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ORTHOGONAL MATRICES WITH RATIONAL COMPONENTS IN COMPOSING TESTS FOR HIGH SCHOOL STUDENTS.

Ruslan A. Sharipov

Abstract. Fermat Last Theorem, which inspired mathematicians during 300 years, is proved by Andrew Wiles. Even among mathematicians there is a narrow circle of specialists, who can read this proof and understand all details. Is it a reason for pessimism? No, since arithmetics if entire numbers contains broad variety of problems with a simple statement, which might be not less intricate. One of them arises in elementary geometry.

1. Elementary problem on pyramid.

Primary education (Elementary School) and secondary education (High School) in Russia are united into one stage that now lasts 11 years (from 6 year old to 17 years old). Mathematics is among disciplines studied during these years. Below we consider a problem, which can be suggested to 10-th or 11-th year students in the course of geometry. It is typical, though is a little more complicated than usual.

Problem on pyramid. *In triangular pyramid $ABCD$ three sides of triangle ABC in its base are given:*

$$|BC| = a, \quad |CA| = b, \quad |AB| = c.$$

From corners A and B two perpendiculars are drown to the faces BCD and ACD respectively. Their lengths are given:

$$|AF| = f, \qquad |BG| = g.$$

Find the length of the segment $[FG]$ connecting feet of these two perpendiculars $[AF]$ and $[BG]$.

Let's consider in brief the steps leading to the solution of this problem. First we draw all three heights in triangle ABC: these are segments $[AH]$, $[BK]$, $[CM]$. It's known that they cross at one point. Denote it by L. For the sake of simplicity we consider the case when triangle ABC is acute-angled. In this case point L lies inside the triangle ABC. Now let's

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apply Pythagor's theorem to rectangular triangles AMC and BMC. As a result we obtain the system of equations with respect to the length of the segment $[AM]$:

$$\begin{cases} |AM|^2+|CM|^2=|AC|^2,\\ (|AB|-|AM|)^2+|CM|^2=|BC|^2. \end{cases} \tag{1.1}$$

Solving the system of equations (1.1), for lengths of $[AM]$ and $[BM]$ we get

$$\begin{aligned} |AM|&=\frac{|AB|^2+|AC|^2-|BC|^2}{2\,|AB|},\\ |BM|&=\frac{|AB|^2+|BC|^2-|AC|^2}{2\,|AB|}. \end{aligned} \tag{1.2}$$

Similar formulas can be obtained for $|AK|$, $|KC|$, $|BH|$, and $|HC|$:

$$\begin{aligned} |AK|&=\frac{|AC|^2+|AB|^2-|BC|^2}{2\,|AC|},\\ |CK|&=\frac{|AC|^2+|BC|^2-|AB|^2}{2\,|AC|},\\ |BH|&=\frac{|BC|^2+|AB|^2-|AC|^2}{2\,|BC|},\\ |CH|&=\frac{|BC|^2+|AC|^2-|AB|^2}{2\,|BC|}. \end{aligned}$$

Let's replace $|AB|-|AM|$ by $|BM|$ in the second equation of the system (1.1). Then we can derive the following formula for the length of segment $[CM]$:

$$|CM|=\sqrt{\frac{|AC|^2+|BC|^2-|AM|^2-|BM|^2}{2}}. \tag{1.3}$$

Similar formulas can be derived for the lengths of segments $[AH]$ and $[BK]$:

$$\begin{aligned} |AH|&=\sqrt{\frac{|AB|^2+|AC|^2-|CH|^2-|BH|^2}{2}},\\ |BK|&=\sqrt{\frac{|AB|^2+|BC|^2-|AK|^2-|CK|^2}{2}}. \end{aligned} \tag{1.4}$$

In order to calculate lengths of segments $[KL]$ and $[HL]$ we use similarity of triangles: $\triangle KLC\sim\triangle MAC$ and $\triangle HLC\sim\triangle MBC$. This yields:

$$|KL|=\frac{|AM|}{|CM|}\,|KC|, \qquad |HL|=\frac{|BM|}{|CM|}\,|HC|. \tag{1.5}$$

Now let's draw segments $[FH]$ and $[GK]$. According to the theorem on three perpendiculars, we have $GK \perp AC$ and $FH \perp BC$. Then, since we already know $|AF|$ and $|BG|$, we can calculate lengths of segments $[FH]$ and $[GK]$:

$$|FH| = \sqrt{|AC|^2 - |CH|^2 - |AF|^2},$$

$$|GK| = \sqrt{|BC|^2 - |CK|^2 - |BG|^2}.$$

In order to derive first of these two expressions we applied Pythagor's theorem to rectangular triangles AHC and AHF. Second expression is derived by Pythagor's theorem applied to triangles BKC and BKG.

Orthogonal projections of the points F and G onto the plane of the base of pyramid belong to the straight lines AH and BK. Denote these projections by $\tilde{F}$ and $\tilde{G}$ respectively. For the sake of simplicity we consider the case when points F and G are above the base of pyramid (i. e. in upper halfspace separated by the plane ABC), and when their projections $\tilde{F}$ and $\tilde{G}$ belong to the segments $[HL]$ and $[KL]$ respectively (see Fig. 1.2 and Fig. 1.3). Due to similarity of triangles $\triangle GK\tilde{G} \sim \triangle KBG$ and $\triangle HF\tilde{F} \sim \triangle FAH$ we derive the following formulas:

$$|G\tilde{G}| = \frac{|BG|\,|GK|}{|BK|}, \qquad |F\tilde{F}| = \frac{|AF|\,|FH|}{|AH|}, \tag{1.6}$$

$$|K\tilde{G}| = \frac{|GK|^2}{|BK|}, \qquad |H\tilde{F}| = \frac{|FH|^2}{|AH|}. \tag{1.7}$$

The length of the segment $[\tilde{G}\tilde{F}]$ (see Fig. 1.4 below) is determined by cosine theorem applied to the triangle $\tilde{G}L\tilde{F}$:

$$|\tilde{G}\tilde{F}| = \sqrt{|L\tilde{G}|^2 + |L\tilde{F}|^2 - 2\,|L\tilde{G}|\,|L\tilde{F}|\,\cos(\widehat{KLH})} \tag{1.8}$$

Note that angles $\angle KLH$ and $\angle KCH$ complete each other to a straight angle. Indeed, triangle LKC is rectangular (see Fig. 1.4 below). Same is true for triangle LHC. Hence for the angles of these two triangles we can write the equalities:

$$\widehat{KLC} + \widehat{KCL} = 90^\circ,$$

$$\widehat{HLC} + \widehat{HCL} = 90^\circ.$$

Adding these equalities and taking into account that $\widehat{KLC}+\widehat{HLC}=\widehat{KLH}$ and $\widehat{KCL}+\widehat{HCL}=\widehat{KCH}$, we get the required equality $\widehat{KLH}+\widehat{KCH}=180^\circ$. As an immediate consequence of this equality we can write the equality for cosines:

$$\cos(\widehat{KLH})=-\cos(\widehat{KCH}).$$

Cosine of the angle $\widehat{KCH}$ can be determined by applying cosine theorem to the triangle ABC, which lies in the base of pyramid $ABCD$:

$$\cos(\widehat{KCH})=\frac{|AC|^2+|BC|^2-|AB|^2}{2\,|AC|\,|BC|}.$$

Lengths of segments $[L\tilde{F}]$ and $[L\tilde{G}]$ in formula (1.8) can be calculated as follows:

$$|L\tilde{F}|=|LH|-|H\tilde{F}|, \qquad\qquad |L\tilde{G}|=|LK|-|K\tilde{G}|.$$

This is obvious from Fig. 1.2 and Fig. 1.3. Now the length of segment $[FG]$, which was to be found, is calculated by Pythagor's theorem (see Fig. 1.5):

$$|FG|=\sqrt{|\tilde{F}\tilde{G}|^2+(|G\tilde{G}|-|F\tilde{F}|)^2}.$$

So **problem on pyramid** is solved. This is typical stereometric problem that can be used to test the knowledge of some basic facts and spatial imagination of students. Its solution considered just above is not tricky. But it is rather huge, and we cannot write simple explicit formula expressing $|FG|$ through parameters a, b, c, f, and g. Therefore we should give numeric values for these parameters, choosing them so that they provide simple numeric values for ultimate result and for results of all intermediate calculations. Thus, another problem arises, problem of choosing proper numeric values for a, b, c, f, and g. We shall consider this problem below.

2. Orthogonal matrices.

Let's apply the coordinate method to the problem on pyramid. Here we have two natural triples of orthogonal vectors. First consists of vectors $\overrightarrow{AF}$, $\overrightarrow{FH}$, $\overrightarrow{HC}$, second is formed by vectors $\overrightarrow{BG}$, $\overrightarrow{GK}$, and $\overrightarrow{KC}$. Let's consider three unitary vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ directed along vectors $\overrightarrow{AF}$, $\overrightarrow{FH}$, and $\overrightarrow{HC}$. Then choose other three unitary vectors directed along vectors $\overrightarrow{BG}$, $\overrightarrow{GK}$, and $\overrightarrow{KC}$. Vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ and $\mathbf{h}_1$, $\mathbf{h}_2$, $\mathbf{h}_3$ form two bases consisting of unitary vectors orthogonal to each other. Such bases are called **orthonormal bases** (ONB). Let's consider the following expansions binding vectors of two ONB's:

$$\mathbf{h}_i = \sum_{i=1}^{3} S^k_i \, \mathbf{e}_k, \qquad \mathbf{e}_k = \sum_{i=1}^{3} T^j_k \, \mathbf{h}_j. \tag{2.1}$$

Coefficients of the expansions (2.1) are usually arranged into square matrices, which are called **transition matrices**:

$$S = \begin{Vmatrix} S^1_1 & S^1_2 & S^1_3 \\ S^2_1 & S^2_2 & S^2_3 \\ S^3_1 & S^3_2 & S^3_3 \end{Vmatrix}, \qquad T = \begin{Vmatrix} T^1_1 & T^1_2 & T^1_3 \\ T^2_1 & T^2_2 & T^2_3 \\ T^3_1 & T^3_2 & T^3_3 \end{Vmatrix}. \tag{2.2}$$

Matrices S and T in (2.2) implement direct and inverse transitions from base to base, they are inverse to each other, i. e. their product is a unitary matrix:

$$S \cdot T = T \cdot S = E.$$

If we treat (2.1) as transition from the base $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ to the base $\mathbf{h}_1$, $\mathbf{h}_2$, $\mathbf{h}_3$, then S is called **direct transition matrix**, while T is called **inverse transition matrix**.

Matrices S and T in our case are binding two ONB's. Therefore components of these matrices are bound by a series of relationships. If we denote by S^t and T^t transposed matrices, i. e. if we denote

$$S^t = \begin{Vmatrix} S^1_1 & S^2_1 & S^3_1 \\ S^1_2 & S^2_2 & S^3_2 \\ S^1_3 & S^2_3 & S^3_3 \end{Vmatrix}, \qquad T^t = \begin{Vmatrix} T^1_1 & T^2_1 & T^3_1 \\ T^1_2 & T^2_2 & T^3_2 \\ T^1_3 & T^2_3 & T^3_3 \end{Vmatrix},$$

then these relationships for components of S and T can be written as follows:

$$S^t \cdot S = E, \qquad T^t \cdot T = E. \tag{2.3}$$

From (2.3) and from $S \cdot T = T \cdot S = E$ we immediately derive $S^t = T$ and $T^t = S$.

Matrices that satisfy the relationships (2.3) are called **orthogonal matrices**. Sum of squares of elements in each column and in each string of orthogonal matrix is equal to 1. So we have the relationships

$$\sum_{i=1}^{3} (S^i_k)^2 = \sum_{i=1}^{3} (S^k_i)^2 = 1 \quad \text{for all} \quad k = 1,\, 2,\, 3. \tag{2.4}$$

Sums of products of elements from different columns and/or different string are equal to zero. This property is expressed by the relationships

$$\sum_{i=1}^{3} S^i_k\,S^i_q=\sum_{i=1}^{3} S^k_i\,S^q_i=0\ \ \text{for}\ \ k\neq q. \tag{2.5}$$

The relationships (2.4) and (2.5) are easily derived from (2.3). Moreover, from (2.3) one can derive the following relationships for determinants of S and T:

$$(\det S)^2=1, \qquad\qquad (\det T)^2=1.$$

Therefore $\det S=\det T=\pm 1$. Looking attentively at Fig. 1, one can note that $\overrightarrow{AF}$, $\overrightarrow{FH}$, $\overrightarrow{HC}$ and $\overrightarrow{BG}$, $\overrightarrow{GK}$, $\overrightarrow{KC}$ are oppositely oriented triples of vectors: first is **left**, while second is **right**. Hence bases $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ and $\mathbf{h}_1$, $\mathbf{h}_2$, $\mathbf{h}_3$ are also oppositely oriented. This fact is reflected by the sign of determinants of transition matrices:

$$\det S=\det T=-1. \tag{2.6}$$

Further we shall be interested in the case when all components of matrices S and T are rational numbers. Components of S belonging to the same column can be brought to common denominator, and hence, they can be written as

$$\begin{aligned}
&S^1_1=\frac{p_1}{d_1}, &&S^2_1=\frac{p_2}{d_1}, &&S^3_1=\frac{p_3}{d_1},\\
&S^1_2=\frac{q_1}{d_2}, &&S^2_2=\frac{q_2}{d_2}, &&S^3_2=\frac{q_3}{d_2},\\
&S^1_3=\frac{r_1}{d_3}, &&S^2_3=\frac{r_2}{d_3}, &&S^3_3=\frac{r_3}{d_3}.
\end{aligned} \tag{2.7}$$

From (2.4) for entire numbers p_1, p_2, p_3, and d_1 in (2.7) we derive the relationship

$$(p_1)^2+(p_2)^2+(p_3)^2=(d_1)^2. \tag{2.8}$$

If four entire numbers satisfy the relationship (2.8), we say that they form **Pythagorean tetrad**. Each column in orthogonal matrix with rational components is related with some Pythagorean tetrad of entire numbers. Thus, in (2.7) we have three Pythagorean tetrads determined by transition matrix S:

$$(p_1,p_2,p_3,d_1), \qquad (q_1,q_2,q_3,d_2), \qquad (r_1,r_2,r_3,d_3). \tag{2.9}$$

Pythagorean tetrads of entire numbers (2.9) are orthogonal to each other in the sense of the following relationships:

$$\begin{aligned}
&p_1\,q_1+p_2\,q_2+p_3\,q_3=0,\\
&p_1\,r_1+p_2\,r_2+p_3\,r_3=0,\\
&r_1\,q_1+r_2\,q_2+r_3\,q_3=0.
\end{aligned}$$

In order to determine an orthogonal matrix it's sufficient to have two orthogonal Pythagorean tetrads, for instance, (p_1, p_2, p_3, d_1) and (q_1, q_2, q_3, d_2). Third Pythagorean tetrad then will be determined by the relationships

$$\frac{r_1}{d_3} = -\frac{\begin{vmatrix} p_2 & p_3 \\ q_2 & q_3 \end{vmatrix}}{d_1\,d_2}, \qquad \frac{r_2}{d_3} = \frac{\begin{vmatrix} p_1 & p_3 \\ q_1 & q_3 \end{vmatrix}}{d_1\,d_2}, \qquad \frac{r_3}{d_3} = -\frac{\begin{vmatrix} p_1 & p_2 \\ q_1 & q_2 \end{vmatrix}}{d_1\,d_2}.$$

This is the consequence of the fact that third vector in orthonormal bases (ONB's) are determined by vector product of first two vectors:

$$\mathbf{e}_3 = -[\mathbf{e}_1,\,\mathbf{e}_2], \qquad\qquad \mathbf{h}_3 = [\mathbf{h}_1,\,\mathbf{h}_2].$$

The difference in sign here is due to the condition (2.6), which expresses difference in orientations of bases $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ and $\mathbf{h}_1$, $\mathbf{h}_2$, $\mathbf{h}_3$.

Returning to the problem on pyramid, we use the fact that vectors $\overrightarrow{AF}$, $\overrightarrow{FH}$ and $\overrightarrow{HC}$ are collinear to base vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$:

$$\overrightarrow{AF} = \alpha_1 \cdot \mathbf{e}_1, \qquad \overrightarrow{FH} = \alpha_2 \cdot \mathbf{e}_2, \qquad \overrightarrow{HC} = \alpha_3 \cdot \mathbf{e}_3. \tag{2.10}$$

Similarly, vectors $\overrightarrow{BG}$, $\overrightarrow{GK}$, and $\overrightarrow{KC}$ are collinear to base vectors $\mathbf{h}_1$, $\mathbf{h}_2$, $\mathbf{h}_3$:

$$\overrightarrow{BG} = \beta_1 \cdot \mathbf{h}_1, \qquad \overrightarrow{GK} = \beta_2 \cdot \mathbf{h}_2, \qquad \overrightarrow{KC} = \beta_3 \cdot \mathbf{h}_3. \tag{2.11}$$

Vector $\overrightarrow{BC}$ is collinear to vector $\mathbf{e}_3$, while vector $\overrightarrow{AC}$ is collinear to vector $\mathbf{h}_3$:

$$\overrightarrow{BC} = \omega \cdot \mathbf{e}_3, \qquad\qquad \overrightarrow{AC} = \sigma \cdot \mathbf{h}_3. \tag{2.12}$$

Let's choose parameters ω and σ in (2.12) to be rational numbers, and then let's apply the relationships (2.1). As a result for vector $\overrightarrow{AC}$ we obtain two expansions:

$$\begin{aligned} &\overrightarrow{AC} = \overrightarrow{AF} + \overrightarrow{FH} + \overrightarrow{HC}, \\ &\overrightarrow{AC} = \sigma \cdot (S^1_3\,\mathbf{e}_1 + S^2_3\,\mathbf{e}_2 + S^3_3\,\mathbf{e}_3). \end{aligned} \tag{2.13}$$

Substituting (2.10) into (2.13) and comparing two expansions (2.13), we obtain

$$\alpha_1 = \sigma\,S^1_3, \qquad \alpha_2 = \sigma\,S^2_3, \qquad \alpha_3 = \sigma\,S^3_3. \tag{2.14}$$

In a similar way, from (2.10) and (2.13) due to (2.1) and due to the expansion $\overrightarrow{BC} = \overrightarrow{BG} + \overrightarrow{GK} + \overrightarrow{KC}$ we can derive the following three relationships:

$$\beta_1 = \omega\,T^1_3, \qquad \beta_2 = \omega\,T^2_3, \qquad \beta_3 = \omega\,T^3_3. \tag{2.15}$$

If components of transition matrix S are rational numbers, then components of inverse transition matrix $T = S^t$ are also rational. Therefore from (2.14) and (2.15) we obtain rationality of numeric coefficients α_2, α_2, α_3 and β_1, β_2, β_3 in

(2.10) and (2.11). This, in turn, provides rationality of lengths of segments $[AC]$, $[BC]$, $[AK]$, $[CK]$, $[BH]$, $[CH]$, $[BG]$, $[GK]$, $[AF]$, and $[FH]$.

Let's consider the vector $\overrightarrow{FG}$, length of which is a final result in the problem on pyramid. For this vector we have an expansion:

$$\overrightarrow{FG}=\overrightarrow{FH}+\overrightarrow{HC}-\overrightarrow{KC}-\overrightarrow{GK}. \tag{2.16}$$

Let's substitute (2.10) and (2.11) into the expansion (2.16). This yields

$$\begin{aligned}\overrightarrow{FG}&=\alpha_2\,\mathbf{e}_2+\alpha_3\,\mathbf{e}_3-\beta_2\,\mathbf{h}_2-\beta_3\,\mathbf{h}_3=\alpha_2\,\mathbf{e}_2+\alpha_3\,\mathbf{e}_3-\\&-\,\beta_2\,(S^1_2\,\mathbf{e}_1+S^2_2\,\mathbf{e}_2+S^3_2\,\mathbf{e}_3)-\beta_3\,(S^1_3\,\mathbf{e}_1+S^2_3\,\mathbf{e}_2+S^3_3\,\mathbf{e}_3).\end{aligned}$$

Now it's easy to see that vector $\overrightarrow{FG}$ has rational coordinates in orthonormal base (ONB) formed by vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$. Therefore its length, in the worst case, is simplest irrational number obtained as a **square root of rational number**. The same is true for lengths of segments $[AB]$, $[AM]$, $[BM]$, $[CM]$, $[KL]$, $[HL]$, $[AH]$, $[BK]$, as well as for the lengths of segments $[F\tilde{F}]$, $[H\tilde{F}]$, $[G\tilde{G}]$, and $[K\tilde{G}]$. For $|AB|$ this follows from the equality $\overrightarrow{AB}=\overrightarrow{AC}-\overrightarrow{BC}$. Further we use formulas (1.2), (1.3), (1.5), (1.4); then formulas (1.6) and (1.7). Main conclusion that we draw from what was said above is the following: **orthogonal matrices with rational components** give the **algorithm** for choosing numeric values of parameters a, b, c, f, g in the problem on pyramid such that we get simple final result in this problem and simple results in all intermediate calculations.

3. Constructing orthogonal matrices with rational components.

Constructing orthogonal matrices with rational components is a separate problem. First we consider regular algorithm for constructing such matrices. It is based on elementary rotations. Let's consider three entire numbers p_1, p_2, d, and suppose that they are bound by the relationship

$$(p_1)^2+(p_2)^2=d^2. \tag{3.1}$$

Such numbers form **Pythagorean triad**. In contrast to Pythagorean tetrads they are well-known. There is a regular algorithm for constructing all Pythagorean triads (see, for instance, [1]). If τ is a greatest common divisor of p_1, p_2, and q, then $p_1=\tau\cdot\tilde{p}_1$, $p_2=\tau\cdot\tilde{p}_2$, $d=\tau\cdot\tilde{d}$. From (3.1) we derive

$$(\tilde{p}_1)^2+(\tilde{p}_2)^2=\tilde{d}^2.$$

If $\tilde{p}_1$ is even and $\tilde{p}_2$ is odd, then $\tilde{d}$ is odd. According to the regular algorithm described in [1], in this case we have the following expressions:

$$\begin{aligned}&\tilde{p}_1=2\,(m^2+m-n^2-n),\\&\tilde{p}_2=4\,m\,n+2\,m+2\,n+1,\\&\tilde{d}=2\,(m^2+m+n^2+n)+1.\end{aligned}$$

Here m and n are two arbitrary entire numbers. So Pythagorean triads are parameterized by three arbitrary entire numbers: m, n, and τ. Suppose that we have some nonzero Pythagorean triad (p_1, p_2, d). Then we can consider two rational numbers p_1/d and p_2/d, sum of their squares being equal to unity:

$$\left(\frac{p_1}{d}\right)^2 + \left(\frac{p_2}{d}\right)^2 = 1.$$

Hence in half-open interval $[0,\, 2\pi)$ there exists some angle φ such that

$$\cos\varphi = \frac{p_1}{d}, \qquad \sin\varphi = \frac{p_2}{d}. \tag{3.2}$$

Angle φ in (3.2) is uniquely determined by numbers p_1, p_2, and d forming Pythagorean triad (p_1, p_2, q). Let's use this angle in order to define four matrices:

$$\begin{aligned} &S^{[x]}_{\varphi} = \left\Vert\begin{matrix} \cos\varphi & \sin\varphi & 0\\ -\sin\varphi & \cos\varphi & 0\\ 0 & 0 & 1\end{matrix}\right\Vert, &&S^{[y]}_{\varphi} = \left\Vert\begin{matrix} \cos\varphi & 0 & \sin\varphi\\ 0 & 1 & 0\\ -\sin\varphi & 0 & \cos\varphi\end{matrix}\right\Vert,\\ &S^{[z]}_{\varphi} = \left\Vert\begin{matrix} 1 & 0 & 0\\ 0 & \cos\varphi & \sin\varphi\\ 0 & -\sin\varphi & \cos\varphi\end{matrix}\right\Vert, &&S^{*} = \left\Vert\begin{matrix} -1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{matrix}\right\Vert. \end{aligned} \tag{3.3}$$

Matrices $S^{[x]}_{\varphi}$, $S^{[y]}_{\varphi}$, $S^{[z]}_{\varphi}$ are geometrically interpreted as matrices of elementary rotations to the angle φ around coordinate axes. They arise as transition matrices in (2.1) in that case when base $\mathbf{h}_1$, $\mathbf{h}_2$, $\mathbf{h}_3$ is got from base $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ by one of such elementary rotations. Matrix S^* in (3.3) is interpreted as a matrix of inversion. It arises as transition matrix in (2.1) when

$$\mathbf{h}_1 = -\mathbf{e}_1, \qquad \mathbf{h}_2 = -\mathbf{e}_2, \qquad \mathbf{h}_3 = -\mathbf{e}_3.$$

All four matrices (3.3) are orthogonal. This can be checked by substituting them into (2.3). If φ is determined by the relationships (3.2), then all components of matrices (3.3) are rational. Product of two orthogonal matrices is an orthogonal matrix (it's well-known that such matrices form a group). Therefore, choosing n Pythagorean triads and determining angles $\varphi_1, \ldots, \varphi_n$ by relationships (3.2), we can consider the following product of corresponding matrices (3.3):

$$S = (S^*)^{\varepsilon} \cdot \prod_{i=1}^{n} (S^{[x]}_{\varphi_i})^{\alpha_i} \cdot (S^{[y]}_{\varphi_i})^{\beta_i} \cdot (S^{[z]}_{\varphi_i})^{\gamma_i}. \tag{3.4}$$

Here ε, α_i, β_i, γ_i are entire numbers either equal to zero or to unity. Formula (3.4) gives an algorithm for constructing orthogonal matrices with rational components in dimension 3. We shall call it a **regular algorithm**.

4. Some generalizations and open questions.

Leonard Euler considered a class of matrices, which is a little more wide than class determined by the relationships (2.3). Euler's class is formed by matrices S with entire components that satisfy the following condition:

$$S^t \cdot S = N \cdot E. \tag{4.1}$$

Here N is some positive entire number. Matrices of Euler's class are called **entire orthogonal matrices**, number N is called a **norm of orthogonality**. If N is a square of entire number, i. e. if $N = M^2$, then matrix $M^{-1} \cdot S$ is an orthogonal matrix with rational components in the sense of standard definition by formulas (2.3). Leonard Euler has suggested an algorithm for constructing entire orthogonal matrices in the dimensions 3 and 4. His algorithm is described in book [2]. In papers [3–5] Euler's algorithm was generalized for $n \times n$ matrices in arbitrary dimension n. Due to the existence of two algorithms we have a series of quite natural questions.

— How do Euler's algorithm relate with regular algorithm, which is expressed by above formula (3.4) ?
— Can we construct an arbitrary orthogonal matrix with rational components by Euler's algorithm ?
— Is there the expansion (3.4) for an arbitrary orthogonal matrix with rational components, i. e. can it be constructed by regular algorithm ?

Answers to these questions are unknown to the author of this paper. Author will be grateful for any information concerning subject of this paper.

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Rabochaya str. 5, 450003, Ufa, Russia
E-mail address: R_Sharipov@ic.bashedu.ru
ruslan-sharipov@usa.net
URL: http://www.geocities.com/CapeCanaveral/Lab/5341

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