sigma} = s_0 = \text{const}, \qquad\qquad \nabla S\,\Big|_{p\in\sigma} = \nu(p)\cdot\mathbf{n}(p). \tag{2.12}$$

Here $\nu = \nu(p)$ is a scalar factor. It is not arbitrary. In order to make (2.12) compatible with the equation (2.4) it should satisfy the following condition:

$$H(p, \nu(p)\cdot\mathbf{n}(p)) = 0. \tag{2.13}$$

Aside from boundary-value problem (2.12) for the equation (2.4), we set up the following initial-value problem for Hamilton equations (2.6):

$$x^i\,\Big|_{t=0} = x^i(p), \qquad\qquad p_i\,\Big|_{t=0} = \nu(p)\cdot n_i(p). \tag{2.14}$$

Initial data of Cauchy problem (2.14) are parametrized by points of hypersurface σ. Function $\nu = \nu(p)$ in (2.14) is the same scalar factor as in (2.12). It is restricted by condition (2.13) above.

Solving Cauchy problem (1.18) for the equations (2.6), we get a family of parametric curves in T^*M. Their projections to M form a family of parametric curves

$$\gamma = \gamma(t, p) \tag{2.15}$$

starting at the points of hypersurface σ. Boundary data (2.12) and initial data (2.14) are called **regular** if function $\nu = \nu(p)$ does not vanish on σ, i. e. if

$$\nu(p) \neq 0, \tag{2.16}$$

and if integral curves (2.15) are transversal to hypersurface σ. Now, using Hamilton function H, let's define the following extended scalar field in M:

$$\Omega = \sum_{i=1}^{n} p_i\, \tilde{\nabla}^i H. \tag{2.17}$$

In terms of Ω transversality condition $\gamma \nparallel \sigma$ is expressed by inequality

$$\Omega(p, \nu(p)\cdot\mathbf{n}(p)) \neq 0. \tag{2.18}$$

Suppose that both regularity conditions (2.16) and (2.18) on hypersurface σ are fulfilled. In this case integral curves (2.15) fill some neighborhood of hypersurface σ and we can formulate the following theorem.

Theorem 2.1. *If boundary data* (2.12) *satisfying condition* (2.13) *are regular, i. e. if both inequalities* (2.16) *and* (2.18) *on σ are fulfilled, then boundary-value problem* (2.12) *for Hamilton-Jacobi equation* (2.4) *has unique solution S in some neighborhood of hypersurface σ. This solution is given by integral*

$$S = s_0 + \int\limits_{\gamma} \sum_{i=1}^{n} p_i \, dx^i = s_0 + \int\limits_{0}^{t} \sum_{i=1}^{n} p_i \, \dot{x}^i \, dt. \tag{2.19}$$

Theorem 2.1 is standard result in the theory of first order partial differential equations. Its proof can be found in paper [1].

3. Wave front dynamics.

Theorem 2.1 gives a method for solving Hamilton-Jacobi equation and finding scalar field $S = S(p)$. Remember that S in (1.5) is a phase of propagating wave. By definition **wave front** is a set of points with constant phase:

$$\sigma(s) = \{p \in M\colon\ S(p) = s = \text{const}\}. \tag{3.1}$$

Hypersurface σ in (2.12) is interpreted as initial wave front, since $\sigma = \sigma(s_0)$. In regular case for s sufficiently close to s_0 wave fronts $\sigma(s)$ in (3.1) are smooth hypersurfaces filling some neighborhood of initial hypersurface σ. Curves (2.15) are interpreted as rays in the limit of geometric optics ($\lambda \to \infty$), while Hamilton equations (2.6) determine the motion of points, starting at the time instant $t = 0$ and moving along these rays. If we fix their positions at some nonzero instant of time t, for sufficiently small value of t we would obtain some smooth hypersurface σ_t close to initial hypersurface σ. Hypersurfaces σ_t form another family filling some neighborhood of initial wave front σ. Should they coincide with $\sigma(s)$? In general, the answer is negative:

$$\sigma_t \neq \sigma(s), \tag{3.2}$$

see Fig. 3.1. Hypersurfaces σ_t are drawn in solid lines, while wave fronts $\sigma(s)$ are drawn in dashed lines. Inequality (3.2) means that Hamilton equations (2.6) do not describe real dynamics of wave fronts exactly. In order to describe this dynamics in terms of ODE's we should change parameter t in (2.6) for another parameter $s = S(t)$. Formula (2.19) for scalar field S yields the relation of these two parameters:

$$\frac{ds}{dt} = \sum_{i=1}^{n} p_i \, \dot{x}^i = \sum_{i=1}^{n} p_i \, \tilde{\nabla}^i H = \Omega. \tag{3.3}$$

Changing t for s according to the formula (3.3), we keep symbol t for denoting this new parameter. Then from (2.6) we derive modified Hamilton equations

$$\dot{x}^i=\frac{\tilde{\nabla}^i H}{\Omega}, \qquad\qquad \nabla_t p_i=-\frac{\nabla_i H}{\Omega}, \tag{3.4}$$

where Ω is given by formula (2.17). Differential equations (3.4) are called **differential equations of wave front dynamics**.

4. Legendre transformation.

Hamilton equations (2.6) and modified Hamilton equations (3.4) are the equations in cotangent bundle T^*M. Now we shall transform them into the equations in tangent bundle TM. Note that for each point $q=(p,\mathbf{p})$ of cotangent bundle T^*M we can consider vector $\mathbf{v}\in T_p(M)$ with components

$$v^i=\tilde{\nabla}^i H. \tag{4.1}$$

Joining together p and $\mathbf{v}$, we get a point $q=(p,\mathbf{v})$ of tangent bundle TM. This determines a map $\lambda^{-1}\colon T^*M\to TM$, which is known as **inverse Legendre transformation**. We shall assume it to be invertible and denote by λ direct Legendre transformation: $\lambda\colon TM\to T^*M$. Suppose that $\mathbf{X}$ is some extended tensor field. According to the definition 2.1, it is tensor-valued function in T^*M. Then composition $\mathbf{X}\circ\lambda$ is tensor-valued function in tangent bundle TM. So we get another definition of extended tensor field.

Definition 4.1. *Extended* tensor field $\mathbf{X}$ of type (r,s) in M is a tensor-valued function that maps each point $q\in G\subseteq TM$ to a tensor of the space $T^r_s(p,M)$, where $p=\pi(q)$. Subset G of TM is a domain of extended tensor field $\mathbf{X}$. If $G=TM$, then $\mathbf{X}$ is called *global* extended tensor field in M.

In the case of arbitrary smooth manifold M definitions 2.1 and 4.1 lead to different theories. But for Riemannian manifold M tangent bundle TM and cotangent bundle T^*M are bound with each other by duality maps

$$\mathbf{g}\colon TM\to T^*M, \qquad\qquad \mathbf{g}^{-1}\colon T^*M\to TM. \tag{4.2}$$

In local coordinates duality maps (4.2) are represented as index lowering and index raising procedures in arguments of $\mathbf{X}$. Due to duality maps (4.2) two objects introduced by definitions 2.1 and 4.1 are the same in essential. We call them *covariant* and *contravariant* representations of extended tensor field $\mathbf{X}$. If we take contravariant representation of $\mathbf{X}$, then formulas (2.9) and (2.10) are rewritten as

$$\tilde{\nabla}_q X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}=\frac{\partial X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}}{\partial p^q}, \tag{4.3}$$

$$\begin{aligned}\nabla_q X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}&=\frac{\partial X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}}{\partial x^q}-\sum_{a=1}^{n}\sum_{b=1}^{n}p^a\,\Gamma^b_{qa}\,\frac{\partial X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}}{\partial p^b}+\\&+\sum_{k=1}^{r}\sum_{a_k=1}^{n}\Gamma^{i_k}_{q\,a_k}\,X^{i_1\ldots\,a_k\ldots\,i_r}_{j_1\ldots\ldots\ldots\,j_s}-\sum_{k=1}^{s}\sum_{b_k=1}^{n}\Gamma^{b_k}_{q\,j_k}\,X^{i_1\ldots\ldots\ldots\,i_r}_{j_1\ldots\,b_k\ldots\,j_s}.\end{aligned} \tag{4.4}$$

Direct and inverse Legendre transformations are other two maps

$$\lambda\colon TM \to T^*M, \qquad\qquad \lambda^{-1}\colon T^*M \to TM \tag{4.5}$$

binding TM and T^*M. Presence of nonlinear maps (4.5) increases the number of representations of extended tensor field $\mathbf{X}$. If $\mathbf{Y} = \mathbf{X} \circ \lambda^{-1}$, then we say that $\mathbf{Y}$ is $\mathbf{p}$-representation or **momentum representation** for $\mathbf{X}$, while $\mathbf{X}$ is called $\mathbf{v}$-representation or **velocity representation** for $\mathbf{Y}$. Total set of representations for extended tensor field is given in the following table:

covariant $\mathbf{p}$-representation	contravariant $\mathbf{p}$-representation
covariant $\mathbf{v}$-representation	contravariant $\mathbf{v}$-representation

The following extended scalar field is called **Lagrange function**:

$$l = \sum_{i=1}^{n} p_i \cdot \tilde{\nabla}^i H - H \tag{4.6}$$

Covariant $\mathbf{p}$ representation is natural for Hamilton function $\mathbf{H}$, while Lagrange function is naturally used in contravariant $\mathbf{v}$-representation: $L = l \circ \lambda$. If Lagrange function (4.6) is already transformed to contravariant $\mathbf{v}$-representation, then direct Legendre transformation λ in (4.5) can be given by formula similar to (4.1):

$$p_i = \tilde{\nabla}_i L \tag{4.7}$$

(see [3] for more details). Note that formulas (2.9), (2.10), (4.3), and (4.4) for ∇ and $\tilde{\nabla}$ are valid either in $\mathbf{p}$-representation and in $\mathbf{v}$-representation. However, in $\mathbf{v}$-representation we should replace p_i by v_i and p^i by v^i in these formulas. Therefore $\tilde{\nabla}$ is called **velocity gradient** in $\mathbf{v}$-representation.

Duality maps (4.2) commutate with covariant differentiations ∇ and $\tilde{\nabla}$. This is expressed by the following equalities:

$$\nabla(\mathbf{X} \circ \mathbf{g}) = (\nabla \mathbf{X}) \circ \mathbf{g}, \qquad\qquad \tilde{\nabla}(\mathbf{X} \circ \mathbf{g}) = (\tilde{\nabla} \mathbf{X}) \circ \mathbf{g}. \tag{4.8}$$

Legendre transformation λ does not commutate with differentiations ∇ and $\tilde{\nabla}$. Unlike (4.8), we have the following equalities, where C is the operation of contraction:

$$\tilde{\nabla}(\mathbf{Y} \circ \lambda) = C(\tilde{\nabla}\mathbf{Y} \circ \lambda \otimes \tilde{\nabla}\tilde{\nabla} L), \tag{4.9}$$

$$\nabla(\mathbf{Y} \circ \lambda) = \nabla \mathbf{Y} \circ \lambda + C(\tilde{\nabla}\mathbf{Y} \circ \lambda \otimes \nabla\tilde{\nabla} L). \tag{4.10}$$

In local coordinates the equalities (4.9) and (4.10) are written as follows:

$$\tilde{\nabla}_q X^{i_1\ldots\, i_r}_{j_1\ldots\, j_s} = \sum_{k=1}^{n} \tilde{\nabla}_q \tilde{\nabla}_k L \cdot \tilde{\nabla}^k Y^{i_1\ldots\, i_r}_{j_1\ldots\, j_s}, \tag{4.11}$$

$$\nabla_q X^{i_1\ldots\, i_r}_{j_1\ldots\, j_s} = \nabla_q Y^{i_1\ldots\, i_r}_{j_1\ldots\, j_s} + \sum_{k=1}^{n} \nabla_q \tilde{\nabla}_k L \cdot \tilde{\nabla}^k Y^{i_1\ldots\, i_r}_{j_1\ldots\, j_s}. \tag{4.12}$$

Similarly, one can write inverse relationships for (4.9) and (4.10):

$$\tilde{\nabla}(\mathbf{X}\circ\lambda^{-1}) = C(\tilde{\nabla}\mathbf{X}\circ\lambda^{-1}\otimes\tilde{\nabla}\tilde{\nabla}H), \tag{4.13}$$

$$\nabla(\mathbf{X}\circ\lambda^{-1}) = \nabla\mathbf{X}\circ\lambda^{-1} + C(\tilde{\nabla}\mathbf{X}\circ\lambda^{-1}\otimes\nabla\tilde{\nabla}H). \tag{4.14}$$

In local coordinates these relationships (4.13) and (4.14) are expressed as

$$\tilde{\nabla}^k Y^{i_1\dots\,i_r}_{j_1\dots\,j_s} = \sum_{q=1}^{n} \tilde{\nabla}^k\tilde{\nabla}^q H\cdot\tilde{\nabla}_q X^{i_1\dots\,i_r}_{j_1\dots\,j_s}, \tag{4.15}$$

$$\nabla_k Y^{i_1\dots\,i_r}_{j_1\dots\,j_s} = \nabla_k X^{i_1\dots\,i_r}_{j_1\dots\,j_s} + \sum_{q=1}^{n} \nabla_k\tilde{\nabla}^q H\cdot\tilde{\nabla}_q X^{i_1\dots\,i_r}_{j_1\dots\,j_s}. \tag{4.16}$$

In (4.11), (4.12), (4.15), and (4.16) we assume that $\mathbf{X}=\mathbf{Y}\circ\lambda$ and $\mathbf{Y}=\mathbf{X}\circ\lambda^{-1}$.

Now let's consider Lagrange function L. Its **p**-representation is given by formula (4.6). Let's calculate $\nabla_k l$ directly by means of formula (4.6):

$$\nabla_k l = \sum_{i=1}^{n} p_i\cdot\nabla_k\tilde{\nabla}^i H - \nabla_k H. \tag{4.17}$$

Then let's apply formula (4.16) and let's calculate this derivative $\nabla_k l$ again:

$$\nabla_k l = \nabla_k L + \sum_{q=1}^{n} \nabla_k\tilde{\nabla}^q H\cdot\tilde{\nabla}_q L. \tag{4.18}$$

Comparing (4.17) and (4.18), and taking into account (4.7), we get:

$$\nabla_k L = -\nabla_k H. \tag{4.19}$$

In coordinate free form this equality (4.19) looks like $\nabla L = -\nabla H\circ\lambda$.

5. Lagrangian representation for the equations of wave front dynamics.

Now we are ready to transform differential equations (3.4) to **v**-representation. Using formula (4.1), we write fist equation (3.4) as

$$\dot{x}^i = \frac{v^i}{\Omega}. \tag{5.1}$$

Extended scalar field Ω in **p**-representation is determined by formula (2.17). Due to (4.1) and (4.7) its **v**-representation is determined by formula

$$\Omega = \sum_{i=1}^{n} v^i\cdot\tilde{\nabla}_i L. \tag{5.2}$$

In order to transform second equation (3.4) we use formulas (4.7) and (4.19):

$$\nabla_t(\tilde{\nabla}_i L) = \frac{\nabla_i L}{\Omega}. \tag{5.3}$$

Written together, (5.1) and (5.3) form Lagrangian representation for the equations of wave front dynamics (3.4). Here are these equations:

$$\dot{x}^i = \frac{v^i}{\Omega}, \qquad\qquad \nabla_t(\tilde{\nabla}_i L) = \frac{\nabla_i L}{\Omega}. \tag{5.4}$$

Though written in terms of Lagrange function, the equations (5.4) are not Euler-Lagrange equations due to the denominator Ω in them.

6. Dynamical systems admitting normal shift.

Note that vector $\mathbf{v}$ with components $v^1, \ldots, v^n$ is not an actual velocity vector for the dynamics described by differential equations (5.4). Let's denote actual velocity vector by $\mathbf{u}$. Then from first equation (5.4) we derive

$$u^i = \frac{v^i}{\Omega}. \tag{6.1}$$

Denominator Ω in (6.1) is not constant, it depends on $x^1, \ldots, x^n$ and on components of velocity vector $\mathbf{v}$ as well. Therefore (6.1) defines nonlinear map

$$\mu\colon TM \to TM \tag{6.2}$$

similar to Legendre map λ determined by (4.7). If this map (6.2) invertible, then one can transform (5.4) to Newtonian form:

$$\dot{x}^i = u^i, \qquad\qquad \nabla_t u^i = F^i(x^1, \ldots, x^n, u^1, \ldots, u^n). \tag{6.3}$$

Components of force vector $\mathbf{F}$ in (6.3) are expressed through Lagrange function L and its derivatives. Now we shall not derive general formula for F^i. But we consider very important special case, when L is fiberwise spherically symmetric extended scalar field in Riemannian manifold M.

Definition 6.1. Extended tensor field $\mathbf{X}$ is called *fiberwise spherically symmetric* if its components depend only on modulus of velocity vector:

$$X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s} = X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}(x^1, \ldots, x^n, v), \text{ where } v = |\mathbf{v}|.$$

This means that $X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}$ is spherically symmetric function within each fiber of tangent bundle TM for each fixed point $p \in M$.

Fiberwise spherically symmetric tensor fields were considered in Chapter VII of thesis [4], see also recent papers [3] and [5]. Now suppose that L is fiberwise spherically symmetric extended scalar field: $L = L(x^1, \ldots, x^n, v)$. Then from (5.2) we derive that Ω is also fiberwise spherically symmetric scalar field:

$$\Omega = |\mathbf{v}| \cdot L'. \tag{6.4}$$

Here L' is partial derivative of L with respect of its last argument $v = |\mathbf{v}|$. Substituting (6.4) into the equality (6.1), we derive the following formulas:

$$u^i = \frac{N^i}{L'}, \qquad\qquad |\mathbf{u}| = \frac{1}{|L'|}. \tag{6.5}$$

By N^i in (6.5) we denote components of unit vector $\mathbf{N}$ directed along vector $\mathbf{v}$. Using its components, we define two operators of orthogonal projection $\mathbf{Q}$ and $\mathbf{P}$:

$$Q^i_k = \frac{v^i\,v_k}{|\mathbf{v}|^2}, \qquad\qquad P^i_k = \delta^i_k - \frac{v^i\,v_k}{|\mathbf{v}|^2}. \tag{6.6}$$

First of them is a projector to the direction of vector $\mathbf{v}$, second is a projector to hyperplane perpendicular to $\mathbf{v}$. Projection operators $\mathbf{Q}$ and $\mathbf{P}$ with components (6.6) are complementary to each other, this means that

$$\mathbf{Q} + \mathbf{P} = \mathbf{1} \quad \text{and} \quad \mathbf{Q} \circ \mathbf{P} = \mathbf{P} \circ \mathbf{Q} = \mathbf{0}.$$

Now let's consider second equation (5.4). For $\nabla_t(\tilde{\nabla}_i L)$ in it we have

$$\nabla_t(\tilde{\nabla}_i L) = \sum_{k=1}^{n} \nabla_k \tilde{\nabla}_i L \cdot \dot{x}^k + \sum_{k=1}^{n} \tilde{\nabla}_k \tilde{\nabla}_i L \cdot \nabla_t v^k. \tag{6.7}$$

By direct calculations for various derivatives in (6.7) we obtain

$$\tilde{\nabla}_i L = L' \cdot \frac{v_i}{|\mathbf{v}|} = L' \cdot N_i, \tag{6.8}$$

$$\nabla_k \tilde{\nabla}_i L = \nabla_k L' \cdot N_i, \tag{6.9}$$

$$\tilde{\nabla}_k \tilde{\nabla}_i L = L'' \cdot Q_{ik} + \frac{L'}{|\mathbf{v}|} \cdot P_{ik}. \tag{6.10}$$

Substituting (6.9) and (6.10) into (6.7), we get the following equality:

$$\nabla_t(\tilde{\nabla}_i L) = \sum_{k=1}^{n} \frac{\nabla_k L'}{L'} \cdot Q^k_i + \sum_{k=1}^{n} \left(L'' \cdot Q_{ik} + \frac{L'}{|\mathbf{v}|} \cdot P_{ik} \right) \cdot \nabla_t v^k. \tag{6.11}$$

Next step is to relate $\nabla_t v^k$ in (6.11) with $\nabla_t u^k$. For this purpose let's apply ∇_t to the first equality in (6.5). As a result of direct calculations we get:

$$\nabla_t u^k = \sum_{s=1}^{n} \left(\frac{1}{|\mathbf{v}| \cdot L'} \cdot P^k_s - \frac{L''}{(L')^2} \cdot Q^k_s \right) \cdot \nabla_t v^s - \sum_{s=1}^{n} \frac{\nabla_s L'}{(L')^3} \cdot Q^{sk}. \tag{6.12}$$

In order to express $\nabla_t v^s$ through $\nabla_t u^k$ we should invert matrix D with components

$$D^k_s = \frac{1}{|\mathbf{v}| \cdot L'} \cdot P^k_s - \frac{L''}{(L')^2} \cdot Q^k_s.$$

This matrix is invertible if and only if $L'' \neq 0$. The condition $L' \neq 0$ is fulfilled since $\Omega = |\mathbf{v}| \cdot L' \neq 0$. This follows from transversality condition (2.18). Inverting matrix D and applying it to (6.12), we get the following expression for $\nabla_t v^k$:

$$\nabla_t v^k = \sum_{s=1}^{n} \left(|\mathbf{v}| \cdot L' \cdot P^k_s - \frac{(L')^2}{L''} \cdot Q^k_s \right) \cdot \nabla_t u^s - \sum_{s=1}^{n} \frac{\nabla_s L'}{L' \cdot L''} \cdot Q^{sk}. \tag{6.13}$$

Now let's substitute (6.13) back to right hand side of (6.11). As a result we get

$$\nabla_t(\tilde{\nabla}_i L) = (L')^2 \sum_{k=1}^{n} (P_{ik} - Q_{ik}) \cdot \nabla_t u^k. \tag{6.14}$$

Then let's substitute (6.14) into second equation (5.4) and let's use formula (6.4) for denominator Ω in it. This leads to the equation for $\nabla_t u^k$:

$$(L')^2 \sum_{k=1}^{n} (P_{ik} - Q_{ik}) \cdot \nabla_t u^k = \frac{\nabla_i L}{|\mathbf{v}| \cdot L'}. \tag{6.15}$$

Equation (6.15) can be explicitly solved. For $\nabla_t u^k$ from (6.15) we derive

$$\nabla_t u^k = \sum_{i=1}^{n} \frac{\nabla_i L}{|\mathbf{v}| \cdot (L')^3} \cdot \bigl(g^{ik} - 2\, N^i\, N^k\bigr). \tag{6.16}$$

Comparing (6.16) with (6.3) we see that right hand side of (6.16) is the expression for F^k. Formula for covariant components F_k is more elegant:

$$F_k = \sum_{i=1}^{n} \frac{\nabla_i L}{|\mathbf{v}| \cdot (L')^3} \cdot \bigl(\delta^i_k - 2\, N^i\, N_k\bigr)\,. \tag{6.17}$$

The only problem now is that $\nabla_i L$, L', and $|\mathbf{v}|$ in right hand side of (6.17) are given in $\mathbf{v}$-representation, while components of force vector in (6.3) should depend on components of actual velocity $\mathbf{u}$. Our last effort is to transform (6.17) to $\mathbf{u}$-representation, using map (6.2).

Let's denote by h Hamilton function H in $\mathbf{v}$-representation, i. e. $h = H \circ \lambda$. Then for h we have the following formula similar to formula (4.6) for l:

$$h = \sum_{i=1}^{n} v^i \cdot \tilde{\nabla}_i L - L = |\mathbf{v}| \cdot L' - L. \tag{6.18}$$

Differentiating (6.18), we obtain

$$\nabla_i h = v \cdot \nabla_i L' - \nabla_i L, \qquad\qquad h' = v \cdot L''. \tag{6.19}$$

Let W be $\mathbf{u}$-representation for h. Denote by u modulus of actual velocity vector. From (6.5) and (6.18) we conclude that W is a fiberwise spherically symmetric scalar field in $\mathbf{u}$-representation: $W = W(x^1, \ldots, x^n, u)$, where $u = |\mathbf{u}|$. Now let's calculate derivatives of W. For W' and $\nabla_i W$ we get:

$$\begin{aligned} W' &= \frac{\partial W}{\partial u} = \frac{\partial h}{\partial v} \cdot \frac{\partial v}{\partial u} = h' \cdot \left(\frac{\partial u}{\partial v}\right)^{-1}, \\ \nabla_i W &= \frac{\partial W}{\partial x^i} = \nabla_i h - h' \cdot \frac{\partial u}{\partial x^i} \cdot \left(\frac{\partial u}{\partial v}\right)^{-1}. \end{aligned} \tag{6.20}$$

Now let's remember that u and v are related by second equality in (6.5). We write it as $u=\varepsilon/L'$, where $\varepsilon=\pm 1$ depending on sign of L'. Then we get

$$\frac{\partial u}{\partial v}=-\frac{\varepsilon\cdot L''}{(L')^2},\qquad\qquad \frac{\partial u}{\partial x^i}=-\frac{\varepsilon\cdot\nabla_i L'}{(L')^2}.\tag{6.21}$$

Combining (6.19), (6.20), and (6.21), we can transform (6.22) as follows:

$$W'=-\varepsilon\cdot|\mathbf{v}|\cdot(L')^2,\qquad\qquad \nabla_i W=-\nabla_i L.\tag{6.22}$$

If we write second equality (6.5) as $|\mathbf{u}|=\varepsilon/L'$, then from (6.22) we derive:

$$|\mathbf{u}|\cdot\frac{\nabla_i W}{W'}=\frac{\nabla_i L}{|\mathbf{v}|\cdot(L')^3}.\tag{6.23}$$

Comparing the equality (6.23) with formula (6.17) for components of force vector $\mathbf{F}$, we can rewrite this formula in the following form:

$$F_k=-|\mathbf{u}|\cdot\sum_{i=1}^{n}\frac{\nabla_i W}{W'}\cdot\left(2\,N^i\,N_k-\delta^i_k\right).\tag{6.24}$$

Now formula (6.24) is completely compatible with (6.3). Its right hand side is given in $\mathbf{u}$-representation as it is required in (6.3).

Formula determines force field of Newtonian dynamical system (6.3) describing the dynamics of wave front for linear wave equation (1.4). But most important fact for us is that formula (6.24) gives the link to the theory of Newtonian dynamical systems admitting normal shift of hypersurfaces. This theory was developed in series of papers [6–21], on the base of which theses [4] and [22] were prepared. According to results of Chapter VII of thesis [4] (see also paper [5]), force field of any Newtonian dynamical system admitting normal shift of hypersurfaces in Riemannian manifold of the dimension $n\geqslant 3$ is given by explicit formula:

$$F_k=\frac{h(W)\cdot N_k}{W'}-|\mathbf{u}|\cdot\sum_{i=1}^{n}\frac{\nabla_i W}{W'}\cdot\left(2\,N^i\,N_k-\delta^i_k\right).\tag{6.25}$$

Here $W=W(x^1,\ldots,x^n,|\mathbf{u}|)$ is arbitrary fiberwise spherically symmetric extended scalar field with $W'\neq 0$, and $h=h(w)$ is arbitrary function of one variable.

7. Conclusions.

Comparing formulas (6.24) and (6.25), we see that they do coincide for the case $h=0$. Term with $h=h(W)$ in (6.25) is responsible for energy dissipation and energy pumping phenomena. In the absence of these phenomena, i. e. for $h=0$, we can state the following main results of present paper:

1) conservative dynamical systems admitting normal shift of hypersurfaces in Riemannian manifold of the dimension $n\geqslant 3$ coincide with those describing wave front dynamics in quasiclassical limit for wave operators with fiberwise spherically symmetric symbol H;
2) these systems are not Hamiltonian, but they are very close to Hamiltonian systems (see equations (3.4) above);
3) they possess first integral, which can be interpreted as energy (or as Hamilton function in $\mathbf{p}$-representation).

Third result that W is a first integral of Newtonian dynamical system with force field (6.24) was already known (see paper [5] or Chapter VII of thesis [4]). But its interpretation was not so clear as it is now.

Note that the equations of wave front dynamics do not exhaust the whole class of Newtonian dynamical systems admitting normal shift of hypersurfaces in M. The problem of proper interpretation for non-conservative term

$$\frac{h(W)\cdot N_k}{W'}$$

in formula (6.25) in the sense of theory from [1] and (23) is still open. This problem should be studied in separate paper. Methods developed in book [24] might appear useful for this purpose.

Remark. Technique based on usage of extended tensor fields is not new (see for instance book [25]). And, as noted by Prof. Mircea Crasmareanu, our terminology above is not standard. However, we do not change this terminology in order to keep integrity of present paper and our previous papers (see [3-22] and some other papers in Electronic Archive at LANL[1]).

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[2]For the convenience of reader we give direct reference to archive file. This is the following URL address: **http://arXiv.org/eprint/math.DG/0002202**.

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[1]Papers [6–21] are arranged here in the order they were written. However, the order of publication not always coincides with the order of writing.