CHAPTER I

INTRODUCTION.

§ 1. Classical constructions of surface transformation in three-dimensional space.

Interest to geometry of surfaces in the beginning of XIX-th century was stimulated by practical tasks of a geodesy. For the military purposes and for calculation of taxes the exact geographical maps were required. It is known, that Gauss since 1816 was engaged in organization of geodesic shooting of the Hanover kingdom, his own measurements were made in 1821-1825, while whole works lasted till 1841 (see [1], chapter I). It was the very time when Gauss obtained his basic results in the theory of surfaces [2] (including the theorem of invariance of Gaussian curvature under the bending of a surface without stretching).

Here we consider later time, when differential geometry has been developed into a separate mathematical discipline. In papers of Bonnet (see [3–5]), Bianchi [6], Sophus Lie [7], Bäcklund [8] and Darboux [9] special transformations of surfaces in three-dimensional Euclidean space were constructed. Totally in these papers four transformations were determined: **Bonnet** transformation, **Bianchi and Lie** transformation, **Bäcklund** transformation and **Darboux** transformation. In each of four above constructions pairs of surfaces S and $\tilde{S}$ connected by a transformation $f\colon S \to \tilde{S}$ are considered. The appropriate points of two surfaces are bound by a segment of straight line: point A on S is bound with a point $\tilde{A} = f(A)$ on $\tilde{S}$. Each transformation is defined by imposing certain limitations on mutual arrangement of surfaces S and $\tilde{S}$ and segments $[A\tilde{A}]$ that connect appropriate points on S and $\tilde{S}$.

Definition 1.1. Bonnet transformation is defined by the following two conditions:

1) segment $[A\tilde{A}]$ is orthogonal to the tangent plane $T_A(S)$ to S at the point A;
2) length of the segment $[A\tilde{A}]$ is the same for all points $A \in S$ (it is a parameter of construction).

Let $t = |A\tilde{A}|$. In Bonnet construction we can choose absolutely arbitrary surface S and arbitrary value of parameter t. Choosing various values of t for the fixed initial surface S, we obtain a family of surfaces S_t parallel to the surface S (as shown on Fig. 1.1). If surface S is compact, then for sufficiently small t surfaces S and S_t do not intersect. And they and are at the distance of $\rho(S, S_t) = t$ apart from each other. In this situation Bonnet transformation yields a set of diffeomorphisms $f_t\colon S \to S_t$. It possess the following property.

THEOREM 1.1. *In Bonnet construction the segment $[A\tilde{A}]$, which connect appropriate points A and $\tilde{A}$ on S and $\tilde{S}$, is orthogonal not only to initial surface of S, but to all intermediate surfaces S_t including the surface $\tilde{S}$.*

Due to this property Bonnet transformation from definition 1.1 is often called *a shift along normal-vector* or *normal shift*.

DEFINITION 1.2. Bianchi-Lie transformation of $f : S \to \tilde{S}$ is defined by the following three conditions:

1) segment $[A\tilde{A}]$ connecting appropriate points A and $\tilde{A}$ on S and $\tilde{S}$ lays in tangent planes $T_A(S)$ and $T_{\tilde{A}}(\tilde{S})$ simultaneously;
2) tangent planes of $T_A(S)$ and $T_{\tilde{A}}(\tilde{S})$ are orthogonal to each other;
3) length of the segment $[A\tilde{A}]$ is the same for all points $A \in S$ (it is a parameter of construction).

DEFINITION 1.3. Bäcklund transformation $f\colon S \to \tilde{S}$ is defined by the following three conditions:

1) segment $[A\tilde{A}]$ connecting appropriate points A and $\tilde{A}$ on S and $\tilde{S}$ lays in tangent planes $T_A(S)$ and $T_{\tilde{A}}(\tilde{S})$simultaneously;
2) the angle γ between tangent planes of $T_A(S)$ and $T_{\tilde{A}}(\tilde{S})$ has the same value for all points of $A \in S$ (it is a parameter of construction);
3) length of the segment $[A\tilde{A}]$ is the same for all points $A \in S$ (it is a parameter of construction).

DEFINITION 1.4. Darboux transformation $f\colon S \to \tilde{S}$ is defined by the following three conditions:

1) segment $[A\tilde{A}]$ connecting appropriate points A and $\tilde{A}$ on S and $\tilde{S}$ forms two angles α and β with tangent planes $T_A(S)$ and $T_{\tilde{A}}(\tilde{S})$, the values of which are same for all points $A \in S$ (they are parameters of construction);
2) the angle γ between tangent planes of $T_A(S)$ and $T_{\tilde{A}}(\tilde{S})$ has the same value for all points of $A \in S$ (it is a parameter of construction);
3) length of the segment $[A\tilde{A}]$ is the same for all points $A \in S$ (it is a parameter of construction).

Despite of presence of one (or two parameters) the Bianchi-Lie transformation and Bäcklund transformation are discrete by their nature. As to Bianchi-Lie transformation this fact is explained by the following theorem.

THEOREM 1.2. *Bianchi-Lie transformation with parameter $\rho = |A\tilde{A}|$ can be realized on an initial surface of S if and only if S is a surface of constant negative Gaussian curvature $K = -\rho^{-2}$, the resulting surface $\tilde{S}$ being also a surface of constant negative Gaussian curvature with the same value of curvature $\tilde{K} = -\rho^{-2}$.*

Similar theorem holds for the Bäcklund transformation.

THEOREM 1.3. *Bäcklund transformation with parameters $\rho = |A\tilde{A}|$ and γ can be realized on an initial surface S if and only if S is a surface of constant negative Gaussian curvature $K = -\sin^2\gamma \cdot \rho^{-2}$. The resulting surface $\tilde{S}$ in this construction is also a surface of constant negative Gaussian curvature with the same value of curvature $\tilde{K} = -\sin^2\gamma \cdot \rho^{-2}$.*

Suppose that surface S of constant negative curvature K is given. By choosing it as an initial surface in Bianchi-Lie construction we fix the value of parameter ρ:

$$\rho = \frac{1}{\sqrt{-K}}.$$

In Bäcklund construction we have additional parameter γ. Therefore we can construct a family of transformations $f_\gamma\colon S \to S_\gamma$ with common initial surface, where

$$\rho = \frac{\sin\gamma}{\sqrt{-K}}.$$

However, in contrast to Bonnet transformation, there is no canonical way to unite transformations f_γ with various values of γ into one construction. Therefore we treat f_γ as separate transformations with fixed values of γ.

In Darboux construction, except for ρ and γ, we have two additional parameters α and β. Let's consider two points A and $\tilde{A} = f(A)$ on S and $\tilde{S}$ related by Darboux transformation with parameters ρ, γ, α and β. Tangent planes $T_A(S)$ and $T_{\tilde{A}}(\tilde{S})$ intersect along the straight line $B\tilde{B}$ (see Fig. 1.2). Points B and $\tilde{B}$ are obtained by dropping perpendiculars from A and $\tilde{A}$ to this line. For the vector of displacement beginning at the point A and ending at the point $\tilde{A} = f(A)$ we have the expansion

$$\overrightarrow{A\tilde{A}} = \overrightarrow{AB} + \overrightarrow{B\tilde{B}} + \overrightarrow{\tilde{B}\tilde{A}}. \tag{1.1}$$

Let's project tangent planes $T_A(S)$ and $T_{\tilde{A}}(\tilde{S})$ shown on Fig. 1.2 onto the plane perpendicular to the line of their intersection. As a result we get Fig. 1.3. Normal vectors $\mathbf{n}$ and $\tilde{\mathbf{n}}$ in this projection can be extended up to the intersection at the point C. Now the expansion (1.1) can be replaced by the following one:

$$\overrightarrow{A\tilde{A}} = \overrightarrow{AC} + \overrightarrow{B\tilde{B}} + \overrightarrow{C\tilde{A}}\,. \tag{1.2}$$

Lengths of the vectors in (1.2) do not depend on the choice of point A on S. On the base of elementary geometry we can calculate them:

$$\begin{aligned} &|AC| = t_1 = \frac{\sin\alpha - \cos\gamma\,\sin\beta}{\sin^2\gamma}\,|A\tilde{A}|,\\ &|C\tilde{A}| = t_2 = \frac{\sin\beta - \cos\gamma\,\sin\alpha}{\sin^2\gamma}\,|A\tilde{A}|,\\ &|B\tilde{B}| = r = \sqrt{|A\tilde{A}|^2 - |AC|^2 - |C\tilde{A}|^2}\,. \end{aligned} \tag{1.3}$$

The vector $\overrightarrow{AC}$ is directed along normal-vector to S, and vector $\overrightarrow{C\tilde{A}}$ is directed along normal-vector to $\tilde{S}$. Therefore taking into account constancy of lengths of segments (1.3) we derive the following theorem.

THEOREM 1.4. *Darboux transformation $f\colon S \to \tilde{S}$ with parameters $\rho = |A\tilde{A}|$, α, β, and γ can be represented as a composition*

$$f = f_{t_2} \circ f_{r\,\gamma} \circ f_{t_1}, \tag{1.4}$$

which consists of two normal shifts (Bonnet transformations) with parameters t_1 and t_2, and one Bäcklund transformation with parameters r and γ, where t_1, t_2, and r are defined by formulas (1.3).

The theorem 1.4 shows that all above classical constructions of surface transformation can be reduced to the following two basic types:

(1) to normal shifts (Bonnet transformation),
(2) to tangent shifts (Bäcklund transformation).

§ 2. Association with nonlinear integrable equations and further generalizations of classical constructions.

In the beginning of XX-th century we observe intensive development of the theory of surface transformations and the theory of congruences[1] (see references in [10]). The latter one was stimulated by its application to the optics in describing light beams in non-homogeneous refracting media. Development of classical constructions from [5–9] in this period was determined by complication of geometry, where

[1] Theory of line families and their envelopes.

they are realized. Thus in paper [11] Bianchi has replaced three-dimensional Euclidean space by three-dimensional spaces of negative constant sectional curvature[1].

Paper of Tzitzeika [14] should also be mentioned here. In this paper he has considered three-dimensional Euclidean space, but with special marked point O, and has constructed the transformation quite different from that of Bianchi-Lie, Bäcklund, and Darboux. His transformation relates two surfaces of negative Gaussian curvature K, which is not constant however. The value of K at the point A of Tzitzeika's surface is defined by formula

$$K = -\frac{\text{const}}{\rho^4}, \tag{2.1}$$

where ρ is a distance from marked point O to the plane $T_A(S)$ tangent to S at the point A.

Further development of differential geometry in XX-th century has displaced interest toward more abstract constructions: connections on bundles, gauge fields etc. A lot of results were forgotten for many years. Nowadays the advent of **inverse scattering method** (see [15]) followed by intensive development of the theory of integrable differential equations and integrable models in classical and quantum physics has resumed interest to the constructions of Bianchi, Lie, Bäcklund, and Darboux. We explain this by taking Bäcklund's construction as an example. This construction is realized on a surface S of constant negative curvature

$$K = -\frac{\sin^2(\gamma)}{\rho^2} \tag{2.2}$$

(see theorem 1.3 above). It is known (see [16]) that on such surface one can choose asymptotic coordinates u^1 and u^2. In asymptotic coordinates the components of metric tensor and the components of second fundamental form are as follows:

$$g_{11} = g_{22} = \rho^2, \qquad g_{12} = g_{21} = \rho^2\,\cos(\varphi). \tag{2.3}$$

$$b_{11} = b_{22} = 0, \qquad b_{12} = b_{21} = \rho\,\sin(\gamma)\,\sin(\varphi). \tag{2.4}$$

Here function $\varphi = \varphi(u^1, u^2)$ determines the angle between asymptotic lines[2]. The condition of constancy of Gaussian curvature K in (2.2) appears to lead to the following partial differential equation for φ:

$$\frac{\partial^2\varphi}{\partial u^1\,\partial u^2} = \sin(\varphi). \tag{2.5}$$

This is well-known Sin-Gordon equation integrable by means of inverse scattering method (see [17], [18]). If we designate by ψ the angle between asymptotic lines on

[1] See definition and detailed description in [12] and [13].

[2] Lines, tangent vector of which make zero the value of second fundamental form, see more details in [16].

the second surface $\tilde{S}$ related to S by Bäcklund transformation, then angles φ and ψ will be bound to each other by the following two differential equations:

$$\begin{aligned}&\frac{\partial}{\partial u^1}\left(\frac{\varphi-\psi}{2}\right)=-C_1\cdot\sin\left(\frac{\varphi+\psi}{2}\right),\\&\frac{\partial}{\partial u^2}\left(\frac{\varphi+\psi}{2}\right)=-C_2\cdot\sin\left(\frac{\varphi-\psi}{2}\right),\end{aligned}\tag{2.6}$$

Here C_1 and C_2 are two constants defined by parameter γ:

$$C_1=\frac{1-\cos(\gamma)}{\sin(\gamma)},\qquad\qquad C_2=\frac{1+\cos(\gamma)}{\sin(\gamma)}.$$

From (2.5) and (2.6) we can derive Sin-Gordon equation for the angle ψ:

$$\frac{\partial^2\psi}{\partial u^1\,\partial u^2}=\sin(\psi).\tag{2.7}$$

The relationships (2.6) are known as Bäcklund transformation for the Sin-Gordon equation. They are used to construct new solutions of this equation on a base of some already known ones. Group analytic treatment of these relationships can be found in [19].

Tzitzeika's surfaces with Gaussian curvature (2.1) are bound with other nonlinear partial differential equation

$$\frac{\partial^2\varphi}{\partial u^1\,\partial u^2}=e^{\varphi}-e^{-2\varphi}.\tag{2.8}$$

Similar to (2.7), this equation is integrable by inverse scattering method. Tzitzeika equation (2.8) was first discovered in [14]. Later on it was rediscovered in [20] and [21]. The analog of geometrical Bäcklund transformation for Tzitzeika surfaces was found in original paper [14]. In form of differential relationships similar to (2.6) Bäcklund transformation for the equation (2.8) was rediscovered in [22][1]. It was studied in details in paper [24]. The relation of geometric construction of Tzitzeika and differential Bäcklund transformation for the equation (2.8) is discussed in [25]. In papers [26] and [27] some special classes of solutions for the equation (2.8) are constructed.

In a series of papers [28], [29], [30] multidimensional generalizations for Bianchi-Lie and Bäcklund transformations in Euclidean and affine spaces. In Euclidean case generalized Bäcklund transformation binds two n-dimensional submanifolds S and $\tilde{S}$ in the space of odd dimension $\mathbb{R}^{2n-1}$. Each of these two submanifolds S and $\tilde{S}$ are described by some system of nonlinear differential equations which is integrable by means of inverse scattering method and therefore can be treated as a generalization of Sin-Gordon equation (2.7).

[1] Other form of differential Bäcklund transformation for the equation (2.8), different from that of [14] and [22], can be found in [23].

In paper [31] Euclidean space $\mathbb{R}^{2n-1}$ was replaced by Riemannian manifold M of constant sectional curvature, where $\dim M = 2n-1$. With respect to the generalized version of Bäcklund transformation from [28] and [29] paper [31] plays the same role as paper [11] respective to classical constructions of Bianchi, Lie, Bäcklund and Darboux. For us it's important to emphasize that in papers [11] and [31] **rectilinear** segment of shift was replaced by a segment of **geodesic line**. In local coordinates geodesic line is described by ordinary differential equations of second order

$$\ddot{x}^k + \sum_{i=1}^{m} \sum_{j=1}^{m} \Gamma^k_{ij}(x^1, \ldots, x^m)\,\dot{x}^i\,\dot{x}^j = 0. \tag{2.10}$$

So this was first event, when rectilinear segment of shift was replaced by curvilinear one. We should note that this replacement did not cause crucial complication in the constructions. The Sin-Gordon equation (2.7) and its multidimensional generalizations from [27] remain unchanged.

§ 3. Bonnet transformation and geodesic normal shift.

As well as Bianchi-Lie and Bäcklund transformations Bonnet transformation from definition 1.1 has its own multidimensional generalization for the case of hypersurfaces in Euclidean space $\mathbb{R}^n$ ($\dim S = n-1$). This multidimensional generalization preserves two properties, which are important for us:

1) multidimensional Bonnet transformation can be defined on arbitrary sufficiently small part of any hypersurface S in $\mathbb{R}^n$ (here we meet no restrictions like constancy of curvature);
2) multidimensional Bonnet transformation is defined for any sufficiently small value of shift parameter $t = |A\tilde{A}|$, and for any such value of t hypersurface S_t is orthogonal to the segment $[A\tilde{A}]$ binding two related points $A \in S$ and $\tilde{A} \in S_t$.

Due to the second property multidimensional Bonnet transformation is **a normal shift**. If we replace Euclidean space $\mathbb{R}^n$ by some n-dimensional Riemannian manifold and rectilinear segments by segments of geodesic lines, we obtain the construction of **geodesic normal shift**.

Definition 3.1. Geodesic normal shift to the distance t in Riemannian manifold is a map $f_t : S \to S_t$, that maps each point A of some hypersurface S onto corresponding point $\tilde{A} = f_t(A)$ of the other hypersurface S_t such that the following conditions are fulfilled:

1) points A and $\tilde{A} = f_t(A)$ are bound by a segment $[A\tilde{A}]$ of geodesic line, which is perpendicular to the tangent plane $T_A(S)$ to S at the point A;
2) length of the geodesic segment $[A\tilde{A}]$ is constant, which is equal to t (it doesn't depend on particular choice of point A on hypersurface S).

Construction of geodesic normal shift in differential geometry is well known. It is used to define *semigeodesic coordinates* on manifold (see in [16] or in [32]). In general relativity the same construction of geodesic normal shift is used to define *synchronous reference system* (see [33], § 97).

Geodesic normal shift in Riemannian manifold M possess same two properties as classical Bonnet transformation in $\mathbb{R}^3$ and its generalizations in multidimensional flat spaces $\mathbb{R}^n$ ($n = 2$ and $n > 3$):

1) geodesic normal shift can be defined on arbitrary sufficiently small part of any hypersurface S in M (here we meet no restrictions like constancy of curvature);
2) geodesic normal shift is defined for any sufficiently small value of shift parameter $t = |A\tilde{A}|$, and for any such value of t hypersurface S_t is orthogonal to the segment $[A\tilde{A}]$ binding two related points $A \in S$ and $\tilde{A} \in S_t$.

§ 4. Normal shift along trajectories of Newtonian dynamical systems.

From what was said in previous three sections §§ 1–3 we conclude that development and generalizations of all constructions considered there came through the same two steps: increase of the dimension and transfer from flat Euclidean spaces to non-flat Riemannian manifolds. In doing last step we replace rectilinear segments by the segments of curved lines. However, we can replace *rectilinear segments* by *curved segments* in flat space too. This idea has played the role of impetus that gave rise to **the theory of dynamical systems admitting the normal shift**. In order realize this idea we was to do the following steps:

1) choose one of the four above constructions (Bianchi-Lie transformation, Bäcklund transformation, Darboux transformation or Bonnet transformation) as a base for generalization;
2) choose class of curves that should be used in generalized construction;
3) choose a space, where the generalized construction should be built.

Our joint paper [34] in cooperation with A. Yu. Boldin (see also [35]) was starting point of the theory of Newtonian dynamical systems admitting the normal shift. In this paper we choose Bonnet transformation as a basic object for generalization, since it is the simplest one and since it can be applied to any hypersurface. The choice of curves was prompted by the equations (2.10). This is the system of differential equations of the second order. We replace them by the autonomous system of differential equations of more general form:

$$\begin{aligned} &\ddot{x}^1 = F^1(x^1, \ldots, x^n, \dot{x}^1, \ldots, \dot{x}^n), \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\ddot{x}^n = F^n(x^1, \ldots, x^n, \dot{x}^1, \ldots, \dot{x}^n). \end{aligned} \tag{4.1}$$

Thus we escape the situation leading to the geodesic normal shift, which was already known, and the same time did not change class of curves drastically. Describing a curve by differential equations (4.1) we not only define them as the set of point, but also fix a parameter on them. As a parameter we take independent variable t that define derivatives $\dot{x}^1, \ldots, \dot{x}^n$ and second derivatives $\ddot{x}^1, \ldots, \ddot{x}^n$ in (4.1).

The choice of space $M = \mathbb{R}^2$ in [35] was determined by the reason of maximal simplicity. This was important in the initial stage of constructing the theory. Hypersurfaces in $\mathbb{R}^2$ are curves, therefore in $\mathbb{R}^2$ we have the displacement of curves along other curves, the latter being trajectories of dynamical system (4.1). The equations of dynamical system (4.1) here can be written in vectorial form:

$$\dot{\mathbf{r}} = \mathbf{v}, \qquad \dot{\mathbf{v}} = \mathbf{F}(\mathbf{r}, \mathbf{v}). \tag{4.2}$$

These are the equations of dynamics for mass point with unit mass $m = 1$ according to Newton's second law. Vector $\mathbf{r} = \mathbf{r}(t)$ in (4.2) is a radius-vector of mass point moving along the trajectory, $\mathbf{v}$ is a vector of velocity tangent to the trajectory, and $\mathbf{F}(\mathbf{r}, \mathbf{v})$ is a vector of force that defines force field of the system (4.2).

In order to draw trajectories of Newtonian dynamical system (4.2) coming out from the curve S (see Fig. 4.1) we should define initial velocity at each point of S, i. e. if curve S is given in parametric form by vectorial function $\mathbf{r} = \mathbf{r}(u)$, then we are to consider the following Cauchy problem for the system of equations (4.2):

$$\mathbf{r}\Big|_{t=0} = \mathbf{r}(u), \qquad \mathbf{v}\Big|_{t=0} = \mathbf{w}(u). \tag{4.3}$$

Here $\mathbf{w}(u)$ is a vector-function that determines the initial velocity for trajectories starting from the curve S. Cauchy problem (4.3) for the equations (4.2) is solvable, for any sufficiently small t its solution defines a map $f_t\colon S \to S_t$ from initial curve S to the curve S_t. This map is to be a curvilinear generalization for classical Bonnet transformation, with trajectories of dynamical system (4.2) being used instead of rectilinear segments $[A\tilde{A}]$ in definition 3.1. Therefore these trajectories should be perpendicular to S. In other words, this means that vector $\mathbf{w}(u)$ in (4.3) should be directed along the normal vector $\mathbf{n}(u)$ of initial curve S:

$$\mathbf{r}\Big|_{t=0} = \mathbf{r}(u), \qquad \mathbf{v}\Big|_{t=0} = \nu(u)\cdot\mathbf{n}(u). \tag{4.4}$$

Scalar function $\nu(u)$ in (4.4) determines the modulus of initial velocity for trajectories starting from S.

As well as (4.3) the Cauchy problem (4.4) is solvable, and it determines a map $f_t\colon S \to S_t$, which has more reasons to be considered as a proper curvilinear generalization of classical Bonnet transformation. It certainly possess first of two properties of Bonnet transformation, which are listed in § 3, but shouldn't ever possess the second one. This means that trajectories of the shift $f_t\colon S \to S_t$, being orthogonal to S due to (4.4), can be **not orthogonal** to S_t for $t \neq 0$. Though for some special choice of initial curve S and some special choice of function $\nu(u)$ in (4.4) they could be orthogonal to S_t for all t. In more details this point is discussed in thesis [36] by

A. Yu. Boldin. There one can find some examples that illustrate both cases: when trajectories of displacement are orthogonal to all curves S_t, and when not as well. Taking into account all what was said above we come to the concept of *normal shift along trajectories of Newtonian dynamical system*. This concept was introduced in paper [35].

Definition 4.1. Let $f_t : S \to S_t$ be a map defined by the displacement of a planar curve $S \subset \mathbb{R}^2$ along trajectories of Newtonian dynamical system (4.2) that start on S according to initial data (4.4). This map is called *the normal shift*, if for all sufficiently small values of t all trajectories of displacement are perpendicular to all curves S_t obtained by displacement.

Suppose that we choose and fix an arbitrary curve S on the plane. Suppose that vector-function $\mathbf{F}(\mathbf{r}, \mathbf{v})$ in (4.2) is also fixed. Then the only arbitrariness that we have at our disposal in defining displacement map $f_t : S \to S_t$ is the choice of function $\nu(u)$ in (4.4). Can we choose this function so that the map $f_t : S \to S_t$ would be a normal shift in the sense of the above definition ? In general case the answer to this question is **negative**. Examples confirming this answer are given in the thesis [36] by A. Yu. Boldin. However, if the curve S or the force field $\mathbf{F}(\mathbf{r}, \mathbf{v})$ is properly chosen, then further choice of function $\nu(u)$ can be made so that the displacement map $f_t : S \to S_t$ would be a normal shift. An example of proper choice of force field is $\mathbf{F}(\mathbf{r}, \mathbf{v}) = 0$. Trajectories of Newtonian dynamical system with identically zero force field are straight lines. In this case for any choice of curve S we choose function $\nu(u)$ being identically equal to unity: $\nu(u) = 1$. Then all trajectories of shift from S to S_t will be segments of straight lines with the same length t. This means that due to our choice we construct a shift $f_t : S \to S_t$, which coincides with classical Bonnet transformation. Therefore it satisfies all conditions needed to be a normal shift.

The above example demonstrates that there are some **force fields** (at least one) such that for any choice of curve S they let define transformation $f_t : S \to S_t$, being a normal shift, at the expense of proper choice of the function $\nu(u)$ in (4.4). Such force fields or (more exactly) dynamical systems with such force fields in paper [35] were called the systems **admitting the normal shift**.

Definition 4.2. Newtonian dynamical system (4.2) on the plane with force field $\mathbf{F}(\mathbf{r}, \mathbf{v})$ is called the system **admitting the normal shift of curves**, if for any sufficiently small part S of any curve on the plane one can find a function ν on S such that the map $f_t : S \to S_t$ defined by the solution of Cauchy problem (4.4) is a normal shift in the sense of definition 4.1.

If force field of Newtonian dynamical system satisfies the condition of definition 4.2, then we can use it to define the construction being curvilinear generalization of Bonnet transformation in $\mathbb{R}^2$. Such construction reproduces both properties of classical construction stated in § 3, i. e. it is applicable to a sufficiently small part of any curve in $\mathbb{R}^2$, and constructed transformation f_t is a normal shift in the sense of definition 4.1.

§ 5. Normality equations.

By means of definition 4.2 we have introduced central object of theory: **class of Newtonian dynamical systems admitting the normal shift**. This class is non-empty, since it contains trivial system with identically zero force field $\mathbf{F}(\mathbf{r}, \mathbf{v}) = 0$. But how broad is this class, does it contain less trivial systems ? In order to find answer to this question we should find another description for dynamical systems of this class, more effective than the definition 4.2. First step in this direction was made in paper [35] for the dimension $n = 2$. There the system of two partial differential equations for the vector-function $\mathbf{F}(\mathbf{r}, \mathbf{v})$ was derived. These equations form sufficient condition for Newtonian dynamical system with force field to belong to the class of systems admitting the normal shift. They were called **the equations of normality**. Description of how these equations were derived an more detailed report on initial period of development of the theory of dynamical systems admitting the normal shift can be found in thesis [36] by A. Yu. Boldin. Here we will only introduce some definitions and notations and then will write the normality equations from [35] themselves.

Planar Newtonian dynamical systems (4.2) describe the motion of the points in $\mathbb{R}^2$. The space $M = \mathbb{R}^2$ is called **configuration space** of such systems, points of this space are marked by radius-vector $\mathbf{r}$. Usually configuration space $M = \mathbb{R}^2$ is completed by one more copy of this space $\mathbb{R}^2$:

$$TM = \mathbb{R}^2 \oplus \mathbb{R}^2. \tag{5.1}$$

Resulting space TM in (5.1) is called **phase space** of planar dynamical system (4.2). Points of TM are marked by pairs of vectors $(\mathbf{r}, \mathbf{v})$. In our case both copies of the space $\mathbb{R}^2$ are assumed to be equipped with standard Euclidean scalar product, which defines lengths of vectors and angles between them. Velocity vector $\mathbf{v}$ is in the second summand $\mathbb{R}^2$ in (5.1). Vector of acceleration $\mathbf{a} = \dot{\mathbf{v}}$ is assumed to be in the second summand too. Therefore we can define the angle between vectors $\mathbf{a}$ and $\mathbf{v}$ (though, if we identify two copies of $\mathbb{R}^2$ in (5.1), we can define the angle between vectors $\mathbf{r}$ and $\mathbf{v}$ too).

Force field $\mathbf{F}(\mathbf{r}, \mathbf{v})$ of dynamical system (4.2) is function of point of phase space, its values are vectors that are in the second copy of $\mathbb{R}^2$ in (5.1). Another example of such function is given by velocity vector: we can treat $\mathbf{v}$ as a map that maps pair of vectors $(\mathbf{r}, \mathbf{v})$ onto the vector $\mathbf{v}$. Let's normalize its length to the unity. As a result we obtain vector $\mathbf{N}$ of unit length:

$$\mathbf{N} = \frac{\mathbf{v}}{|\mathbf{v}|}. \tag{5.2}$$

In two dimensional space $\mathbb{R}^2$ we can rotate unit vector (5.2) by the angle 90° (clockwise or counter-clockwise). Let's fix on of these two direction of rotation and denote by $\mathbf{M}$ unit vector obtained by such rotation. Like vector $\mathbf{v}$, both vectors $\mathbf{N}$ and $\mathbf{M}$ can be interpreted as vector-functions on a phase space TM with values in second copy of $\mathbb{R}^2$ in (5.1). The only peculiarity of these two functions $\mathbf{N}$ and $\mathbf{M}$ is that they aren't defined everywhere in TM, one should exclude that points, where $\mathbf{v} = 0$. Pair of vectors $\mathbf{N}$ and $\mathbf{M}$ forms an orthogonal and normalized frame in $\mathbb{R}^2$. Therefore one can define force field $\mathbf{F}(\mathbf{r},\mathbf{v})$ by the expansion

$$\mathbf{F}=A(\mathbf{r},\mathbf{v})\cdot\mathbf{N}+B(\mathbf{r},\mathbf{v})\cdot\mathbf{M}. \tag{5.3}$$

Here A and B are scalar functions on phase space TM. First summand in the expansion (5.3) is a vector directed along trajectory of dynamical system (4.2), this is **tangential** component of force field. Second summand is directed perpendicular to the trajectory toward the curvature center, this is **centripetal** component of force field $\mathbf{F}$.

Functions A and B depend on four scalar arguments, two components r^1 and r^2 of radius vector $\mathbf{r}$ and two components v^1 and v^2 of the velocity vector $\mathbf{v}$. Partial derivatives

$$\frac{\partial A}{\partial r^1}\ \text{ and }\ \frac{\partial A}{\partial r^2}$$

form the vector of **spatial gradient** of the function A, we denote it by ∇A. Other two partial derivatives

$$\frac{\partial A}{\partial v^1}\ \text{ and }\ \frac{\partial A}{\partial v^2}$$

form the vector of **velocity gradient** of the function A, which we denote by $\tilde{\nabla} A$. Similarly we can define gradients ∇B and $\tilde{\nabla} B$. Let's consider the following expansions for the vectors of gradients ∇A, $\tilde{\nabla} A$, ∇B, and $\tilde{\nabla} B$:

$$\begin{aligned}&\nabla A=\alpha_1\cdot\mathbf{N}+\alpha_2\cdot\mathbf{M}, &&\tilde{\nabla} A=\alpha_3\cdot\mathbf{N}+\alpha_4\cdot\mathbf{M},\\ &\nabla B=\beta_1\cdot\mathbf{N}+\beta_2\cdot\mathbf{M}, &&\tilde{\nabla} B=\beta_3\cdot\mathbf{N}+\beta_4\cdot\mathbf{M},\end{aligned} \tag{5.4}$$

These expansions are analogous to the expansion (5.3) for the vector of force $\mathbf{F}$. Coefficients of the expansions (5.4) can be calculated in form of scalar products:

$$\begin{aligned}&\alpha_1=(\nabla A\,|\,\mathbf{N}), &&\alpha_2=(\nabla A\,|\,\mathbf{M}),\\ &\alpha_3=(\tilde{\nabla} A\,|\,\mathbf{N}), &&\alpha_4=(\tilde{\nabla} A\,|\,\mathbf{M}),\\ &\beta_1=(\nabla B\,|\,\mathbf{N}), &&\beta_2=(\nabla B\,|\,\mathbf{M}),\\ &\beta_3=(\tilde{\nabla} B\,|\,\mathbf{N}), &&\beta_4=(\tilde{\nabla} B\,|\,\mathbf{M}).\end{aligned}$$

Now we are able to write down the system of **normality equations** derived in [35]:

$$\begin{cases} B=-|\mathbf{v}|\,\alpha_4,\\[2ex] \dfrac{B\,A}{|\mathbf{v}|^2}-\beta_1-\beta_3\dfrac{A}{|\mathbf{v}|}-\beta_4\,\dfrac{B}{|\mathbf{v}|}=\alpha_2-\alpha_3\,\dfrac{B}{|\mathbf{v}|}.\end{cases} \tag{5.5}$$

Parameters α_2, α_3, α_4, β_1, β_3, β_4 in (5.5) can be expressed through gradients of A and B. Therefore the equations (5.5) form the system of partial differential equations with respect to the coefficients A and B in the expansion (5.3) of the force field.

THEOREM 5.1. *Planar Newtonian dynamical system* (4.2) *with force field* (5.3) *satisfying normality equations is a system admitting normal in the sense of definition 4.2.*

The first equation in (5.5) expresses B through vector of velocity gradient $\tilde{\nabla}A$ in explicit form: $B = -|\mathbf{v}|\,\alpha_4 = -|\mathbf{v}|\,(\tilde{\nabla}A\,|\,\mathbf{M})$. By substituting this expression into second equation (5.5) and taking into account above formulas for α_2, α_3, α_4, β_1, β_3, β_4 we reduce the system of equations (5.5) to one nonlinear partial differential equation for the function $A(\mathbf{r}, \mathbf{v})$. Some simples particular solutions of this equation and corresponding dynamical systems were found in [35]. Their existence showed us that class of dynamical systems admitting the normal shift is **non-trivial**. And this class is worth for further study. List of examples was substantially enlarged in [37] on the base of systematic analysis of the above normality equations. Here we will not consider examples from [35] and [37], since they are analyzed in the other thesis [36].

§ 6. Generalization for n-dimensional case.

Further progress in theory of dynamical systems admitting the normal shift was bound with the growth of dimension. In paper [38] two-dimensional space $\mathbb{R}^2$ was replaced by $\mathbb{R}^n$ (see also [34] section 5). The equations of Newtonian dynamical system in $\mathbb{R}^n$ is written in a form

$$\dot{\mathbf{r}} = \mathbf{v}, \qquad \dot{\mathbf{v}} = \mathbf{F}(\mathbf{r}, \mathbf{v}) \tag{6.1}$$

quite similar to (4.2). But phase space for dynamical system (6.1) is a sum

$$TM = \mathbb{R}^n \oplus \mathbb{R}^n. \tag{6.2}$$

One can easily reformulate the definitions 4.1 and 4.2 for multidimensional case: simply curves should be replaced by hypersurfaces. On sufficiently small part of any hypersurface S in $\mathbb{R}^n$ one can define normal vector of unit length such that it is smooth function of the points of S. This vector is orthogonal to S in standard scalar product of Euclidean space $\mathbb{R}^n$. We can set up the following Cauchy problem for the equations (6.1):

$$\mathbf{r}\Big|_{t=0} = \mathbf{r}(p), \qquad \mathbf{v}\Big|_{t=0} = \nu(p) \cdot \mathbf{n}(p). \tag{6.3}$$

Here p is a point on S, $\mathbf{n}(p)$ — is a normal vector at p, and $\mathbf{r} = \mathbf{r}(p)$ is a vectorial parametric equation of hypersurface S. Solution of Cauchy problem (6.1) is given by vector-function $\mathbf{r} = \mathbf{r}(t, p)$. If we map point p with radius-vector $\mathbf{r}(p)$ onto the point $\mathbf{r}(t, p)$, we get the displacement $f_t : S \to S_t$ that maps hypersurface S onto another hypersurface S_t.

DEFINITION 6.1. Let $f_t\colon S \to S_t$ be a displacement of hypersurface $S \subset \mathbb{R}^n$ along trajectories of Newtonian dynamical system (6.1) defined by the solution of Cauchy problem (6.3). This displacement is called *the normal shift*, if for all sufficiently small values of t all trajectories of displacement are perpendicular to all hypersurfaces S_t obtained by this displacement.

DEFINITION 6.2. Newtonian dynamical system (6.1) with force field $\mathbf{F}(\mathbf{r}, \mathbf{v})$ is called the system **admitting the normal shift**, if for any sufficiently small part S of any hypersurface in $\mathbb{R}^n$ one can find a function ν on S such that the map $f_t\colon S \to S_t$ defined by the solution of Cauchy problem (6.3) is a normal shift in the sense of definition 6.1.

The derivation of normality equations in multidimensional case has some peculiarities compared to that of two-dimensional case. In order to describe these peculiarities we should consider some details of such derivation. Let $\mathbf{r}(t, p)$ be the solution of Cauchy problem (6.3) for the equations (6.1). If we choose local (curvilinear) coordinates $u^1, \ldots, u^{n-1}$ on S, then $\mathbf{r}(t, p)$ is represented by vector-function

$$\mathbf{r} = \mathbf{r}(t, u^1, \ldots, u^{n-1}). \tag{6.4}$$

For fixed (and sufficiently small) t function (6.4) can be understood as a vectorial parametric equation of hypersurface S_t, which is close to S. In this case displacement $f_t\colon S \to S_t$ is local diffeomorphism that transfer local coordinates $u^1, \ldots, u^{n-1}$ from S to S_t. Partial derivatives

$$\boldsymbol{\tau}_i = \frac{\partial \mathbf{r}}{\partial u^i}, \quad i = 1, \ldots, n-1, \tag{6.5}$$

form a base in tangent hyperplane to S at the point p with coordinates $u^1, \ldots, u^{n-1}$.

Now let's unfix parameter t in (6.4). Varying of t corresponds to the motion of the point with radius vector (6.4) along trajectory of dynamical system (6.1). Varying parameters $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_{n-1}$ means that we transfer from one trajectory to another. Therefore in this context vectors $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_{n-1}$ are called the **vectors of variation** of trajectory. Let's denote by $\varphi_1, \ldots, \varphi_{n-1}$ the following scalar products:

$$\varphi_i = (\boldsymbol{\tau}_i \,|\, \mathbf{v}), \quad i = 1, \ldots, n-1. \tag{6.6}$$

In situation of normal shift all functions $\varphi_i(t, u^1, \ldots, u^{n-1})$ are identically zero, and conversely, if they are not zero, this means that we deviate from the situation of normal shift. Therefore functions $\varphi_i(t, u^1, \ldots, u^{n-1})$ will be called **functions of deviation**.

Let's consider initial data (6.3). According to (6.3) vector of initial velocity is directed along the normal vector to S. Therefore all functions of deviation on initial hypersurface S are zero:

$$\varphi_i\Big|_{t=0} = 0, \quad i = 1, \ldots, n-1. \tag{6.7}$$

In situation of normal shift their derivatives should be zero as well:

$$\dot\varphi_i\Big|_{t=0} = 0, \quad i = 1, \ldots, n-1. \tag{6.8}$$

We can hope to satisfy conditions (6.8) by means of proper choice of function $\nu(p) = \nu(u^1, \ldots, u^{n-1})$ in (6.3) (in two-dimensional case this is really so). However, vanishing of second and third derivatives and identical vanishing of all functions of deviation is possible only due to some special properties of force field $\mathbf{F}(\mathbf{r}, \mathbf{v})$. Note that (6.7) and (6.8) can be treated as initial data of some Cauchy problem for the ordinary differential equation of the second order for the function φ_i in t. If this equation is linear and homogeneous

$$\ddot{\varphi}_i - \mathcal{A}\,\dot{\varphi}_i - \mathcal{B}\,\varphi_i = 0, \tag{6.9}$$

then initial data (6.7) and (6.8) provide identical vanishing of φ_i. These considerations form the motivation for the following definition.

Definition 6.3. We say that newtonian dynamical system (6.1) in $\mathbb{R}^n$ satisfies **weak normality condition**, if for each trajectory of this system there is an ordinary differential equation $\ddot{\varphi} = \mathcal{A}(t)\,\dot{\varphi} + \mathcal{B}(t)\,\varphi$ such that arbitrary function of deviation on this trajectory satisfies this differential equation.

The term "arbitrary function of deviation" in definition 6.3 requires special comment. The matter is that Cauchy problem (6.3) with some special function $\nu(p)$ on hypersurface S is not the only way of defining the family of trajectories for dynamical system (6.1). If instead of (6.4) we consider an arbitrary parametric family of trajectories given by a vector-function $\mathbf{r}(t, u)$, and if we calculate partial derivative $\boldsymbol{\tau} = \partial\mathbf{r}/\partial u$, then the scalar product $\varphi = (\boldsymbol{\tau}\,|\,\mathbf{v})$ will be that "arbitrary function of deviation".

On a base of weak normality condition in [38] **weak normality equations** were derived. They can be understood as multidimensional generalizations for the equations (5.5). We shall write down these equations later, when required notations will be introduced. Now let's come back to the condition (6.8). Taking into account (6.5), (6.6) and the equations of dynamical system (6.1) we can bring (6.8) to the form of Pfaff equations for the function $\nu(p) = \nu(u^1, \ldots, u^{n-1})$:

$$\frac{\partial\nu}{\partial u^i} = \psi_i(\nu, u^1, \ldots, u^{n-1}), \quad i = 1, \ldots, n-1. \tag{6.10}$$

Functions $\psi_i(\nu, u^1, \ldots, u^{n-1})$ in the right hand side of (6.10) are determined by force field $\mathbf{F}(\mathbf{r}, \mathbf{v})$ and by hypersurface S, i. e. by vector-function $\mathbf{r} = \mathbf{r}(u^1, \ldots, u^{n-1})$. We shall not give their explicit expressions (see in [38] or [34]). Its the equations (6.10) that substantially differ multidimensional case from two-dimensional case. **In two-dimensional case** we had only one equation (6.10), which is ordinary differential equation of the first order respective one independent variable $u = u^1$. Such equation is always solvable (at least locally). **In multidimensional case** the equations (6.10) form complete system of Pfaff equations, solvability of this system depends on its compatibility (see definition of compatibility below in Chapter V, § 8, definition 8.1). If we want that the equations (6.10) would be compatible for any choice of hypersurface S, then we come to the following definition.

DEFINITION 6.4. We say that Newtonian dynamical system (6.1) in $\mathbb{R}^n$ with $n \geqslant 3$ satisfies **additional normality condition**, if for any hypersurface S system of Pfaff differential equations (6.10) for the function $\nu(p)$ is compatible.

On a base of additional normality condition in [38] (see also [34], section 5) **additional normality equations** were derived. In order to write these equations, and to write weak normality equations as well, we need to introduce special "fiber spherical coordinates" in phase space (6.2). Cartesian coordinates $r^1, \ldots, r^n$ in first copy ob $\mathbb{R}^n$ in (6.2) remain unchanged. Cartesian coordinates $v^1, \ldots, v^n$ in second copy of $\mathbb{R}^n$ will be replaced by spherical coordinates $v, u^1, \ldots, u^{n-1}$ (this means that we introduce spherical coordinates in **velocity space**). Transfer from spherical coordinates $v, u^1, \ldots, u^{n-1}$ back to cartesian coordinates $v^1, \ldots, v^n$ is defined by vector-function $\mathbf{v} = v\cdot\mathbf{N}(u^1, \ldots, u^{n-1})$, where $\mathbf{N}$ is a unit vector directed along the vector of velocity, or, in other words, $\mathbf{N}$ is a radius-vector of a point on unit sphere in velocity space. Taking $\mathbf{v} = \mathbf{N}(u^1, \ldots, u^{n-1})$ for the vectorial parametric equation of unit sphere in velocity space we can define tangent vectors to this sphere:

$$\mathbf{M}_i = \frac{\partial \mathbf{N}}{\partial u^i}, \quad i = 1, \ldots, n-1.$$

Vectors $\mathbf{N}, \mathbf{M}_1, \ldots, \mathbf{M}_{n-1}$ form moving frame of spherical coordinates. One can expand force vector by this frame:

$$\mathbf{F} = A\cdot\mathbf{N} + \sum_{i=1}^{n-1} B^i\cdot\mathbf{M}_i. \tag{6.11}$$

Coefficients $A, B^1, \ldots, B^{n-1}$ in the expansion (6.11) depend on spatial cartesian coordinates $r^1, \ldots, r^n$ and on spherical coordinates $v, u^1, \ldots, u^{n-1}$ in velocity space. Their spatial gradients can be expanded by the frame of vectors $\mathbf{N}, \mathbf{M}_1, \ldots, \mathbf{M}_{n-1}$:

$$\begin{aligned} \nabla A &= a\cdot\mathbf{N} + \sum_{k=1}^{n-1} \alpha^k\cdot\mathbf{M}_k,\\ \nabla B^i &= b^i\cdot\mathbf{N} + \sum_{k=1}^{n-1} \beta^{ik}\cdot\mathbf{M}_k. \end{aligned} \tag{6.12}$$

Denote by G_{ij} components of metric tensor of standard Euclidean metric in moving frame $\mathbf{M}_1, \ldots, \mathbf{M}_{n-1}$ of spherical coordinates on unit sphere:

$$G_{ij} = (\mathbf{M}_i \,|\, \mathbf{M}_j).$$

Let G^{ij} be components of dual metric tensor, and let ϑ^k_{ij} be component of metric connection for metric G_{ij} in spherical coordinates. Cartesian components of vectors $\mathbf{N}, \mathbf{M}_1, \ldots, \mathbf{M}_{n-1}$, and component G_{ij}, G^{ij}, ϑ^k_{ij} of metric tensors and metric connection are the functions of the following variables: $u^1, \ldots, u^{n-1}$. Upon choosing some particular spherical coordinates in velocity space one can calculate them explicitly. Coefficients of expansions (6.12) are calculated as follows:

$$
\begin{aligned}
&a = (\nabla A\,|\,\mathbf{N}), &&\alpha^k = \sum_{q=1}^{n-1} G^{kq}\,(\nabla A\,|\,\mathbf{M}_q),\\
&b^i = (\nabla B^i\,|\,\mathbf{N}), &&\beta^{ik} = \sum_{q=1}^{n-1} G^{kq}\,(\nabla B^i\,|\,\mathbf{M}_q).
\end{aligned}
\tag{6.13}
$$

Now we can write weak normality equations, which were mentioned above:

$$
\begin{cases}
B^i = -\displaystyle\sum_{k=1}^{n-1} G^{ik}\,\frac{\partial A}{\partial u^k},\\[2ex]
\alpha^i - \dfrac{B^i\,A}{v^2} + \displaystyle\sum_{k=1}^{n-1}\frac{\bar\nabla_k B^i\,B^k}{v^2} + b^i + \frac{A}{v}\,\frac{\partial B^i}{\partial v} - \frac{B^i}{v}\,\frac{\partial A}{\partial v} = 0,
\end{cases}
\tag{6.14}
$$

And we can write additional normality equation too. Here are they:

$$
\begin{cases}
\dfrac{1}{v}\,\dfrac{\partial B^i}{\partial v}\,B^k - \beta^{ik} = \dfrac{1}{v}\,\dfrac{\partial B^k}{\partial v}\,B^i - \beta^{ki},\\[2ex]
\bar\nabla_k B^i = \displaystyle\sum_{q=1}^{n-1}\frac{\bar\nabla_q B^q}{n-1}\,\delta^q_k.
\end{cases}
\tag{6.15}
$$

Covariant derivatives $\bar\nabla_k B^i$ in (6.14) and (6.15) are defined by relationships

$$
\bar\nabla_k B^i = \frac{\partial B^i}{\partial u^k} + \sum_{q=1}^{n-1}\vartheta^i_{kq}\,B^q.
$$

These are covariant derivatives with respect to spherical coordinates $u^1, \ldots, u^{n-1}$ calculated in metric connection ϑ in velocity space. Substituting (6.13) into (6.14) and (6.15) we see that weak and additional normality equations are partial differential equations for the components of force vector $\mathbf{F}$ in expansion (6.11).

THEOREM 6.1. *If force field of Newtonian dynamical system* (6.1) *in Euclidean space* $\mathbb{R}^n$ *satisfies both weak and additional normality equations* (6.14) *and* (6.15)*, then this system admits normal shift in the sense of definition 6.2.*

§ 7. The condition of normalization. Strong and complete normality.

Let's unite weak normality condition from definition 6.3 and additional normality condition from definition 6.4 into one condition. It is called **complete normality condition**. Theorem 6.1, that was proved in [38], asserts that complete normality condition is **enough** for the dynamical system to admit the normal shift of hypersurfaces in $\mathbb{R}^n$, i. e. for to satisfy **normality condition** from definition 6.2. Let's study how far is it from being **necessary** condition too. In order to do this we consider two examples.

Simplest example of dynamical system in $\mathbb{R}^n$ that satisfies normality condition from definition 6.2 is a system with identically zero force field $\mathbf{F}(\mathbf{r},\mathbf{v}) = 0$. Normal shift along trajectories of such system is reduced to multidimensional generalization of Bonnet construction, For to provide normality of displacement $f_t\colon S \to S_t$ in this case one can always choose $\nu(p) = 1$ in (6.3). Therefore $|\mathbf{v}| = 1$ on all trajectories of displacement, which are rectilinear segments in this case. This means that in process of displacement $f_t\colon S \to S_t$ we use not all points of phase space (6.2). From this fact we derive the idea of another example:

$$\mathbf{F}(\mathbf{r},\mathbf{v}) = \begin{cases} 0 & \text{for } \ |\mathbf{v}| < 2, \\ \mathbf{f}(\mathbf{r},\mathbf{v}) & \text{for } \ |\mathbf{v}| \geqslant 2. \end{cases} \tag{7.1}$$

Its easy to understand that for any choice of $\mathbf{f}(\mathbf{r},\mathbf{v})$ in (7.1) dynamical system with such force field admits the normal shift in the sense of definition 6.2. So, normality condition from definition 6.2 is not very rigid. It doesn't contradict to the existence of gaps in phase space, where force field can be given by absolutely arbitrary function. In such gaps normality equations (6.14) and (6.15) can be broken.

Thus we can draw conclusion: complete normality isn't necessary for the normality condition from definition 6.2 to be satisfied. This is not good. In order to improve this situation in [39] we introduced the normalization for the function $\nu(p)$:

$$\nu(p_0) = \nu_0. \tag{7.2}$$

Now definition 6.2 can be reformulated as follows.

Definition 7.1. Newtonian dynamical system (6.1) with force field $\mathbf{F}(\mathbf{r},\mathbf{v})$ is called the system admitting the normal shift **in strong sense**, if for any sufficiently small part S of any hypersurface in $\mathbb{R}^n$, for any point p_0 on S, and for arbitrary nonzero number ν_0 one can find a function ν on S normalized by the condition (7.2) such that the map $f_t\colon S \to S_t$ defined by the solution of Cauchy problem (6.3) is a normal shift in the sense of definition 6.1.

Definition 7.1 excludes the existence of lacunas like in second example above. The matter is that any nonzero vector $\mathbf{v} \neq 0$ at any point p_0 in $\mathbb{R}^n$ can be represented as $\mathbf{v} = \nu_0 \cdot \mathbf{n}(p_0)$, where $\mathbf{n}(p_0)$ is a normal vector of some hypersurface passing through the point p_0. Hence each point of phase space can be involved into the process of displacement for some hypersurface.

Condition stated in definition 7.1 is called **strong normality condition**. The diagram on Fig 7.1 (see next page) shows the mutual relation for various normality conditions. Implication 1 in this diagram is obvious. Implications 4, 5, 6, and 7 are proved during derivation of normality equation (6.14) and (6.15). Implications

2 and 3 are expressed by the following theorem proved in [40], Implication 3 in this theorem is obvious.

THEOREM 7.1. *Conditions of* **complete** *and* **strong** *normality for Newtonian dynamical systems are equivalent.*

Condition of $\mathbf{v} \neq 0$ is very important. In definition 7.1 it is represented by $\nu_0 \neq 0$. In implicit form it is present in definition 6.4 too, Indeed, functions $\psi_i(\nu, u^1, \ldots, u^{n-1})$ in right hand side of Pfaff equations (6.10), when written in explicit form, are not defined for $\nu = 0$. Point with $\mathbf{v} = 0$ are singular point on the trajectory of dynamical system (6.1). Use of trajectories with singular points is unreasonable, since it may cause some extra difficulties.

§ 8. Problem of metrizability and the test for non-triviality.

Normal shift of hypersurfaces in $\mathbb{R}^n$ along trajectories of special Newtonian dynamical systems generalizes classical construction of Bonnet transformation. But is this generalization non-trivial? Or, may be, it coincides with generalizations which were already known? Particularly with geodesic normal shift for some metrics in $\mathbb{R}^n$? This question was studied in papers [41] and [42]. The answer appeared favorable for the theory of dynamical systems admitting the normal shift. In general case normal shift along trajectories of Newtonian dynamical system does not reduce to geodesic normal shift. Moreover:

1) complete description of those systems, for which normal shift along their trajectories reduces to geodesic normal shift, was obtained;
2) some explicit examples of dynamical systems, for which normal shift can't be reduced to geodesic normal shift, were given.

First was done in [41], second in [42] respectively.

Suppose that $f_t\colon S \to S_t$ is a normal shift of hypersurface S along trajectories of dynamical system (6.1). For this shift to coincide with geodesic normal shift of some Riemannian metric $\mathbf{g}$ in $\mathbb{R}^n$ the following conditions should be fulfilled:

1) trajectories of shift should coincide (as sets of points) with geodesic lines for the metric $\mathbf{g}$ (though they can differ in parametrization);
2) right angles between trajectories of dynamical system and hypersurfaces S_t measured in standard Euclidean metric in $\mathbb{R}^n$ should be right angles in Riemannian metric $\mathbf{g}$ too.

If we want that coincidence of $f_t\colon S \to S_t$ and geodesic normal shift for Riemannian metric $\mathbf{g}$ not to be a casual event, we should require the conditions 1) and 2) to be fulfilled for any hypersurface S. Note that right angle defined by any two vectors can be represented as an angle between a tangent vector for some hypersurface S and the unit normal vector of this hypersurface. Condition 2) then means that each right angle in Euclidean metric in $\mathbb{R}^n$ is a right angle in Riemannian metric $\mathbf{g}$. Here we can apply the following obvious lemma.

LEMMA 8.1. *If each Euclidean right angle in $\mathbb{R}^n$ is a right angle in Riemannian metric $\mathbf{g}$, then $\mathbf{g}$ is a conformally Euclidean metric: $\mathbf{g} = e^{-2f}\,\boldsymbol{\delta}$.*

By $f = f(\mathbf{r})$ in Lemma 8.1 we denote some scalar function in $\mathbb{R}^n$. In Cartesian coordinates for the components of metric $\mathbf{g}$ we have

$$g_{ij} = e^{-2f}\,\delta_{ij}, \ \text{ where } \ \delta_{ij} = \begin{cases} 1 & \text{for } \ i = j, \\ 0 & \text{for } \ i \neq j. \end{cases} \tag{8.1}$$

Metric (8.1) generates a geodesic flow, which can be represented as Newtonian dynamical system (6.1) with force field

$$\mathbf{F}_0 = -|\mathbf{v}|^2 \cdot \nabla f + 2\,(\nabla f\,|\,\mathbf{v}) \cdot \mathbf{v}. \tag{8.2}$$

Trajectories of dynamical system with force field (8.2) are geodesic lines for the metric (8.1).

Let's consider pair of Newtonian dynamical systems with force fields $\mathbf{F}(\mathbf{r},\mathbf{v})$ and $\mathbf{F}_0(\mathbf{r},\mathbf{v})$ in n-dimensional Euclidean space $\mathbb{R}^n$:

$$\dot{\mathbf{r}} = \mathbf{v}, \qquad \dot{\mathbf{v}} = \mathbf{F}(\mathbf{r},\mathbf{v}), \tag{8.3}$$

$$\dot{\mathbf{r}} = \mathbf{v}, \qquad \dot{\mathbf{v}} = \mathbf{F}_0(\mathbf{r},\mathbf{v}). \tag{8.4}$$

Trajectories of these two dynamical systems are defined by initial data that fix starting point and initial velocity for them:

$$\mathbf{r}\,\Bigr|_{t=0} = \mathbf{r}_0, \qquad \mathbf{v}\,\Bigr|_{t=0} = \mathbf{v}_0, \tag{8.5}$$

$$\mathbf{r}\,\Bigr|_{t=0} = \mathbf{r}_0, \qquad \mathbf{v}\,\Bigr|_{t=0} = \mathbf{w}_0. \tag{8.6}$$

Denote by $\mathbf{r} = \mathbf{R}_1(t,\mathbf{r}_0,\mathbf{v}_0)$ solution of Cauchy problem (8.5) for the equations (8.3). Let $\mathbf{r} = \mathbf{R}_2(t,\mathbf{r}_0,\mathbf{w}_0)$ be a solution of Cauchy problem (8.6) for the equations (8.4).

If normal shift along trajectories of dynamical system (8.3) coincides with geodesic normal shift in metric (8.1), then according to above condition 1) each trajectory of such shift should coincide with some trajectory of other dynamical system (8.4) up to a change of parametrization. Since hypersurface S, point p_0 on S, and numeric parameter $\nu_0 \neq 0$ in normalization condition (7.2), we are in a situation described by the following definition.

Definition 8.1. We say that dynamical system (8.3) **inherits** trajectories of the system (8.4), if for any pair of vectors $\mathbf{r}_0$ and $\mathbf{v}_0 \neq 0$ one can find a vector $\mathbf{w}_0$ and twice differentiable function $T(\tau)$ such that $T(0) = 0$ and the following equality

$$\mathbf{R}_2(T(\tau), \mathbf{r}_0, \mathbf{w}_0) = \mathbf{R}_1(\tau, \mathbf{r}_0, \mathbf{v}_0),$$

holds identically for τ in some neighborhood of zero.

definition 8.2. Two Newtonian dynamical systems in $\mathbb{R}^n$ are called **trajectory equivalent**, if they inherit trajectories of each other in the sense of definition 8.1.

Note that above definition of trajectory equivalence differs from the definition of papers [43–49], which was used for topological classification of integrable Hamiltonian systems. Our definition is more specific, it is applicable only for Newtonian dynamical systems with common phase space (6.2). Similar definition was stated in paper [50] by Levi-Civita, and was used in [51–54]. In geometry the concept of geodesic equivalence for two affine connections appears to be a specialization of definition 8.2 (see [55–57]).

Definition 8.3. Newtonian dynamical system in $\mathbb{R}^n$ is called **metrizable**, if it inherits trajectories of geodesic flow for some conformally Euclidean metric (8.1).

In paper [41] was considered the class of Newtonian dynamical systems, which are both **metrizable** and **admitting the normal shift**[1] simultaneously. The following result was obtained.

Theorem 8.2. *Newtonian dynamical system* (8.3) *admitting the normal shift in* $\mathbb{R}^3$ *is metrizable if its force field is given by formula*

$$\mathbf{F}(\mathbf{r}, \mathbf{v}) = -|\mathbf{v}|^2 \cdot \nabla f + 2\,(\nabla f \,|\, \mathbf{v}) \cdot \mathbf{v} + \frac{\mathbf{v}}{|\mathbf{v}|} \cdot H(|\mathbf{v}| e^{-f}) e^{f}, \tag{8.7}$$

where $f = f(\mathbf{r}) = f(r^1, \ldots, r^n)$ *and* $H = H(v)$ *are arbitrary functions. Being metrizable, such dynamical system performs geodesic normal shift for metric* (8.1).

Theorem 8.2 solves the problem of metrizability for dynamical systems admitting the normal shift and gives examples of force fields $\mathbf{F}(\mathbf{r}, \mathbf{v}) \neq 0$, satisfying normality equations (6.14) and (6.15) in multidimensional case. But force fields (8.7) are assumed to be trivial, since their existence is predictable on a base of simple geometric considerations. The normal shift defined by such systems is reduced to the construction of geodesic normal shift, which was already known in geometry. Therefore

[1] Systems admitting the normal shift here and in what follows are understood in strong sense, i. e. in the sense of definition 7.1.

we have a problem to find some other solutions of normality equations that correspond to **non-metrizable** dynamical systems. The importance of this problem was pointed out by academician A. T. Fomenko during my report in the seminar at Moscow State University in the beginning of 1994. Fortunately solution of this problem appeared to be relatively simple. It was solved in 1994 in paper [42]. Force field that was found in paper [42] is the following:

$$\mathbf{F}(\mathbf{x},\mathbf{v})=\frac{A(|\mathbf{v}|)}{|\mathbf{v}|^2}\left(2\,(\mathbf{m}\,|\,\mathbf{v})\cdot\mathbf{v}-\mathbf{m}\cdot|\mathbf{v}|^2\right). \tag{8.8}$$

Here $A(v)$ is some arbitrary function of one variable, and $\mathbf{m}$ is an arbitrary constant vector. Existence of the solution (8.8) showed that, in spite of being overdetermined, system of equations (6.14) and (6.15) has nontrivial solutions in multidimensional case. So in multidimensional case we also have the examples of dynamical systems that can perform normal shift of hypersurfaces along their trajectories, being different from geodesic normal shift.

§ 9. Generalization for Riemannian geometry. Tensorial form of equations.

Next step in development of the theory of dynamical systems admitting the normal shift was due to the transfer to Riemannian geometry. This was done in [58], [39], and [59]. In these papers configuration space $M=\mathbb{R}^n$ of dynamical system (6.1) was replaced by an arbitrary n-dimensional Riemannian manifold M. Newton's second law, by its origin, is written for separate mass point or for system of several mass interacting mass point in $\mathbb{R}^3$. Configuration space for the system of N mass points is n-dimensional space $\mathbb{R}^n$, where $n=3N$. Continuous rigid body can be modeled by N mass points bound with each other by hard weightless rods. If we take into account these bounds we see that they diminish the degree of freedom from $n=3N$ to $n=6$. This means that in n-dimensional space $\mathbb{R}^n$ we mark six-dimensional manifold M isomorphic to $\mathbb{R}^3\times\mathrm{SO}(3,\mathbb{R})$. This is a configuration space for moving rigid body. Here we have 3 degrees of freedom for translational movement and 3 degrees of freedom for rotational movement. This situation may get more complicated if some bounds are flexible, e. g. if some hard rods are replaced by cardan joint, thumbscrew, tooth gearing etc. But even in this case we have some manifold M embedded into $\mathbb{R}^n$. Thus we can draw the following conclusion: dynamical systems on manifolds are more typical in classical mechanics than systems in Euclidean space $\mathbb{R}^n$.

Suppose that M is some manifold, being configuration space for some system of rigid bodies with flexible bounds. Local coordinates $x^1,\,\ldots\,x^n$ from some map of such manifold in mechanics are called *generalized coordinates*, their derivatives $\dot x^1,\,\ldots\,,\,\dot x^n$ are called *generalized velocities* (see [60]). **Kinetic energy** is a very important characteristics of mechanical system. Its dependence on generalized coordinates $x^1,\,\ldots\,,\,x^n$ may be very complicated, but (for realistic mechanical systems) it is always a quadratic function of generalized velocities. Let's denote kinetic energy of dynamical system by G and let's write it in the following form:

$$G=\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}g_{ij}\,\dot{x}^i\,\dot{x}^j. \tag{9.1}$$

Coefficients $g_{ij}=g_{ij}(x^1,\ldots,x^n)$ in quadratic form (9.1) define a metric in M arranging the structure of Riemannian manifold.

Dynamical systems that appears in classical mechanics are often *conservative*. This means that their behavior is described by two functions: kinetic energy (9.1) and *potential energy* $\mathit{\Pi}=\mathit{\Pi}(x^1,\ldots,x^n)$. They are used to define *the function of Lagrange* $L=\mathit{\Pi}-G$. Then differential equations of dynamics are derived from variational principle and written as Euler-Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{x}^k}\right)-\frac{\partial L}{\partial x^k}=0,\quad\text{where}\quad k=1,\,\ldots\,,n.$$

Afterwards these equations are usually examined for the existence of stationary solutions, for some asymptotics near such stationary solutions, for stability or, may be, for complete integrability. Kinetic energy by itself (and associated Riemannian metric on M) appears to be of less importance on this stage. In the theory of dynamical system admitting the normal shift, to the contrary, it plays central role, while conservativity or non-conservativity of the system doesn't matter at all.

One can consider Newtonian dynamical systems in non-Riemannian manifolds as well. In local coordinates such systems are written in form of systems of $2n$ ordinary differential equations

$$\dot{x}^k=v^i,\qquad\qquad \dot{v}^k=\Phi^k(x^1,\ldots,x^n,v^1,\ldots,v^n), \tag{9.2}$$

where $k=1,\,\ldots\,,n$. But in such unstructured manifolds one cannot define the force vector. Functions $\Phi^1,\,\ldots\,,\Phi^1$ in (9.2) have no direct vectorial interpretation. They form a subset of coordinates for some vector field on tangent bundle TM:

$$\mathbf{\Phi}=v^1\,\frac{\partial}{\partial x^1}+\ldots+v^n\,\frac{\partial}{\partial x^n}+\mathit{\Phi}^1\,\frac{\partial}{\partial v^1}+\ldots+\mathit{\Phi}^n\,\frac{\partial}{\partial v^n}. \tag{9.3}$$

If we have Riemannian metric $\mathbf{g}$ in M, then geodesic flow of this metric defines another vector field of the form (9.3) on TM:

$$\tilde{\mathbf{\Phi}}=v^1\,\frac{\partial}{\partial x^1}+\ldots+v^n\,\frac{\partial}{\partial x^n}-\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\Gamma^k_{ij}\,v^i\,v^j\,\frac{\partial}{\partial v^k}. \tag{9.4}$$

Difference of vectors (9.3) and (9.4) is a vector tangent to the fiber in TM:

$$\mathbf{\Phi}-\tilde{\mathbf{\Phi}}=\sum_{k=1}^{n}\left(\mathit{\Phi}^k+\sum_{i=1}^{n}\sum_{j=1}^{n}\Gamma^k_{ij}\,v^i\,v^j\right)\frac{\partial}{\partial v^k}. \tag{9.5}$$

By means of canonical vertical lift (see below in § 3 of Chapter III) we can identify (9.5) with tangent vector $\mathbf{F}$ on initial manifold M:

$$\mathbf{F}=\sum_{k=1}^{n}\left(\varPhi^k+\sum_{i=1}^{n}\sum_{j=1}^{n}\Gamma^k_{ij}\,v^i\,v^j\right)\frac{\partial}{\partial x^k}. \tag{9.6}$$

Vector $\mathbf{F}$ from (9.6) generalizes the concept of **force vector** of classical mechanics for the case of Newtonian dynamics on an abstract Riemannian manifold. In terms of components of such vector the equations (9.2) are written as

$$\dot{x}^k=v^k, \qquad\qquad \dot{v}^k+\sum_{i=1}^{n}\sum_{j=1}^{n}\Gamma^k_{ij}\,v^i\,v^j=F^k, \tag{9.7}$$

where $k=1,\,\ldots\,,n$. Force vector is a tangent vector to M. However, its components in (9.7) depend not only on local coordinates $x^1,\,\ldots\,,x^n$ of the point on M, but they depend also on the components of velocity vector $v^1,\,\ldots\,,v^n$, which are local coordinates on the fiber of tangent bundle TM. In trying to comprehend this situation in [58] we came to the concept of **extended vector field**.

Definition 9.1. Vector valued function $\mathbf{F}$ on TM that for each point $q=(p,\mathbf{v})$ on TM puts into correspondence some vector of tangent space $T_p(M)$ at the point $p=\pi(q)$ on M is called the **extended vector field** on M.

Extended scalar fields, extended covector fields and other extended tensor field are defined in a similar way, they form extended algebra of tensor fields on the manifold M (see below in Chapter II). The concept of extended algebra of tensor fields is not new, such fields are intensively used in Finslerian geometry (see [61]). In the theory of dynamical systems admitting the normal shift in Riemannian manifolds they appear to be to the point too.

In extended algebra of tensor field on Riemannian manifold we have two operation of covariant differentiation: **spatial gradient** ∇ and **velocity gradient** $\tilde{\nabla}$ (see Chapter III below). Weak normality equations in Riemannian geometry are written as the equations for the components of force field $\mathbf{F}$ in terms of gradients (covariant differentiations ∇ and $\tilde{\nabla}$):

$$\begin{cases}\displaystyle\sum_{i=1}^{n}\left(v^{-1}\,F_i+\sum_{j=1}^{n}\tilde{\nabla}_i\left(N^j\,F_j\right)\right)P^i_k=0,\\[2ex] \displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\nabla_iF_j+\nabla_jF_i-2\,v^{-1}\,F_i\,F_j\right)N^j\,P^i_k+\\[2ex] \displaystyle+\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{F^j\,\tilde{\nabla}_jF_i}{v}-\sum_{r=1}^{n}\frac{N^r\,N^j\,\tilde{\nabla}_jF_r}{v}\,F_i\right)P^i_k=0.\end{cases} \tag{9.8}$$

Here $v=|\mathbf{v}|$ is the modulus of velocity vector, $\mathbf{N}$ is the unit vector directed along

$\mathbf{v}$, and $\mathbf{P}$ is the operator field composed of orthogonal projectors to the hyperplanes perpendicular to the vector of velocity $\mathbf{v}$. Components of $\mathbf{P}$ are denoted by P^i_k, components of vector $\mathbf{N}$ are denoted by $N^1, \ldots, N^n$, and finally $N^1, \ldots, N^n$ are covariant components of force field. They are obtained by lowering the upper index:

$$F_i = \sum_{j=1}^{n} g_{ij}\, F^j.$$

Additional normality equations are also written in terms of covariant differentiations ∇ and $\tilde{\nabla}$ (spatial and velocity gradients):

$$\begin{cases} \displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n} P^i_\varepsilon\, P^j_\sigma \left(\sum_{m=1}^{n} N^m\, \frac{F_i\, \tilde{\nabla}_m F_j}{v} - \nabla_i F_j\right) = \\ \qquad = \displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n} P^i_\varepsilon\, P^j_\sigma \left(\sum_{m=1}^{n} N^m\, \frac{F_j\, \tilde{\nabla}_m F_i}{v} - \nabla_j F_i\right), \\ \displaystyle\sum_{i=1}^{n}\sum_{j=1}^{n} P^j_\sigma\, \tilde{\nabla}_j F^i\, P^\varepsilon_i = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{m=1}^{n} \frac{P^j_m\, \tilde{\nabla}_j F^i\, P^m_i}{n-1}\, P^\varepsilon_\sigma. \end{cases} \tag{9.9}$$

The above form of normality equations (9.8) and (9.9) implies that we choose local coordinates $x^1, \ldots, x^n$ in M and corresponding local coordinates $v^1, \ldots, v^n$ in fibers of tangent bundle TM. However, transfer from one system of local coordinates to another doesn't change the form of these equations. This is their difference from the equations (6.14) and (6.15), which are strictly bound to Cartesian coordinates in $\mathbb{R}^n$. Direct recalculation of (9.8) and (9.9) to the form (6.14) and (6.15) for the case $M = \mathbb{R}^n$ was done in [62].

The equations (9.8) and (9.9) were derived in [58] and [39]. Transfer of basic definitions of strong and complete normality (definitions 6.3, 6.4, and 7.1 above) to the Riemannian geometry did not require much efforts (details see in Chapter V below). Diagram of implications on Fig. 7.1 remained unchanged thereby. One should note that implication 2 in that diagram was proved immediately for the case of Riemannian manifolds skipping preliminary steps of $\mathbb{R}^2$ and $\mathbb{R}^n$.

In paper [59] the problem of metrizability by means of conformally equivalent metric $\tilde{\mathbf{g}} = e^{-2f}\,\mathbf{g}$ for the dynamical systems on Riemannian manifold with metric $\mathbf{g}$ was considered. Within the class of dynamical systems admitting the normal shift subclass of metrizable systems was distinguished. Force fields for such dynamical systems are given by the following explicit formula:

$$\mathbf{F} = -|\mathbf{v}|^2 \cdot \nabla f + 2\,(\nabla f\,|\,\mathbf{v}) \cdot \mathbf{v} + \frac{\mathbf{v}}{|\mathbf{v}|} \cdot H(|\mathbf{v}|e^{-f})e^f. \tag{9.10}$$

Here $H(v)$ is an arbitrary function of one variable, and $f = f(x^1, \ldots, x^n)$ is a scalar field on M. By ∇f in (9.10) we denote the vector field with components

$$\nabla^i f = \sum_{j=1}^{n} g^{ij}\,\frac{\partial f}{\partial x^j}.$$

Formula (9.10) appeared to coincide with formula (8.7) that corresponds to special case $M = \mathbb{R}^n$.

In § 4 of Chapter VI of this thesis wider treatment of the problem of metrizability is suggested. The condition of conformal equivalence $\mathbf{g}$ and $\tilde{\mathbf{g}}$ in its statement is eliminated. Moreover, there Newtonian dynamical systems that inherit trajectories of geodesic flow of an arbitrary affine connection Γ in M are considered. But even in such deliberate approach, if we require that dynamical system should admit the normal shift in metric $\mathbf{g}$, then Γ appears to be a metric connection for some conformally equivalent metric $\tilde{\mathbf{g}} = e^{-2f}\,\mathbf{g}$.

§ 10. Reduction of normality equations for the case $n \geqslant 3$.

In the dimension $\dim M = n \geqslant 3$ complete system of normality equations consists of weak normality equations (9.8) and additional normality equations (9.9). It is highly overdetermined, becoming more and more overdetermined with the growth of dimension. In the study of overdetermined systems of equations we encounter the problem of their compatibility. The existence of the solution (9.10) an the solution (8.8) in the flat space $M = \mathbb{R}^n$ make this problem less urgent as it could be. But this doesn't not solve the problem. The matter is that systems of differential equations of some definite order one can derive additional equations being differential consequences of initial ones. Sometimes the order of such differential consequences is equal to, or even less than maximal order of initial equations in the system. For instance, well known Korteweg-de Vries equation $u_t = 6\,u\,u_x - u_{xxx}$ is the differential consequence in the system of two Lax equations

$$\begin{cases} -\psi_{xx} + u\,\psi = \lambda\,\psi, \\ \psi_t = -4\,\psi_{xxx} + 6\,u\,\psi_x + 3\,u_x\,\psi, \end{cases}$$

where $\lambda = \text{const}$. Complete analysis of the system of differential equations implies that one should find all differential consequences of the order less or equal to the maximal order of the equations in the system, and possibly simplify the system by means of such differential consequences. Such analysis for the complete system of normality equations (9.8) and (9.9) was done in [63].

Writing normality equations (6.14) and (6.15) in Euclidean space $M = \mathbb{R}^n$ we used the expansion (6.11) for vector of force. In this expansion we have one component $A \cdot \mathbf{N}$ directed along the vector of velocity, other components in this expansion $B^1 \cdot \mathbf{M}_1, \ldots, B^{n-1} \cdot \mathbf{M}_{n-1}$ are perpendicular to $\mathbf{v}$. Due to the first equation (6.14) coefficients $B^1, \ldots, B^{n-1}$ of perpendicular components are expressed through the coefficient A. Something like this we have in general case for dynamical systems in

Riemannian manifolds. Here for the force vector we can write the expansion

$$F^k = A\,N^k + \sum_{j=1}^{n} P^k_j\,F^j, \tag{10.1}$$

similar to (6.11). Substituting (10.1) into first normality equation from (9.8) we derive the following expressions for the components of force vector **F**:

$$F_k = A\,N_k - |\mathbf{v}|\sum_{i=1}^{n} P^i_k\,\tilde\nabla_i A. \tag{10.2}$$

Formula (10.2) is called the **scalar substitution**, since it expresses force vector **F** through one scalar field A from extended algebra of tensor field in M.

Let's substitute (10.2) into (9.8). As a result first equation in (9.8) arrears to be identically fulfilled, while the second equation takes the form

$$\begin{aligned}\sum_{s=1}^{n}\Bigg(\nabla_s A + |\mathbf{v}|\sum_{q=1}^{n}\sum_{r=1}^{n} P^{qr}\,\tilde\nabla_q A\,\tilde\nabla_r\tilde\nabla_s A\,-\\ -\sum_{r=1}^{n} N^r\,A\,\tilde\nabla_r\tilde\nabla_s A - |\mathbf{v}|\sum_{r=1}^{n} N^r\,\nabla_r\tilde\nabla_s A\Bigg)\,P^s_k = 0.\end{aligned} \tag{10.3}$$

Making scalar substitution into first equation in (9.9) we get

$$\begin{aligned}\sum_{s=1}^{n}\sum_{r=1}^{n} P^r_\sigma\,P^s_\varepsilon\left(\nabla_r\tilde\nabla_s A + \sum_{q=1}^{n}\tilde\nabla_r A\,N^q\,\tilde\nabla_q\tilde\nabla_s A\right) =\\ = \sum_{s=1}^{n}\sum_{r=1}^{n} P^r_\sigma\,P^s_\varepsilon\left(\nabla_s\tilde\nabla_r A + \sum_{q=1}^{n}\tilde\nabla_s A\,N^q\,\tilde\nabla_q\tilde\nabla_r A\right).\end{aligned} \tag{10.4}$$

Reduced form of second equation in the system (9.9) is especially remarkable:

$$\sum_{r=1}^{n}\sum_{s=1}^{n} P^r_\sigma\,\tilde\nabla_r\tilde\nabla_s A\,P^{s\varepsilon} = \lambda\,P^\varepsilon_\sigma. \tag{10.5}$$

Here λ is a scalar parameter that can be determined by formula

$$\lambda = \sum_{r=1}^{n}\sum_{s=1}^{n}\frac{P^{rs}\,\tilde\nabla_r\tilde\nabla_s A}{n-1}. \tag{10.6}$$

This equation was analyzed in [63]. There the following theorem was proved.

THEOREM 10.1. *Extended scalar field A on riemannian manifold M satisfies normality equations* (10.5) *if and only if it is given by formula*

$$A = a + \sum_{i=1}^{n} b_i\, v^i, \tag{10.7}$$

where a and $\mathbf{b}$ are special scalar and covectorial fields of extended algebra such that they depend only on the modulus of velocity vector $v = |\mathbf{v}|$.

Then the expression (10.7) for scalar field A should be substituted into normality equations (10.3) and (10.4). This yields

$$\left(\frac{\partial}{\partial x^s} + b_s\,\frac{\partial}{\partial v}\right) a = \left(a\,\frac{\partial}{\partial v}\right) b_s, \tag{10.8}$$

$$\left(\frac{\partial}{\partial x^s} + b_s\,\frac{\partial}{\partial v}\right) b_r = \left(\frac{\partial}{\partial x^r} + b_r\,\frac{\partial}{\partial v}\right) b_s. \tag{10.9}$$

Unfortunately by substituting (10.7) into (10.4) in [63] we made a mistake. As a result nonlinear terms in equation (10.9) were lost, and we obtained erroneous equations. Further study of erroneous equations has led us to the equation

$$y'' = H_y(y' + 1) + H_x, \ \text{ where } \ H = H(x, y). \tag{10.10}$$

Now we know that the equation (10.10) has no relation to the theory of dynamical systems admitting the normal shift. When this wasn't yet known, in [64-67] we undertook broad study of the following class of ordinary differential equations

$$y'' = P(x, y) + 3\,Q(x, y)\,y' + 3\,R(x, y)\,{y'}^2 + S(x, y)\,{y'}^3$$

that includes the equations (10.10) as a subclass. Now, due to recent news, the results of [64-67] are excluded from this thesis.

Displeasing error found in [63] now is corrected. Corrected results are given below in Chapter VII. Correct system of equations (10.8) and (10.9) appears to be explicitly solvable. General solution of these equations is defined by two arbitrary functions. First is a function of $(n+1)$ variables $W = W(x^1, \ldots, x^n, v)$ with

$$W_v = \frac{\partial W}{\partial v} = \sum_{i=1}^{n} N^i\ \tilde{\nabla}_i W \neq 0. \tag{10.11}$$

Second is a function of one variable $h = h(v)$. Then a and b_k are the following:

$$\begin{aligned} a &= \frac{h(W(x^1, \ldots, x^n, v))}{\partial W(x^1, \ldots, x^n, v)/\partial v},\\ b_k &= -\frac{\partial W(x^1, \ldots, x^n, v)/\partial x^k}{\partial W(x^1, \ldots, x^n, v)/\partial v}. \end{aligned} \tag{10.12}$$

Function W is interpreted as a scalar field of extended algebra of tensor fields on M depending only on the modulus of velocity vector $v = |\mathbf{v}|$. Substituting (10.12) into (10.7), and then substituting (10.7) into (10.2) we make scalar substitution (10.2) more specific. As a result we get the following theorem proved in Chapter VII.

THEOREM 10.2. *Newtonian dynamical system on n-dimensional Riemannian manifold M with $n \geqslant 3$ admits the normal shift of hypersurfaces in M if and only if its force field $\mathbf{F}$ is given by the formula*

$$F_k = \frac{h(W)\,N_k}{W_v} - |\mathbf{v}| \sum_{i=1}^{n} \frac{\nabla_i W}{W_v} \left(2\,N^i\,N_k - \delta^i_k\right). \tag{10.13}$$

Here W is arbitrary extended scalar field depending only on $v = |\mathbf{v}|$ and satisfying condition (10.11)*; $h = h(v)$ is arbitrary scalar function of one variable.*

Explicit formula (10.13) for the force field $\mathbf{F}$ makes possible detailed analysis of mechanism of normal shift for any hypersurface (see in § 7 of Chapter VII below). The inequality $n \geqslant 3$ in theorem 10.2 is essential. In two-dimensional case situation is quite different (see [36]).

§ 11. Some generalizations: generalization for Finslerian geometry and higher order dynamical systems.

Formula (10.13) yields explicit and complete description of force fields for all Newtonian dynamical systems admitting the normal shift in Riemannian manifolds of the dimension greater than 2. However, one shouldn't consider it as a final point of the theory in whole. It was in 1993, when in [68] theory of dynamical systems admitting the normal shift was generalized for the systems on Finslerian manifolds. Paper [68] was submitted to one of journals, was being refereed for the long time, but, nevertheless, it wasn't published. Results of this paper is given in Chapter VIII of this thesis. Here we give brief description of these results.

Finslerian metric $\mathbf{g}$ in the manifold M, in contrast to the Riemannian metric, is extended tensor field on M. Its components depend not only on the point $p \in M$, but on the components of velocity vector $\mathbf{v}$ too. This doesn't influence the properties of scalar product of two tangent vectors

$$(\mathbf{X}\,|\,\mathbf{Y}) = \sum_{i=1}^{n} \sum_{j=1}^{n} g_{ij}(p, \mathbf{v})\,X^i\,Y^j, \tag{11.1}$$

except for the case when one of these two vectors $\mathbf{X}$ or $\mathbf{Y}$ coincides with the vector of velocity $\mathbf{v}$. It such case scalar product is nonlinear respective to one of its multiplicands. And this is the very situation that arises when we define the normal shift in Finslerian geometry. Despite to this difficulty in [68] the proper generalization of the theory was found. There all normality equations were rederived and the diagram of implications from Fig. 7.1 was completely reproduced.

Its remarkable that weak normality equations and additional normality equations in Finslerian manifolds have exactly the same form as the equations (9.8) and (9.9) in Riemannian geometry. Scalar substitution (10.2) here is also applicable. However, this is the end point of coincidences. The matter is that proof of theorem 10.2 in [63] is based on the fact that Riemannian metric **g** in M induces flat Euclidean metric in each fiber of tangent bundle TM. For Finslerian metric this is not true. Therefore problem of reducing normality equations in the dimension $n \geqslant 3$ for Finslerian manifolds still remains open.

Another possible way of generalization for the theory of dynamical systems admitting the normal shift consist in considering higher order equations. Even in flat space $\mathbb{R}^n$ we can replace Newtonian dynamics (6.1) by the equations of k-th order:

$$\frac{d^k\mathbf{r}}{dt^k} = \mathbf{F}(\mathbf{r}, \mathbf{v}_1, \ldots, \mathbf{v}_{k-1}), \text{ where } \mathbf{v}_i = \frac{d^i\mathbf{r}}{dt^i}, \quad i = 1, \ldots, k-1.$$

These are the equations of **higher** (non-Newtonian) dynamical system. By analogy with Newtonian systems vector-function **F** is called the **force field** of higher dynamical system. Similar higher order dynamical systems can be defined on an arbitrary Riemannian manifolds. This was done in [40]. In this paper one version of the theory of higher dynamical systems admitting the normal shift was suggested. All necessary definitions were stated and normality equations for the force field **F** were derived. However, for $k > 2$ complete system of normality equations appeared to be sufficient condition for the dynamical system to admit the normal shift. But it is far from being necessary condition too. And we have no examples of force fields satisfying all normality equations from [40] simultaneously. Therefore all results for $k > 2$ are not included into this thesis.

§ 12. Unsolved problems, possible applications and further prospects.

Explicit formula (10.13) for the force field **F** closes the problem of studying normality equations for $n \geqslant 3$ in Riemannian geometry[1]. But at the same time this formula substantially simplifies the use of dynamical systems admitting the normal shift for further generalization of classical Bianchi-Lie, Bäcklund, and Darboux constructions and their multidimensional analogs from [28–31]. This is the nearest and most realistic prospect of application for our theory.

We have some prospects within the theory of dynamical systems admitting the normal shift. First is the study of normality equations in Finslerian geometry and the theory of higher order systems. Then one can replace hypersurfaces by submanifolds of higher codimension. How does it change the theory? This question is still open. Note also the problem of inheriting. Suppose, that M is Riemannian or Finslerian manifold equipped with dynamical system admitting the normal shift, and let $M' \subset M$ be a submanifold in M. Can we restrict dynamical system from M to M'? Will this restriction be admitting the normal shift or not?

As a further prospect we should mark the desire to join this theory with the theory of relativity and with other geometric theories popular in modern physics. Finslerian version of our theory developed in [68] may be very opportunely here (see for instance [69–73]).

[1] In Finslerian geometry such problem still remains open.

§ 13. Structure of thesis: distribution of topics by chapters.

In this thesis we state the theory of dynamical systems admitting the normal shift starting from that point when it was generalized for the case of systems on arbitrary Riemannian manifolds. Chapter II, Chapter III, and IV contain some preliminaries. **In Chapter II** we derive the equations of Newtonian dynamics on Riemannian manifolds proceeding from mechanics of a system of mass points with holonomic bounds. Here we explain the origin of Riemannian metric in configuration space of realistic mechanical systems. Moreover in Chapter II we introduce the concept of extended algebra of tensor fields which is a basic tool for all other Chapters.

In Chapter III we build the calculus of differentiations in extended algebra of tensor fields. Here the covariant differentiations ∇ and $\tilde{\nabla}$ are introduced and their properties are studied. Structural theorem for arbitrary differentiation in extended algebra of tensor fields is proved.

In Chapter IV we consider the curves and one-parametric families of curves on Riemannian manifolds. Vector of variation is defined and tensor fields on curves are studied.

Chapter V contains main results of thesis. Here basic definitions of the theory of dynamical systems admitting the normal shift are given and normality equations are derived. And here we prove all implications shown on Fig. 7.1.

In Chapter VI problem of metrizability is studied. Explicit formula for the force field of metrizable dynamical systems admitting the normal shift is derived, and the existence of non-metrizable systems is shown. **In Chapter VII** we study the complete system of normality equations on Riemannian manifolds, and find explicit formula for their general solution in the dimension $n \geqslant 3$.

In Chapter VIII theory of dynamical systems admitting the normal shift is generalized for the case of Finslerian manifolds.

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CHAPTER II

DYNAMICAL SYSTEMS ON RIEMANNIAN MANIFOLDS.

§ 1. Newton's equations and the change of variables in them.

Let's consider a system of N rigid bodies in classical mechanics. If their sizes are small in comparison with the distances between them, then these bodies can be considered as mass-points, and their positions can be given by their radius-vectors. In such model we have N radius-vectors $\mathbf{r}_1, \ldots, \mathbf{r}_N$. They depend on t, describing the motion of these mass-points in the space. According to Newton's second law vector-functions $\mathbf{r}_1(t), \ldots, \mathbf{r}_N(t)$ are determined as the solutions of the system of N vectorial ordinary differential equations of the second order:

$$\begin{aligned} m_1\cdot\ddot{\mathbf{r}}_1 &= \mathbf{F}_1(\mathbf{r}_1,\ldots,\mathbf{r}_N,\dot{\mathbf{r}}_1,\ldots,\dot{\mathbf{r}}_N,t),\\ &\cdots\cdots\cdots\cdots\cdots\cdots\\ m_N\cdot\ddot{\mathbf{r}}_N &= \mathbf{F}_N(\mathbf{r}_1,\ldots,\mathbf{r}_N,\dot{\mathbf{r}}_1,\ldots,\dot{\mathbf{r}}_N,t). \end{aligned}\tag{1.1}$$

Here $m_1, \ldots, m_N$ is masses of points and $\mathbf{F}_1, \ldots, \mathbf{F}_N$ are forces acting on them. Let's choose some system of cartesian coordinates (which are possibly scalene). Then each of the vectors $\mathbf{r}_1, \ldots, \mathbf{r}_N$ can be represented by its three coordinates, the whole system of vectors being given by the set of numbers $x^1, \ldots, x^n$, where $n = 3N$. Lets divide force vectors $\mathbf{F}_1, \ldots, \mathbf{F}_N$ by the masses of the points. Resulting set of acceleration-vectors we denote by $\Phi^1, \ldots, \Phi^n$:

$$\mathbf{\Phi}_1=\frac{\mathbf{F}_1}{m_1},\ \mathbf{\Phi}_2=\frac{\mathbf{F}_2}{m_2},\ \ldots,\ \mathbf{\Phi}_N=\frac{\mathbf{F}_N}{m_N}.$$

Then we can rewrite the equations (1.1) as follows:

$$\begin{aligned} \ddot{x}^1 &= \Phi^1(x^1,\ldots,x^n,\dot{x}^1,\ldots,\dot{x}^n,t),\\ &\cdots\cdots\cdots\cdots\cdots\cdots\\ \ddot{x}^n &= \Phi^n(x^1,\ldots,x^n,\dot{x}^1,\ldots,\dot{x}^n,t). \end{aligned}\tag{1.2}$$

In this form of equations (1.1) we loose the individuality of each mass-point, we need not know even their masses. Equations (1.2) coincides with the equations describing the dynamics of one hypothetical mass-point of unit mass $m = 1$ in n-dimensional space. Therefore we say that the equations (1.2) define a **Newtonian dynamical system** in $\mathbb{R}^n$. Space $\mathbb{R}^n$ in this case is called **configuration space** of dynamical system (1.2), while variables $x^1, \ldots, x^n$ are coordinates in configuration space $\mathbb{R}^n$.

Let's do linear change of variables in configuration space, i. e. in instead of $x^1, \ldots, x^n$ we introduce new variables $\tilde{x}^1, \ldots, \tilde{x}^n$ defined as follows:

$$\tilde{x}^i = \sum_{j=1}^{n} T^i_j\, x^j, \quad \text{where} \quad i = 1, \ldots, n. \tag{1.3}$$

Here T^i_j are constants, they can be arranged into a non-degenerate square matrix T. Non-degeneracy of T means that change of variables (1.3) is invertible, and

$$x^j = \sum_{q=1}^{n} S^j_q\, \tilde{x}^q, \quad \text{where} \quad j = 1, \ldots, n. \tag{1.4}$$

By S^j_q in (1.4) we denote components of inverse matrix $S = T^{-1}$. Traditionally matrix S is called the matrix of **direct transition** and T is called the matrix of **inverse transition** (see [12], [32], [76], or [77]).

Relying on (1.3) and (1.4), one can do the change of variables in equations (1.2). Then we get the system of equations of the same form as the initial system (1.2):

$$\begin{aligned} &\ddot{\tilde{x}}^1 = \tilde{\Phi}^1(\tilde{x}^1, \ldots, \tilde{x}^n, \dot{\tilde{x}}^1, \ldots, \dot{\tilde{x}}^n, t),\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ &\ddot{\tilde{x}}^n = \tilde{\Phi}^n(\tilde{x}^1, \ldots, \tilde{x}^n, \dot{\tilde{x}}^1, \ldots, \dot{\tilde{x}}^n, t). \end{aligned} \tag{1.5}$$

Right hand sides of the equations (1.2) and (1.5) are bound by the relationships

$$\Phi^k = \sum_{m=1}^{n} S^k_m\, \tilde{\Phi}^m \tag{1.6}$$

that correspond to transformation rule for the components of n-dimensional vector by the change of base (see [78]). Therefore vector with components $\Phi^1, \ldots, \Phi^n$ is called the **force vector** or **force field** of Newtonian dynamical system (1.2).

Kinetic energy is an important characteristics of the system of N mass-points that is described by the equations (1.2):

$$G = \sum_{i=1}^{N} \frac{m_i\, |\dot{\mathbf{r}}_i|^2}{2}. \tag{1.7}$$

This is scalar quantity defined by velocities of mass-points. When expressed in coordinates $x^1, \ldots, x^n$ kinetic energy (1.7) can be written as follows:

$$G = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} g_{ij}\, \dot{x}^i\, \dot{x}^j. \tag{1.8}$$

Matrix g in (1.8) is block-diagonal. It is composed of of N copies of Gram matrix of that base, which is used to represent radius-vectors of mass-points. Change of variables (1.3) breaks the block structure of the matrix g. So we have

$$G=\frac{1}{2}\sum_{p=1}^{n}\sum_{q=1}^{n}\tilde{g}_{pq}\,\dot{\tilde{x}}^p\,\dot{\tilde{x}}^q, \tag{1.9}$$

where $\tilde{g}$ is not block-diagonal. Matrix $\tilde{g}$ in (1.9) is symmetric, its components are bound with the components of matrix g in (1.8) by the relationships

$$g_{ij}=\sum_{p=1}^{n}\sum_{q=1}^{n}T^p_i\,T^q_j\,\tilde{g}_{pq} \tag{1.10}$$

that correspond to transformation rule for the components of quadratic form by the change of base (see [78]). Quadratic form of kinetic energy G is positively definite, since masses of points in (1.7) are positive constants.

Now let's do nonlinear change of variables in the equations (1.2). This is the same as if we introduce curvilinear coordinates in configuration space $\mathbb{R}^n$:

$$\begin{aligned}&\tilde{x}^1=\tilde{x}^1(x^1,\ldots,x^n),\\&\ldots\ldots\ldots\ldots\ldots\\&\tilde{x}^n=\tilde{x}^n(x^1,\ldots,x^n).\end{aligned} \tag{1.11}$$

Supposing the change of variables (1.11) to be invertible and non-singular, we consider Jacoby matrix T for this change of variables:

$$T^i_j=\frac{\partial\tilde{x}^i}{\partial x^j}. \tag{1.12}$$

For non-singular change of variables matrix T with components (1.12) is non-degenerate. It is called **inverse transition** matrix. **Direct transition** matrix $S=T^{-1}$ can be obtained as Jacoby matrix for inverse change of variables:

$$\begin{aligned}&x^1=x^1(\tilde{x}^1,\ldots,\tilde{x}^n),\\&\ldots\ldots\ldots\ldots\ldots\\&x^n=x^n(\tilde{x}^1,\ldots,\tilde{x}^n).\end{aligned} \tag{1.13}$$

For the components of direct transition matrix $S=T^{-1}$ then we can write

$$S^i_j=\frac{\partial x^i}{\partial\tilde{x}^j}. \tag{1.14}$$

Let's apply nonlinear change of variables (1.11) to the equations (1.2). Transformed equations in curvilinear coordinates $\tilde{x}^1,\ \ldots,\ \tilde{x}^n$ have the same form as (1.5). However, relationships binding $\Phi^1,\ \ldots,\ \Phi^n$ and $\tilde{\Phi}^1,\ \ldots,\ \tilde{\Phi}^n$ now is different from (1.6).

Instead of (1.6) here we have the following relationships for $\Phi^1, \ldots, \Phi^n$:

$$\Phi^k=\sum_{m=1}^{n} S^k_m\,\tilde\Phi^m-\sum_{m=1}^{n}\sum_{i=1}^{n}\sum_{j=1}^{n} S^k_m\,\frac{\partial T^m_i}{\partial x^j}\,\dot x^i\,\dot x^j. \tag{1.15}$$

Let's compare the relationships (1.15) with the transformation rules for the components of affine connection in differential geometry:

$$\Gamma^k_{ij}=\sum_{m=1}^{n}\sum_{p=1}^{n}\sum_{q=1}^{n} S^k_m\,T^p_i\,T^q_j\,\tilde\Gamma^m_{pq}+\sum_{m=1}^{n} S^k_m\,\frac{\partial T^m_i}{\partial x^j} \tag{1.16}$$

(see [12], [32], [76], or [77]). We should take into account that

$$\dot{\tilde x}^p=\sum_{i=1}^{n} T^p_i\,\dot x^i,\qquad\qquad \dot x^i=\sum_{q=1}^{n} S^i_q\,\dot{\tilde x}^q. \tag{1.17}$$

Let's multiply (1.16) y $\dot x^i\,\dot x^j$ and let's sum up with respect to the indices i and j. As a result from (1.16) and (1.17) we derive

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\Gamma^k_{ij}\,\dot x^i\,\dot x^j=\sum_{m=1}^{n}\sum_{p=1}^{n}\sum_{q=1}^{n} S^k_m\,\tilde\Gamma^m_{pq}\,\dot{\tilde x}^p\,\dot{\tilde x}^q+\sum_{m=1}^{n}\sum_{i=1}^{n}\sum_{j=1}^{n} S^k_m\,\frac{\partial T^m_i}{\partial x^j}\,\dot x^i\,\dot x^j.$$

Last terms in (1.15) and in the above relationships differ only in sign. Hence, if we add these two relationships, last terms in them will cancel each other:

$$\Phi^k+\sum_{i=1}^{n}\sum_{j=1}^{n}\Gamma^k_{ij}\,\dot x^i\,\dot x^j=\sum_{m=1}^{n} S^k_m\left(\tilde\Phi^m+\sum_{p=1}^{n}\sum_{q=1}^{n}\tilde\Gamma^m_{pq}\,\dot{\tilde x}^p\,\dot{\tilde x}^q\right).$$

Denote by F^k left hand side of the above relationship:

$$F^k=\Phi^k+\sum_{i=1}^{n}\sum_{j=1}^{n}\Gamma^k_{ij}\,\dot x^i\,\dot x^j. \tag{1.18}$$

The relationships similar to (1.18) can be written in curvilinear coordinates too. Here we define the quantities $\tilde F^q$. Then

$$F^k=\sum_{q=1}^{n} S^k_q\,\tilde F^q. \tag{1.19}$$

The relationships (1.19) coincide with transformation rule for the components of some vector $\mathbf{F}$ by the change of base.

Quantities $F^1, \ldots, F^n$ introduced in (1.18) are used to simplify the relationships (1.15), which are now written as (1.19). But to define them we need an affine connection. Where could we get it from? The answer to this question is found in quadratic form of **kinetic energy** (1.7). It is positively defined, therefore it defines Riemannian metric $\mathbf{g}$ in configuration space $\mathbb{R}^n$ of dynamical system (1.2). Riemannian metric is always associated with metric connection of Levi-Civita. Its components are the following:

$$\Gamma^k_{ij} = \frac{1}{2}\sum_{s=1}^{n} g^{ks}\left(\frac{\partial g_{sj}}{\partial x^i} + \frac{\partial g_{is}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^s}\right) \tag{1.20}$$

(see more details in [12], [32], [76], or [77]). In Cartesian coordinates components of matrix g are constants. Therefore components of metric connection (1.20) in Cartesian coordinates are identically zero. Hence formula (1.18) reduces to $F^k = \Phi^k$.

In curvilinear coordinates components of metric connection (1.20) are nonzero, and formula (1.18) here is non-trivial. The equations of dynamical system (1.2) in curvilinear coordinates are written as follows:

$$\begin{gathered} \ddot{x}^1 + \sum_{i=1}^{n}\sum_{j=1}^{n} \Gamma^1_{ij}\,\dot{x}^i\,\dot{x}^j = F^1, \\ \dots\dots\dots\dots\dots\dots\dots \\ \ddot{x}^n + \sum_{i=1}^{n}\sum_{j=1}^{n} \Gamma^n_{ij}\,\dot{x}^i\,\dot{x}^j = F^n. \end{gathered} \tag{1.21}$$

Quantities $F^1, \dots, F^n$ in (1.21) are components of some n-dimensional vector $\mathbf{F}$, which is called the **force vector**. time derivatives $\dot{x}^1, \dots, \dot{x}^n$ in (1.21) are also the components of some n-dimensional vector $\mathbf{v}$, which is called the **velocity vector**. For the force vector $\mathbf{F}$ in (1.21) we have

$$\mathbf{F} = \mathbf{F}(x^1, \dots, x^n, \mathbf{v}, t). \tag{1.22}$$

In other words, this means that force vector $\mathbf{F}$ can depend on the point of configuration space (i. e. on coordinates $x^1, \dots, x^n$), it can depend on velocity vector $\mathbf{v}$, and it can contain explicit dependence on time variable t.

§ 2. Dynamical systems with constraints.

Writing the equations of the system of N bodies in form of (1.21), we unify these equations and introduce the structure of Riemannian manifold in configuration space of the system. But geometry of this manifold is trivial, since Euclidean metric (1.8) generates metric connection (1.20) of zero curvature. Much less trivial situation arises in considering mechanical system with constraints. These constraints can be hard (like welding joint) or flexible (like cardan joint, thumbscrew, tooth gearing, hinge and so on). System of N mass-points has $3N$ degrees of freedom. Constraints imposed on this system diminish the number of degrees of freedom.

Definition 2.1. Constraint imposed on the system of N mass-points is called **holonomic** constraint, if it is described by one or several equations of the form $f(\mathbf{r}_1, \ldots, \mathbf{r}_N) = 0$, where $\mathbf{r}_1, \ldots, \mathbf{r}_N$ are radius-vectors of mass-points.

Let's consider the system of N mass-points with holonomic constraints. Denote by $\mathbf{r}$ the radius-vector of a point in configuration space $\mathbb{R}^n$ of the system of free mass-points ($n = 3N$). Its components are $x^1, \ldots, x^n$. According to the definition 2.1 constraints are given by the equations

$$\begin{aligned} &f_1(x^1, \ldots, x^n) = 0,\\ &\dots\dots\dots\dots\dots\\ &f_K(x^1, \ldots, x^n) = 0. \end{aligned} \tag{2.1}$$

In general case the equations (2.1) define some submanifold M of the dimension $m = 3N - K$ in $\mathbb{R}^n$. It is called the **configuration space** of the system with constraints.

Let $y^1, \ldots, y^m$ be local coordinates in some map on of the manifold M. Suppose that embedding of M into $\mathbb{R}^n$ is described by vector-function $\mathbf{r} = \mathbf{R}(y^1, \ldots, y^m)$:

$$\begin{aligned} &x^1 = R^1(y^1, \ldots, y^m),\\ &\dots\dots\dots\dots\dots\\ &x^n = R^n(y^1, \ldots, y^m), \end{aligned} \tag{2.2}$$

Function (2.2) are the coordinate form of the vector-function $\mathbf{r} = \mathbf{R}(y^1, \ldots, y^m)$. If we substitute them into (2.1), these equations would be be identically fulfilled.

In order to calculate the n-dimensional vector of velocity for the system in $\mathbb{R}^n$ let's differentiate the function $\mathbf{r} = \mathbf{R}(y^1, \ldots, y^m)$ in t:

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \sum_{i=1}^{m} \frac{\partial \mathbf{R}}{\partial y^i}\,\dot{y}^i = \sum_{i=1}^{m} \mathbf{E}_i\,\dot{y}^i. \tag{2.3}$$

Partial derivatives

$$\mathbf{E}_i(y^1, \ldots, y^m) = \frac{\partial \mathbf{R}}{\partial y^i} \tag{2.4}$$

that arise in (2.3) form the base in tangent space to M embedded into $\mathbb{R}^n$.

Quadratic form of kinetic energy (1.7) defines scalar product $(\mathbf{X}\,|\,\mathbf{Y}) = 2\,G(\mathbf{X}, \mathbf{Y})$ in $\mathbb{R}^n$ and it induces Riemannian metric in submanifold M. Components of metric tensor $\mathbf{g}$ in local coordinates $y^1, \ldots, y^m$ on M are calculated by formula

$$g_{ij} = (\mathbf{E}_i\,|\,\mathbf{E}_j). \tag{2.5}$$

Let's differentiate vector $\mathbf{E}_i$ from (2.4) with respect to y_j and expand the derivative into a sum of two components, one of them being in tangent space to M, and second being perpendicular to the manifold M:

$$\frac{\partial \mathbf{E}_i}{\partial y^j} = \sum_{k=1}^{m} \Gamma^k_{ij}\, \mathbf{E}_k + \mathbf{N}_{ij}. \tag{2.6}$$

In (2.6) tangent component is expanded in a base of vectors $\mathbf{E}_1, \ldots, \mathbf{E}_m$, while perpendicular component is denoted by $\mathbf{N}_{ij}$. Formulas like (2.6) are known as **derivational formulas of Weingarten** (see [32], [76], or [77]). Coefficients in right hand side of such formulas coincide with components of metric connection for induced Riemannian metric (2.5) on M:

$$\Gamma^k_{ij} = \frac{1}{2} \sum_{s=1}^{m} g^{ks} \left(\frac{\partial g_{sj}}{\partial x^i} + \frac{\partial g_{is}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^s} \right). \tag{2.7}$$

Now let's calculate time derivative of velocity vector $\mathbf{v}$, taking its components $\dot{x}^1, \ldots, \dot{x}^n$ from (2.3). Then let's substitute $\ddot{x}^1, \ldots, \ddot{x}^n$ into the equations (1.2):

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \sum_{i=1}^{m} \mathbf{E}_i\, \ddot{y}^i + \sum_{i=1}^{m} \sum_{j=1}^{m} \frac{\partial \mathbf{E}_i}{\partial y^j}\, \dot{y}^i\, \dot{y}^j. \tag{2.8}$$

Taking into account derivational formulas (2.6), we can bring (2.8) to the form

$$\mathbf{a} = \sum_{k=1}^{m} \left(\ddot{y}^k + \sum_{i=1}^{m} \sum_{j=1}^{m} \Gamma^k_{ij}\, \dot{y}^i\, \dot{y}^j \right) \mathbf{E}_k + \sum_{i=1}^{m} \sum_{j=1}^{m} \mathbf{N}_{ij}\, \dot{y}^i\, \dot{y}^j. \tag{2.9}$$

Vector of n-dimensional acceleration $\mathbf{a}$ in (2.9) is expanded into a sum of two components: one is tangent to the manifold M, second is perpendicular to it. Perpendicular component of $\mathbf{a}$ given by the formula

$$\mathbf{a}_{\perp} = \sum_{i=1}^{m} \sum_{j=1}^{m} \mathbf{N}_{ij}\, \dot{y}^i\, \dot{y}^j \tag{2.10}$$

is called **centripetal acceleration**, and tangent component $\mathbf{a}_{\parallel}$ is called **tangent acceleration**. Force vector $\mathbf{\Phi}$ with components $\Phi^1, \ldots, \Phi^n$ from (1.2) can also be represented as a sum of tangent and perpendicular components: $\mathbf{\Phi} = \mathbf{\Phi}_{\parallel} + \mathbf{\Phi}_{\perp}$. Tangent component in turn can be expanded in the base of vectors $\mathbf{E}_1, \ldots, \mathbf{E}_m$ in tangent space to the manifold M:

$$\mathbf{\Phi}_{\parallel} = \sum_{k=1}^{m} F^k\, \mathbf{E}_k. \tag{2.11}$$

Taking into account (2.9), (2.10), and (2.11), we can rewrite the equations of dynamics (1.2) in form of two systems of equations. First arises from $\mathbf{a}_{\parallel} = \mathbf{\Phi}_{\parallel}$:

$$\ddot{y}^k + \sum_{i=1}^{m} \sum_{j=1}^{m} \Gamma^k_{ij}\, \dot{y}^i\, \dot{y}^j = F^k, \quad \text{where} \quad k = 1, \ldots, m. \tag{2.12}$$

This is the system of m scalar differential equations of the second order with respect to local coordinates $y^1, \ldots, y^m$. Second system is written as an equality

$$\sum_{i=1}^{m}\sum_{j=1}^{m}\mathbf{N}_{ij}\,\dot{y}^i\,\dot{y}^j = \mathbf{\Phi}_\perp \tag{2.13}$$

of two vectors, which are perpendicular to M. Therefore it is equivalent to $n-m$ scalar equations.

The equations (2.12) play the main role in description of dynamics for mechanical systems with constraints. As for the equations (2.13), they are often omitted at all. The matter is that in right hand sides of initial Newton's equations (1.1), apart from forces defined by the nature of interacting mass-points, we have **constraint forces**. In most these are maintaining forces that provide the equations (2.1) to be fulfilled. For instance, if we have two bodies bound by a hard rod, then in dynamics there appears the force that prevents tension and contraction of this rod. Let's denote by $\mathbf{Q}$ the vector of constraint forces, and let's expand it into a sum of two components: $\mathbf{Q} = \mathbf{Q}_\parallel + \mathbf{Q}_\perp$. Then the equations (2.13) are written as follows:

$$-\mathbf{Q}_\perp = \mathbf{\Phi}_\perp - \sum_{i=1}^{m}\sum_{j=1}^{m}\mathbf{N}_{ij}\,\dot{y}^i\,\dot{y}^j. \tag{2.14}$$

Force $Q_\perp$ is purely maintaining component of constraint force $\mathbf{Q}$. And (2.14) is the very equation that defines the value of this component. Vector

$$\mathbf{N} = -\sum_{i=1}^{m}\sum_{j=1}^{m}\mathbf{N}_{ij}\,\dot{y}^i\,\dot{y}^j,$$

differs from (2.10) only by sign, it is called **centrifugal force**. One shouldn't be surprised that in our notations forces and accelerations are measured by the same units. This is because we reduced initial Newton's equations (1.1) to the form (1.2) that corresponds to n-dimensional dynamics of a point with mass $m = 1$.

Force $\mathbf{Q}_\parallel$ is usually a dissipative component of constraint force $\mathbf{Q}$. It is due to the friction in joints (in cardan joints, thumbscrews, tooth gearings, hinges and so on). It is defined by the load of joints (i. e. by the force$Q_\perp$) and by relative velocities of various parts of joints. Therefore we can write

$$\mathbf{Q}_\parallel = \mathbf{K}(y^1, \ldots, y^n, \dot{y}^1, \ldots, \dot{y}^n, \mathbf{Q}_\perp). \tag{2.15}$$

Explicit expression for the function $\mathbf{K}$ in (2.15) depends on particular structure of joints and materials they are made of.

So, in order to describe mechanical system with constraints we should find vector $\mathbf{Q}_\perp$ by means of the equation (2.14), then we should substitute it into (2.15) and calculate total vector of constraint force $\mathbf{Q} = \mathbf{Q}_\parallel + \mathbf{Q}_\perp$. It appears to be depending on local coordinates $y^1, \ldots, y^m$ and their time derivatives $\dot{y}^1, \ldots, \dot{y}^m$.

Let's expand vector $\mathbf{Q}_{\parallel}$ in the base of vectors $\mathbf{E}_1, \ldots, \mathbf{E}_m$ in tangent space to the manifold M embedded into the space $\mathbb{R}^n$:

$$\mathbf{Q}_{\parallel} = \sum_{k=1}^{m} Q^k\,\mathbf{E}_k. \tag{2.16}$$

Taking into account constraint force in the form of (2.16), we can write the equations of dynamics (2.12) as

$$\ddot{y}^k + \sum_{i=1}^{m}\sum_{j=1}^{m} \Gamma^k_{ij}\,\dot{y}^i\,\dot{y}^j = F^k + Q^k, \quad \text{where} \quad k = 1, \ldots, m. \tag{2.17}$$

Here F^k are the components of force vector other than constraint force. Force components F^k and Q^k in right hand sides of the equations (2.17) differ only by their physical origin. Adding Q^k to F^k, we do not change the structure of differential equations in whole, since

$$\begin{aligned} &F^k = F^k(y^1, \ldots, y^n, \dot{y}^1, \ldots, \dot{y}^n, t),\\ &Q^k = Q^k(y^1, \ldots, y^n, \dot{y}^1, \ldots, \dot{y}^n). \end{aligned} \tag{2.18}$$

Due to (2.18) we can redesignate $Q^k + F^k$ by F^k, and then we come back to the equations (2.12). However, the study of constraint forces, as described above, is important from conceptual point of view. It explains how the system of n equations (1.2) is reduced to the system of m equations (2.12).

§ 3. Newtonian dynamical systems in Riemannian manifolds.

As we see in § 1 and § 2, each mechanical system with n degrees of freedom is associated with some n-dimensional Riemannian manifold M, which is called **configuration space** of the system. This takes place for the system of free mass-points and for the system with holonomic constraints as well. Let $x^1, \ldots, x^n$ be local coordinates in some map on the manifold M. Kinetic energy of mechanical system is defined by metric tensor $\mathbf{g}$ according to the formula

$$G = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} g_{ij}\,\dot{x}^i\,\dot{x}^j, \tag{3.1}$$

its dynamics is defined by Newton's equations written as

$$\ddot{x}^k + \sum_{i=1}^{n}\sum_{j=1}^{n} \Gamma^k_{ij}\,\dot{x}^i\,\dot{x}^j = F^k, \quad \text{where} \quad k = 1, \ldots, n. \tag{3.2}$$

Here Γ^k_{ij} are the components of metric connection, which can be calculated by standard formula (1.20).

Each solution of the equations (3.2) is a set of n functions $x^1(t), \dots, x^n(t)$ that can be interpreted as parametric equations o some curve in M. This curve is called the **trajectory**. Let's recalculate the equations trajectory to local coordinates in some other map on M by means of change of variables (1.11). Resulting set of functions $\tilde{x}^1(t), \dots, \tilde{x}^n(t)$ forms the solution of the equations analogous to (3.2):

$$\ddot{\tilde{x}}^k + \sum_{i=1}^{n}\sum_{j=1}^{n} \tilde{\Gamma}^k_{ij}\, \dot{\tilde{x}}^i\, \dot{\tilde{x}}^j = \tilde{F}^k, \quad \text{where} \quad k = 1, \dots, n. \tag{3.3}$$

Definition 3.1. Systems of equations (3.2) and (3.3) related to the pair of overlapping maps $(U, x^1, \dots, x^n)$ and $(\tilde{U}, \tilde{x}^1, \dots, \tilde{x}^n)$ on M are called **concordant systems**, if in the domain of intersection of maps right hand sides of these equations are bound by the relationships (1.19).

Definition 3.2. Newtonian dynamical system on Riemannian manifold M is a set of concordant systems of differential equations of the form (3.2) associated to each map on this manifold.

Simplest example of Newtonian dynamical system is a **geodesic flow**, when right hand sides of the equations (3.2) for all maps are identically zero. Trajectories of geodesic flow are geodesic lines on M.

Let's study the time dependence of kinetic energy (3.1) on the trajectory of some dynamical system (3.2):

$$G(t_2) - G(t_1) = \int\limits_{t_1}^{t_2} \frac{dG}{dt}\, dt. \tag{3.4}$$

Time derivative under the sign of integral in (3.4) can be evaluated by direct calculation. Taking into account the symmetry of components of metric $\mathbf{g}$, we get:

$$\begin{aligned}\frac{dG}{dt} &= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(g_{ij}\,\ddot{x}^i\,\dot{x}^j + g_{ij}\,\dot{x}^i\,\ddot{x}^j + \sum_{k=1}^{n}\frac{\partial g_{ij}}{\partial x^k}\,\dot{x}^i\,\dot{x}^j\,\dot{x}^k\right) = \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} g_{ij}\,F^i\,\dot{x}^j + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\nabla_k g_{ij}\,\dot{x}^i\,\dot{x}^j\,\dot{x}^k.\end{aligned}$$

Second term in the obtained expression vanishes due to vanishing of covariant derivative $\nabla_k g_{ij}$. The latter fact is the consequence of concordance of metric and connection (see [12], [32], [76], or [77]). Therefore we have

$$\frac{dG}{dt} = \sum_{i=1}^{n}\sum_{j=1}^{n} g_{ij}\,F^i\,\dot{x}^j. \tag{3.5}$$

Right hand side of (3.5) is a scalar product of two tangent vectors on M: force vector $\mathbf{F}$ and velocity vector $\mathbf{v}$. In inner geometry of the manifold M they are represented by two first order linear differential operators:

$$\mathbf{F} = \sum_{i=1}^{n} F^i \frac{\partial}{\partial x^i}, \qquad \mathbf{v} = \sum_{i=1}^{n} \dot{x}^i \frac{\partial}{\partial x^i}. \tag{3.6}$$

If M is embedded into some Euclidean space (as it usually happens for mechanical systems with constraints), then vectors (3.6) in exterior geometry are represented by (2.11) and (2.3). Regardless to the form of representation the vectors $\mathbf{F}$ and $\mathbf{v}$ the relationship (3.4) can be written as

$$G(t_2) - G(t_1) = \int_{t_1}^{t_2} (\mathbf{F} \,|\, \mathbf{v})\, dt. \tag{3.7}$$

Integral in right hand side of (3.7) is called the work of the force $\mathbf{F}$ on the trajectory of dynamical system.

Special subclass of dynamical systems (3.2) is formed by system for which the value of integral in (3.7) depends only on initial an final points of trajectory. Force $\mathbf{F}$ for such systems is determined by gradient of some scalar function $\mathit{\Pi}$ on M:

$$F^i = -\sum_{k=1}^{n} g^{ik}\, \nabla_k \mathit{\Pi} = -\sum_{k=1}^{n} g^{ik} \frac{\partial \mathit{\Pi}}{\partial x^k}. \tag{3.8}$$

Function $\mathit{\Pi} = \mathit{\Pi}(x^1, \ldots, x^n)$ in (3.8) is called **potential energy** of the system, and sum $E = G + \mathit{\Pi}$ is called **total energy** of the system. From (3.7) and (3.8) we can easily derive **conservation law** for total energy of the system. Dynamical systems with potential energy obeying the conservation law for their total energy are called **conservative** systems.

For conservative mechanical systems (3.2) one can define the Lagrange function $L = G - \mathit{\Pi}$. The equations (3.2) of dynamical system in this case are written in form of Euler-Lagrange equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^k}\right) - \frac{\partial L}{\partial x^k} = 0, \quad \text{where} \quad k = 1, \ldots, n. \tag{3.9}$$

We shall not come deep into the Lagrangian mechanics any more (see books [60] or [79]). Instead of this we shall formulate the main conclusion from all what was said in §§ 1–3: *Newtonian dynamical systems on Riemannian manifolds appears in a quite natural way in the study of real mechanical systems. Corresponding manifolds for such systems are obtained as their configuration spaces, while metric are defined by kinetic energy of these systems.*

§ 4. Tangent bundle and the force field of Newtonian dynamical system.

Having solved the problem of the origin of Newtonian dynamical systems on Riemannian manifolds, let's proceed with the description of mathematical structures related to such systems. Suppose that some particular Newtonian dynamical system (3.2) is fixed. Denote by $v^1, \ldots, v^n$ components of velocity vector $\mathbf{v}$ in (3.6). Then by means of $v^1, \ldots, v^n$ we can rewrite the equations (3.2) as a system of $2n$ differential equations of the first order:

$$\dot{x}^k = v^k, \qquad \dot{v}^k + \sum_{i=1}^{n}\sum_{j=1}^{n} \Gamma^k_{ij}\, v^i\, v^j = F^k. \tag{4.1}$$

Here k runs from 1 to n. The equations (4.1) correspond to the dynamical system on tangent bundle TM of the manifold M. Let's recall that **tangent bundle** TM is a set of pairs $(p, \mathbf{v})$, where p is a point of M, and $\mathbf{v}$ is a vector from tangent space $V = T_p(M)$ of manifold M at the point p. Denote by $\pi : TM \to M$ canonical projection that maps the pair $(p, \mathbf{v})$ onto the point p. Structure of manifold in tangent bundle TM is defined by maps

$$(\pi^{-1}(U), x^1, \ldots, x^n, v^1, \ldots, v^n), \tag{4.2}$$

where $(U, x^1, \ldots, x^n)$ is a local map on base manifold M. In other words, $x^1, \ldots, x^n$ are local coordinates of the point p from pair $(p, \mathbf{v})$ in the map $(U, x^1, \ldots, x^n)$, and $v^1, \ldots, v^n$ are components of the vector $\mathbf{v}$ in the expansion

$$\mathbf{v} = \sum_{i=1}^{n} v^i\, \frac{\partial}{\partial x^i}. \tag{4.3}$$

Suppose that two maps $(U, x^1, \ldots, x^n)$ and $(\tilde{U}, \tilde{x}^1, \ldots, \tilde{x}^n)$ are overlapping, i. e. $U \cap \tilde{U} \neq \varnothing$, and suppose that transition functions are given:

$$\begin{aligned} &\tilde{x}^1 = \tilde{x}^1(x^1, \ldots, x^n), && x^1 = x^1(\tilde{x}^1, \ldots, \tilde{x}^n),\\ &\ldots\ldots\ldots\ldots\ldots && \ldots\ldots\ldots\ldots\ldots\\ &\tilde{x}^n = \tilde{x}^n(x^1, \ldots, x^n), && x^n = x^n(\tilde{x}^1, \ldots, \tilde{x}^n). \end{aligned} \tag{4.4}$$

Then corresponding maps $\pi^{-1}(U)$ and $\pi^{-1}(\tilde{U})$ of the form (4.2) on TM are also overlapping. Transition functions for such maps are given by formulas

$$\begin{cases} \tilde{x}^k = \tilde{x}^k(x^1, \ldots, x^n),\\[1ex] \tilde{v}^k = \displaystyle\sum_{i=1}^{n} T^k_i\, v^i, \end{cases} \qquad \begin{cases} x^k = x^k(\tilde{x}^1, \ldots, \tilde{x}^n),\\[1ex] v^k = \displaystyle\sum_{i=1}^{n} S^k_i\, \tilde{v}^i, \end{cases} \tag{4.5}$$

where k runs from 1 to n. Transition matrices S and $T = S^{-1}$ in (4.5) are composed by partial derivatives of transition functions from (4.4):

$$S^k_i = \frac{\partial x^k}{\partial \tilde{x}^i}, \qquad T^k_i = \frac{\partial \tilde{x}^k}{\partial x^i}. \tag{4.6}$$

Tangent bundle TM corresponds to physical concept of **phase space**. Its dimension is twice as great than the dimension of **configuration space**.

Now let's consider again the equations (4.1). According to the results of §§ 1–3 the quantities $F^1, \ldots, F^n$ in (4.1) are the components of force vector $\mathbf{F}$ tangent to M at the point p with local coordinates $x^1, \ldots, x^n$. This vector depend on point p, on velocity vector $\mathbf{v}$, and it can depend on t explicitly:

$$\mathbf{F} = \mathbf{F}(p, \mathbf{v}, t). \tag{4.7}$$

Definition 4.1. Newtonian dynamical system (4.1) is called the **stationary system**, if its force field doesn't depend on time variable explicitly.

Further we shall consider only stationary Newtonian dynamical systems. Here instead of the formula (4.7) we have

$$\mathbf{F} = \mathbf{F}(p, \mathbf{v}). \tag{4.8}$$

Vectors depending on the points of manifold usually correspond to the concept of vector fields. In order to be more precise let's remember standard definition of vector field on the manifold.

Definition 4.2. Vector field $\mathbf{F}$ on the manifold M is a vector-valued function that to each point p of M puts into correspondence some vector from tangent space $T_p(M)$ at this point p.

Now its clear that definition 4.2 describes only some special functions (4.8) when force doesn't depend on the velocity: $\mathbf{F} = \mathbf{F}(p)$. In general case argument of the function (4.8) is a point of tangent bundle TM. However these functions cannot be treated as vector fields on TM too. Their values are tangent vectors to M, not to TM. Therefore we need to formulate special definition to describe force fields of Newtonian dynamical systems.

Definition 4.3. **Extended** vector field $\mathbf{F}$ on the manifold M is a vector-valued function that to each point $q = (p, \mathbf{v})$ of tangent bundle TM puts into correspondence some vector from tangent space $T_p(M)$ at the point p of base manifold M being the image of q by canonical projection: $p = \pi(q)$.

Schematically the concept of extended vector field is shown on Fig. 4.1. Let $q = (p, \mathbf{v})$ be the point of tangent bundle TM. By definition it is the pair including the point $p = \pi(q)$ and the vector $\mathbf{v}$ tangent to M at the point p. Function that maps $q = (p, \mathbf{v})$ into the vector $\mathbf{v}$ is an example of extended vector field on M. It is called the **velocity field** on M. Velocity field is defined on any manifold regardless to the existence of Riemannian metric or Newtonian dynamical system on it. As for the Riemannian manifolds, here stationary Newtonian dynamical systems (4.1) are in one-to-one correspondence with extended vector fields, i. e. in order to define such dynamical system we should only define its force field $\mathbf{F}$.

§ 5. Extended algebra of tensor fields.

Definition 4.3 extends the concept of vector field. Similarly we can extend the concept of scalar field, of covectorial field, and of arbitrary tensor field as well.

Definition 5.1. **Extended** scalar field φ on the manifold M is a scalar function on its tangent bundle TM.

Definition 5.2. **Extended** covectorial field $\mathbf{F}$ on the manifold M is a covector-valued function that to each point $q = (p, \mathbf{v})$ of tangent bundle TM puts into correspondence some vector from cotangent space $T^*_p(M)$ at the point p of base manifold M being the image of q by canonical projection: $p = \pi(q)$.

In order to define extended tensor field first we consider the following tensor products of r copies of tangent space and s copies of cotangent space:

$$T^r_s(p, M) = \overbrace{T_p(M) \otimes \ldots \otimes T_p(M)}^{r \text{ times}} \otimes \underbrace{T^*_p(M) \otimes \ldots \otimes T^*_p(M)}_{s \text{ times}}$$

Tensor product $T^r_s(p, M)$ is known as a space of (r, s)-tensors at the point p of the manifold M. Pair of integer numbers (r, s) determines the type of tensors. Elements of the space $T^r_s(p, M)$ are called *r-times contravariant and s-times covariant tensors*, or tensors of the type (r, s), or for brevity (r, s)-tensors.

Definition 5.3. **Extended** tensor field $\mathbf{X}$ of the type (r, s) on the manifold M is a tensor-valued function that to each point $q = (p, \mathbf{v})$ of tangent bundle TM puts into correspondence some tensor from tensor space $T^r_s(p, M)$ at the point p on M.

Tensor field of the type $(1, 0)$ is a vector field, tensor field of the type $(0, 1)$ is a covector field, and tensor field of the type $(0, 0)$ is a scalar field. This is the consequence of the following relationships:

$$T^1_0(p, M) = T_p(M), \qquad T^0_1(p, M) = T^*_p(M), \qquad T^0_0(p, M) = \mathbb{R}.$$

Ordinary tensor fields form a subset in the set of extended tensor fields. Most of properties of ordinary tensor fields are available for extended fields too. First of all we consider algebraic properties. In the set of tensors we have summation, multiplication by real numbers, and the operation of tensor product:

(1) $T^r_s(p, M) + T^r_s(p, M) \longrightarrow T^r_s(p, M)$;
(2) $\mathbb{R} \cdot T^r_s(p, M) \longrightarrow T^r_s(p, M)$;
(3) $T^r_s(p, M) \otimes T^a_c(p, M) \longrightarrow T^{r+a}_{s+c}(p, M)$.

Due to the equality $T^0_0(p, M) = \mathbb{R}$ second operation is a special case of the third one. Now let's consider the sum

$$\mathbf{T}(p, M) = \bigoplus_{r=0}^{\infty} \bigoplus_{s=0}^{\infty} T^r_s(p, M). \tag{5.1}$$

Operations (1)–(3) equip the set of tensors (5.1) with the structure of algebra over the field of real numbers. For extended tensor fields these operations are implemented pointwise in each point $p = \pi(q)$. Suppose that $T^r_s(M)$ is a set of extended tensor fields of the type (r, s). It is equipped with the structure of module over the ring of extended scalar fields, while the following sum

$$\mathbf{T}(M) = \bigoplus_{r=0}^{\infty} \bigoplus_{s=0}^{\infty} T^r_s(M) \tag{5.2}$$

is a graded algebra over that ring. Algebra (5.2) in what follows will be called the **extended algebra of tensor fields** on the manifold M.

Extension of the concept of tensor fields in the sense of definition 5.3 isn't of common use in geometry. Probably it was first used in Finslerian geometry (see [61]), and here it is presently in use (see [69–73]). Concept of extended tensor field can have various generalizations, one of them is considered in [40]. The concept of **jets** is close relative to the concept extended tensor fields, it is much more general and more popular now.

§ 6. Extended tensor fields in local coordinates.

Let's choose some map $(U, x^1, \ldots, x^n)$ on the manifold M. In definition 5.3, apart from M, we have another manifold, it's tangent bundle TM. Therefore to describe extended tensor field $\mathbf{X} \in T^r_s(M)$ we might have to choose tho maps: one for to map point $q \in TM$, and other for to express components of tensor $\mathbf{X}(q) \in T^r_s(p, M)$. But in our terminology extended tensor field $\mathbf{X}$ is treated as a field on M. Therefore we will choose only one map $(U, x^1, \ldots, x^n)$ on M and will construct associated map on tangent bundle TM according to (4.2). Local coordinates $x^1, \ldots, x^n, v^1, \ldots, v^n$ of such map will be called **canonically associated coordinates** for the coordinates $x^1, , \ldots, x^n$ on M. Base in tangent space $T_p(M)$ to M consists of vectors

$$\frac{\partial}{\partial x^1}, \ \ldots, \ \frac{\partial}{\partial x^n}, \tag{6.1}$$

while base in dual space $T^*_p(M)$ consists of covectors

$$dx^1, \ \ldots, \ dx^n. \tag{6.2}$$

Two bases (6.1) and (6.2) are dual to each other. Base in tensor space $T^r_s(p, M)$ is enumerated by $r + s$ indices:

$$\mathbf{E}^{j_1 \ldots j_s}_{i_1 \ldots i_r} = \frac{\partial}{\partial x^{i_1}} \otimes \ldots \otimes \frac{\partial}{\partial x^{i_r}} \otimes dx^{j_1} \otimes \ldots \otimes dx^{j_s}, \tag{6.3}$$

each index running from 1 to n. For the tensor field $\mathbf{X}$ from the definition 5.3 we have the following expansion:

$$\mathbf{X}=\sum_{i_1=1}^{n}\ldots\sum_{i_r=1}^{n}\sum_{j_1=1}^{n}\ldots\sum_{j_r=1}^{n}X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}\,\mathbf{E}^{j_1\ldots\,j_s}_{i_1\ldots\,i_r}. \tag{6.4}$$

Here $X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}$ are the components of the field $\mathbf{X}$ in the base (6.3). They are numeric functions of canonically associated coordinates $x^1,\,\ldots\,,\,x^n,v^1,\,\ldots\,,\,v^n$. These functions are called **components** of extended tensor field $\mathbf{X}$ in the map $(U,x^1,\ldots\,,x^n)$.

Suppose that two maps $(U,x^1,\ldots\,,x^n)$ and $(\tilde{U},\tilde{x}^1,\ldots\,,\tilde{x}^n)$ on M are overlapping. Then in the domain $U\cap\tilde{U}$ components of tensor field $\mathbf{X}$ related to these two maps are bound with each other by standard relationships

$$X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}=\sum_{\substack{h_1,\,\ldots\,,\,h_r\\ k_1,\,\ldots\,,\,k_s}}\ldots\sum S^{i_1}_{h_1}\ldots\,S^{i_r}_{h_r}T^{k_1}_{j_1}\ldots\,T^{k_s}_{j_s}\,\tilde{X}^{h_1\ldots\,h_r}_{k_1\ldots\,k_s}, \tag{6.5}$$

which are known as transformation rule for the components of tensor field under the change or local coordinates. Components of transition matrices S and $T=S^{-1}$ in (6.5) are defined by (4.6). Operating with components of tensor fields in the base (6.3) and using canonically associated coordinates for to map their arguments, we could reach our goals without mention of tangent bundle TM at all.

§ 7. Contraction of extended tensor fields.

Cotangent space $T^*_p(M)$ consists of linear functionals on the tangent space $T_p(M)$. Choosing vector $\mathbf{X}$ from $T_p(M)$ and covector $\mathbf{h}$ from $T^*_p(M)$, we can apply linear functional $\mathbf{h}$ to the vector $\mathbf{X}$, and we can write this as follows:

$$\mathbf{h}(\mathbf{X})=\langle\mathbf{h}\,|\,\mathbf{X}\rangle=C(\mathbf{X},\mathbf{h}). \tag{7.1}$$

According to (7.1) we can write $\mathbf{h}(\mathbf{X})$ in form of scalar product $\langle\mathbf{h}\,|\,\mathbf{X}\rangle$, this style of writing arose in quantum mechanics (see [80] or [81]). In contrast to (2.5) here we use angular brackets. Functional form $C(\mathbf{X},\mathbf{h})$ of writing $\mathbf{h}(\mathbf{X})$ in (7.1) defines the function C bilinear in both its arguments, one of which is a vector while the other is a covector. It can be extended up to the linear map

$$C\colon\,T_p(M)\otimes T^*_p(M)\longrightarrow\mathbb{R}. \tag{7.2}$$

The map (7.2) is known as the **contraction** of tensors of the type $(1,1)$. In tensor product of several copies of $T_p(M)$ and $T^*_p(M)$

$$T_p(M)\otimes\ldots\,\underbrace{T_p(M)\ldots\otimes T_p(M)\otimes T^*_p(M)\otimes\ldots\,T^*_p}(M)\ldots\otimes T^*_p(M)$$

one can define contraction of any copy of $T_p(M)$ with any copy of $T^*_p(M)$. This is done by the map $C\colon T^{r+1}_{s+1}(p,M)\longrightarrow T^r_s(p,M)$ that in local coordinates is given by

$$C(\mathbf{X})^{i_1\dots\,i_r}_{j_1\dots\,j_s}=\sum_{\rho=1}^{n}X^{i_1\dots\,i_{m-1}\,\rho\,i_m\dots\,i_r}_{j_1\dots\,j_{k-1}\,\rho\,j_k\dots\,j_s}. \tag{7.3}$$

This map is known as **contraction** of tensor $\mathbf{X}$ by its m-th upper and k-th lower index. By contracting extended tensor field this operation is executed pointwise in each point $p=\pi(q)$ of the manifold M. Therefore

$$C\left(\bigotimes_{i=1}^{r+1}\mathbf{X}_i\otimes\bigotimes_{j=1}^{s+1}\mathbf{h}_j\right)=\langle\mathbf{h}_k\,|\,\mathbf{X}_m\rangle\cdot\left(\bigotimes_{i\neq m}^{r+1}\mathbf{X}_i\otimes\bigotimes_{j\neq k}^{s+1}\mathbf{h}_j\right). \tag{7.4}$$

Here $\mathbf{X}_1,\,\dots\,,\,\mathbf{X}_{r+1}$ are some arbitrary vector fields, and $\mathbf{h}_1,\,\dots\,,\,\mathbf{h}_{s+1}$ are arbitrary covectorial fields. Formula (7.4) yields one more way of determining the contraction map (7.3).

CHAPTER III

DIFFERENTIATION IN EXTENDED ALGEBRA OF TENSOR FIELDS.

§ 1. Differentiation of extended tensor fields.

We shall construct the theory of differentiation for extended tensor fields following the scheme of [12], where it was done for ordinary (non-extended) tensor fields. The operation of differentiation for tensor fields assumes sufficient smoothness of objects to be differentiated. Suppose that M is smooth real manifold.

Definition 1.1. Extended scalar field φ is **smooth**, if it is represented by smooth function $\varphi(x^1,\ldots,x^n,v^1,\ldots,v^n)$ in any local map U of M.

Remember that $x^1,\ldots,x^n,v^1,\ldots,v^n$ are local coordinates in tangent bundle TM canonically associated with local coordinates $x^1,\ldots,x^n$ in the map U (see formula (4.2) in § 4 of Chapter II). Extended tensor field $\mathbf{X}$ of the type (r,s) in map U is represented by its components $X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}$ in the expansions like (6.4) in Chapter II.

Definition 1.2. Extended tensor field $\mathbf{X}$ is **smooth**, if it is represented by smooth functions $X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}(x^1,\ldots,x^n,v^1,\ldots,v^n)$ in any local map U on M.

Starting from here we shall understand $T^r_s(M)$ to be the space of smooth extended tensor fields on M, and the sum of such spaces will be called the **extended algebra of tensor fields** (see formula (5.2) in Chapter II).

Definition 1.3. The map $D\colon \mathbf{T}(M)\to\mathbf{T}(M)$ is called the **differentiation** of extended algebra of tensor fields, if the following conditions are fulfilled:

(1) concordance with grading: $D(T^r_s(M))\subset T^r_s(M)$;
(2) $\mathbb{R}$-linearity: $D(\mathbf{X}+\mathbf{Y})=D(\mathbf{X})+D(\mathbf{Y})$ and $D(\lambda\mathbf{X})=\lambda D(\mathbf{X})$ for $\lambda\in\mathbb{R}$;
(3) commutation with contractions: $D(C(\mathbf{X}))=C(D(\mathbf{X}))$;
(4) Leibniz rule: $D(\mathbf{X}\otimes\mathbf{Y})=D(\mathbf{X})\otimes\mathbf{Y}+\mathbf{X}\otimes D(\mathbf{Y})$.

The subset $T^0_0(M)$ in extended algebra of tensor fields is closed with respect to the operation of tensor product. It coincides with the ring of smooth scalar functions on tangent bundle TM, we denote this ring by $\mathfrak{F}(TM)$. Here $\mathfrak{F}$ is gothic version of the letter F. Tensor product in $T^0_0(M)$ coincides with ordinary multiplication of functions in the ring $\mathfrak{F}(TM)$. Total set of extended tensor fields $T(M)$ is an algebra over this ring.

Let's consider the set of differentiations of extended algebra of tensor fields $\mathbf{T}(M)$. One can easily check that

(1) the sum of two differentiations is a differentiation of the algebra $\mathbf{T}(M)$;

(2) product of any differentiation and any function from the ring $\mathfrak{F}(TM)$ is a differentiation of the algebra $\mathbf{T}(M)$.

Let's denote by $\mathfrak{D}(M)$ the total set of differentiations of extended algebra of tensor fields $\mathbf{T}(M)$. Here $\mathfrak{D}$ is a gothic version of the letter D. Above two properties mean that $\mathfrak{D}(M)$ is a module over the ring of functions $\mathfrak{F}(TM)$. Composition of two differentiations is not a differentiation. However, their commutator

$$[D_1,\, D_2] = D_1 \circ D_2 - D_2 \circ D_1$$

is a differentiation. Therefore we can say that $\mathfrak{D}(M)$ is a Lie algebra.

Suppose that D is some differentiation from Lie algebra $\mathfrak{D}(M)$. Denote by δ and ξ restrictions of the map $D\colon \mathbf{T}(M) \to \mathbf{T}(M)$ to the spaces $T^0_0(M)$ and $T^1_0(M)$:

$$\delta\colon T^0_0(M) \to T^0_0(M), \qquad\qquad \xi\colon T^1_0(M) \to T^1_0(M). \tag{1.1}$$

Due to the equality $T^0_0(M) = \mathfrak{F}(TM)$ first map in (1.1) is a differentiation of the ring of smooth functions on the tangent bundle TM:

$$\delta(\varphi \cdot \psi) = \delta(\varphi) \cdot \psi + \varphi \cdot \delta(\psi). \tag{1.2}$$

It is known (see [12], Chapter I, § 1) that any differentiation of the ring $\mathfrak{F}(TM)$ is determined by some vector field $\mathbf{Z}$ in TM: $\delta(\varphi) = \mathbf{Z}\varphi$. In local coordinates it is written in the following form:

$$\delta(\varphi) = \mathbf{Z}\varphi = \sum_{i=1}^{n} Z^i \frac{\partial \varphi}{\partial x^i} + \sum_{i=1}^{n} W^i \frac{\partial \varphi}{\partial v^i}. \tag{1.3}$$

From (1.3) we easily derive the following property of locality for the map δ.

Lemma 1.1. *If the values of two functions φ and ψ coincide in some neighborhood of the point $q \in TM$, then the functions $\delta(\varphi)$ and $\delta(\psi)$ are equal at that point q.*

In other words, all changes of value of φ outside of some arbitrarily small neighborhood of the point q do not change the value of $\delta(\varphi)$ at the point q. Similar property of locality takes place for the initial map D too, for which δ is a restriction to the space $T^0_0(M)$.

Lemma 1.2. *If two extended tensor fields $\mathbf{X}$ and $\mathbf{Y}$ coincide in some neighborhood of the point $q \in TM$, then the values of extended tensor fields $D(\mathbf{X})$ and $D(\mathbf{Y})$ at the point q are equal.*

Proof. Denote by $\mathbf{V}$ the difference of fields $\mathbf{X}$ and $\mathbf{Y}$. Field $\mathbf{V}$ equals to zero identically in some neighborhood O of the point q. Let's choose some smooth scalar function φ on TM such that it is equal to 1 outside the neighborhood O and equal to 0 at the point q. Then $\mathbf{V} = \varphi \cdot \mathbf{V} = \varphi \otimes \mathbf{V}$. Therefore

$$D(\mathbf{V})(q) = D(\varphi)(q) \cdot \mathbf{V}(q) + \varphi(q) \cdot D(\mathbf{V})(q).$$

We know that $\mathbf{V}(q) = 0$ and $\varphi(q) = 0$, hence $D(\mathbf{V})(q) = 0$. This yields the required equality $D(\mathbf{X})(q) = D(\mathbf{Y})(q)$ at the point q. □

Now let's consider the second map in (1.1). It is $\mathbb{R}$-linear map from $T^1_0(M)$ to $T^1_0(M)$ bounded with δ by the following relationship

$$\xi(\varphi \cdot \mathbf{X}) = \delta(\varphi) \cdot \mathbf{X} + \varphi \cdot \xi(\mathbf{X}). \tag{1.4}$$

Denote by ζ the restriction of D to the space $T^0_1(M)$. Then $\zeta : T^0_1(M) \to T^0_1(M)$ is $\mathbb{R}$-linear map satisfying the relationship $\zeta(\varphi \cdot \mathbf{h}) = \delta(\varphi) \cdot \mathbf{h} + \varphi \cdot \zeta(\mathbf{h})$, which is analogous to (1.4). Moreover, δ, ξ, and ζ are bound by the relationship

$$\delta(\langle \mathbf{h} \,|\, \mathbf{X} \rangle) = \langle \zeta(\mathbf{h}) \,|\, \mathbf{X} \rangle + \langle \mathbf{h} \,|\, \xi(\mathbf{X}) \rangle, \tag{1.5}$$

which is the consequence of commutation of D with contractions (see definition 1.3).

Suppose that for some differentiation D of extended algebra of tensor fields $\mathbf{T}(M)$ we know only its restrictions δ and ξ to $T^0_0(M)$ and $T^1_0(M)$ respectively. By means of (1.5) we can reconstruct its restriction ζ to the space $T^0_1(M)$. Then we can reconstruct the whole map D by setting

$$\begin{aligned} &D(\mathbf{X}_1 \otimes \ldots \otimes \mathbf{X}_r \otimes \mathbf{h}_1 \otimes \ldots \otimes \mathbf{h}_s) = \\ &= \sum_{m=1}^{r} \mathbf{X}_1 \otimes \ldots \xi(\mathbf{X}_m) \ldots \otimes \mathbf{X}_r \otimes \mathbf{h}_1 \otimes \ldots \otimes \mathbf{h}_s + \\ &+ \sum_{m=1}^{s} \mathbf{X}_1 \otimes \ldots \otimes \mathbf{X}_r \otimes \mathbf{h}_1 \otimes \ldots \zeta(\mathbf{h}_m) \ldots \otimes \mathbf{h}_s \end{aligned} \tag{1.6}$$

for its action upon the tensor fields from $T^r_s(M)$. This formula (1.6) is easily derived from Leibniz rule by induction in the numbers of multiplicands r and s. Above arguments form the proof of the following theorem.

Theorem 1.1. *Each differentiation D of extended algebra of tensor fields $\mathbf{T}(M)$ is uniquely fixed by its restrictions to the spaces $T^0_0(M)$ and $T^1_0(M)$.*

Theorem 1.1 means that if the restrictions of two differentiations D_1 and D_2 to $T^0_0(M)$ and $T^1_0(M)$ coincide, then $D_1 = D_2$. This result can be strengthened as follows.

Theorem 1.2. *Any two $\mathbb{R}$-linear maps δ and ξ from* (1.1) *satisfying the relationships* (1.2) *and* (1.4) *define some differentiation D of extended algebra of tensor fields $\mathbf{T}(M)$ on the manifold M.*

Theorem 1.2 is proved by direct construction of the map D relying on the formulas (1.5) and (1.6). Then one should only to check up the properties (1)–(4) from the definition 1.3. Theorem 1.1 and 1.2 can be united into one theorem.

Theorem 1.3. *Defining the differentiation D in extended algebra of tensor fields $\mathbf{T}(M)$ is equivalent to defining the vector field $\mathbf{Z}$ in TM and some $\mathbb{R}$-linear map $\xi \colon T^1_0(M) \to T^1_0(M)$ satisfying the relationship $\xi(\varphi \cdot \mathbf{X}) = \mathbf{Z}\varphi \cdot \mathbf{X} + \varphi \cdot \xi(\mathbf{X})$ for any extended scalar field $\varphi \in T^0_0(M)$ and for any extended vector field $\mathbf{X} \in T^1_0(M)$.*

DEFINITION 1.4. We say that the differentiation D of extended algebra of tensor fields $\mathbf{T}(M)$ is **degenerate**, if its restriction δ to $T^0_0(M)$ is identically zero map.

For degenerate differentiation D corresponding vector field $\mathbf{Z}$ is identically zero, and the relationship (1.4) is written as

$$\xi(\varphi\cdot\mathbf{X})=\varphi\cdot\xi(\mathbf{X}). \tag{1.7}$$

Relying on (1.7) we conclude that the map $\xi\colon T^1_0(M)\to T^1_0(M)$ is the endomorphism of the module $T^1_0(M)$ over the ring $\mathfrak{F}(TM)$. It is known (see [12], Chapter I, § 2 and § 3) that any such endomorphism is determined by some extended tensor field $\mathbf{S}$:

$$\xi(\mathbf{X})=C(\mathbf{S}\otimes\mathbf{X}). \tag{1.8}$$

Here $\mathbf{S}\in T^1_1(M)$, and C is the operation of contraction. From (1.8) and (1.5) we derive how the degenerate differentiation D acts upon covectorial field $\mathbf{h}\in T^0_1(M)$:

$$\zeta(\mathbf{h})=C(\mathbf{h}\otimes\mathbf{S}). \tag{1.9}$$

The relationships (1.8) and (1.9) together with $\delta=0$ and the formula (1.6) yield complete description of all degenerate differentiations of extended algebra of tensor fields on the manifold M.

§ 2. Covariant differentiations.

The set of differentiations $\mathfrak{D}(M)$ of the extended algebra of tensor fields on the manifold M is equipped the structure of module over the ring $\mathfrak{F}(TM)$ of smooth functions on tangent bundle TM. The set of extended vector fields $T^1_0(M)$ is equipped with the same structure of module over the ring $\mathfrak{F}(TM)$. This coincidence motivates the following definition.

DEFINITION 2.1. **Covariant differentiation** ∇ in the algebra of extended vector fields $\mathbf{T}(M)$ is the homomorphism of $\mathfrak{F}(TM)$-modules $\nabla : T^1_0(M)\to\mathfrak{D}(M)$. Image of vector field $\mathbf{Y}$ under such homomorphism denoted by $\nabla_{\mathbf{Y}}$ is called **covariant differentiation along the vector field** Y.

The set of differentiations $\mathfrak{D}(M)$ has natural grading that is defined by the grading of extended algebra of tensor fields $\mathbf{T}(M)$:

$$\mathfrak{D}(M)\subsetneq\bigoplus_{r=0}^{\infty}\bigoplus_{s=0}^{\infty}\mathfrak{D}^r_s(M). \tag{2.1}$$

Therefore each covariant differentiation ∇ generates the series of homomorphisms of $\mathfrak{F}(TM)$-modules numerated by grading numbers r and s:

$$\nabla\colon T^1_0(M)\to\mathfrak{D}^r_s(M). \tag{2.2}$$

According to the theorem 1.1 only two homomorphisms in the series (2.2) play key role in defining covariant differentiation $\nabla\colon T^1_0(M) \to \mathfrak{D}(M)$. These are

$$\nabla\colon T^1_0(M) \to \mathfrak{D}^0_0(M), \tag{2.3}$$

$$\nabla\colon T^1_0(M) \to \mathfrak{D}^1_0(M). \tag{2.4}$$

Let's consider the homomorphism (2.3). It maps extended vector field $\mathbf{Y}$ into the differentiation of the ring $\mathfrak{F}(TM)$, which, as we mentioned in § 1, is defined by some vector field $\mathbf{Z}$ on the tangent bundle TM. Therefore we can treat the map (2.3) as a homomorphism f from $\mathfrak{F}(TM)$-module of extended vector fields on M into $\mathfrak{F}(TM)$-module of ordinary vector fields on TM. Suppose that $q = (p, \mathbf{v})$ is a point of tangent bundle TM, and $p = \pi(q)$ is its projection to M. Extended vector field Y maps q to the vector $\mathbf{Y} = \mathbf{Y}(q)$ tangent to M at the point p. Homomorphism f maps $\mathbf{Y}$ to the vector $\mathbf{Z} = f(\mathbf{Y})$ tangent to TM at the point q. Vector function $\mathbf{Z} = f(\mathbf{Y})$ is linear in its argument.

Definition 2.2. Suppose that at each point q of tangent bundle TM we have $\mathbb{R}$-linear map $f\colon T_p(M) \to T_q(TM)$ from tangent space $T_p(M)$ at the point $p = \pi(q)$ to the tangent space $T_q(TM)$ at the point q. Such construction is called the **lift** of vectors from M to TM.

Definition 2.3. Lift of vectors f from definition 2.2 is called **smooth lift**, if it maps each smooth extended vector field on M to the smooth vector field on TM.

Let's compare the definition 2.2 with previous analysis of homomorphism (2.3). By this comparison we draw **very important conclusion** that defining homomorphism of $\mathfrak{F}(TM)$-modules $\nabla : T^1_0(M) \to \mathfrak{D}^0_0(M)$ is equivalent to defining some smooth lift of vectors from M to tangent bundle TM.

Now consider canonical projection $\pi : TM \to M$. Differential of this map π_* acts in the direction opposite to any lift of vectors: $\pi_*\colon T_q(TM) \to T_p(M)$. Therefore composition $\pi_* \circ f$ is a map from $T_p(M)$ to $T_p(M)$. Such composition defines operator field (field of type $(1, 1)$) in extended algebra of tensor fields on M.

Definition 2.4. Lift of vectors f from M to TM is called **vertical lift**, if composition $\pi_* \circ f$ is identically zero: $\pi_* \circ f = 0$.

Definition 2.5. Lift of vectors f from M to TM is called **horizontal lift**, if composition $\pi_* \circ f$ is the field of identical operators on M, i. e. $\pi_* \circ f = \mathrm{id}$.

Tangent bundle TM is naturally subdivided into the fibers over the points of base manifold M. Vectors tangent to the fiber form n-dimensional subspace $V_q(TM)$ in $2n$-dimensional space $T_q(TM)$. It is called **vertical** subspace in $T_q(TM)$. Vertical lift f defines the map from $T_p(M)$ to vertical space $V_q(TM)$.

Lemma 2.1. *Difference of two horizontal lifts is a vertical lift.*

If we take two horizontal lifts f_1 and f_2, then $\pi_* \circ (f_1 - f_2) = \pi_* \circ f_1 - \pi_* \circ f_2 = \mathrm{id} - \mathrm{id} = 0$. This means that the difference $f_1 - f_2$ is vertical lift according to the definition 2.4.

Definition 2.6. Covariant differentiation ∇ in extended algebra of tensor fields on M is called **horizontal differentiation** (or **vertical differentiation**), if corresponding lift of vectors is **horizontal** (or **vertical**).

Lemma 2.2. *Difference of two horizontal covariant differentiations is vertical differentiation in extended algebra of tensor fields on M.*

Lemma 2.2 is an obvious consequence of lemma 2.1.

§ 3. Canonical vertical lift of vectors and the velocity gradient.

On any manifold M one can define infinitely many vertical lifts, but there is only one which is canonical. We define it as described below. Let $\mathbf{Y}$ be some fixed vector at the point p in M. Suppose that $q = (p, \mathbf{v})$ is a point of tangent bundle TM from the fiber over the point p. Let's choose numeric parameter t and define parametric curve $q = q(t)$ in TM as follows:

$$q(t) = (p, \mathbf{v} + t \cdot \mathbf{Y}). \tag{3.1}$$

Curve (3.1) lies within the fiber over the point p and passes through the point $q = (p, \mathbf{v})$, when $t = 0$. Denote by $\mathbf{Z}$ tangent vector to the curve (3.1) at the point q, and let's take it for the lift of vector $\mathbf{Y}$:

$$\mathbf{Z} = w(\mathbf{Y}). \tag{3.2}$$

Vector $\mathbf{Z}$ can be treated as differential operator acting at the point q upon the functions from the ring $\mathfrak{F}(TM)$. Then

$$\mathbf{Z}\varphi = \left.\frac{d\varphi(p, \mathbf{v} + t \cdot \mathbf{Y})}{dt}\right|_{t=0}. \tag{3.3}$$

Definition 3.1. Vertical lift w from (3.2) defined by formula (3.3) is called **canonical vertical lift** of vectors from M to TM.

One can easily write canonical vertical lift of vectors in local coordinates. Suppose that vector $\mathbf{Y} \in T_p(M)$ is given by formula

$$\mathbf{Y} = Y^1 \frac{\partial}{\partial x^1} + \ldots + Y^n \frac{\partial}{\partial x^n}, \tag{3.4}$$

in local coordinates $x^1, \ldots, x^n$. Then its lift is defined by formula

$$\mathbf{Z} = w(\mathbf{Y}) \in T_q(TM) = Y^1 \frac{\partial}{\partial v^1} + \ldots + Y^n \frac{\partial}{\partial v^n}, \tag{3.5}$$

where $x^1, \ldots, x^n, v^1, \ldots, v^n$ are coordinates in TM canonically associated to local coordinates $x^1, \ldots, x^n$. Now one can see that canonical vertical lift w, which takes vector (3.4) to the vector (3.5), is one-to-one map from $T_p(M)$ to the vertical subspace $V_q(TM)$ in $T_q(TM)$.

Canonical vertical lift $w\colon T_p(M) \to T_q(TM)$ corresponds to some homomorphism of $\mathfrak{F}(TM)$-modules of the form (2.3). Let's denote it by $\tilde{\nabla}$. This homomorphism can be extended up to a covariant differentiation of the whole extended algebra of tensor fields $\mathbf{T}(M)$. In order to do it we should define homomorphism $\tilde{\nabla}\colon T^1_0(M) \to \mathfrak{D}^1_0(M)$ of the form (2.4). Suppose that $\mathbf{X}$ and $\mathbf{Y}$ are two vector fields from $T^1_0(M)$. We should put them into correspondence third vector field $\hat{\mathbf{X}} = \tilde{\nabla}_{\mathbf{Y}}\mathbf{X}$. Let's choose a point $p \in M$ and a point $q = (p, \mathbf{v}) \in TM$, consider the curve (3.1) passing through the point q, and then define vector $\hat{\mathbf{X}}$ as a differential operator in the ring of smooth functions on M by means of the following formula:

$$\hat{\mathbf{X}}\varphi = \left.\frac{d\,(\mathbf{X}(q(t))\varphi)}{dt}\right|_{t=0}. \tag{3.6}$$

Here $\varphi \in \mathfrak{F}(M)$ is a smooth function on M, and $\mathbf{X}(q(t))\varphi$ is its derivative along the vector $\mathbf{X}(q(t) \in T_p(M)$. If we know components of vector fields $\mathbf{X}$ and $\mathbf{Y}$ in some map $(U, x^1, \ldots, x^n)$ on M, then we can calculate components of vector field $\hat{\mathbf{X}} = \tilde{\nabla}_{\mathbf{Y}}\mathbf{X}$ in (3.6) by means of formula

$$\hat{X}^k = \sum_{m=1}^{n} Y^m\,\frac{\partial X^k}{\partial v^m}, \tag{3.7}$$

where $x^1, \ldots, x^n, v^1, \ldots, v^n$ are local coordinates on TM canonically associated to $x^1, \ldots, x^n$ on M. Formula (3.7) can be extended for the case of arbitrary tensor field $\mathbf{X} \in T^r_s(M)$ with components $X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}$ in local map $(U, x^1, \ldots, x^n)$. For the components of the field $\hat{\mathbf{X}} = \tilde{\nabla}_{\mathbf{Y}}\mathbf{X}$ in that map we have

$$\hat{X}^{i_1\ldots\,i_r}_{j_1\ldots\,j_s} = \sum_{m=1}^{n} Y^m\,\frac{\partial X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}}{\partial v^m}. \tag{3.8}$$

Covariant differentiation $\tilde{\nabla}$ given by the relationships (3.3) and (3.6), and expressed by the formula (3.8) in local coordinates is called **canonical vertical covariant differentiation** in extended algebra of tensor fields on the manifold M. It is also called **velocity covariant differentiation** or **velocity gradient**. The latter is motivated by formulas (3.7) and (3.8), which contain partial derivatives in components of velocity vector only.

§ 4. Horizontal lifts of vectors and dynamic connections.

Canonical vertical lift of vectors considered in section § 3 above is related to canonical covariant differentiation $\tilde{\nabla}$. In this section we consider covariant differentiations related to horizontal lifts of vectors (see definition 2.5). Suppose that ∇ is one of such differentiations, and let f be corresponding lift of vectors from M to TM. Horizontality of f means that image of linear map $f: T_p(M) \to T_q(TM)$ is some n-dimensional space $H_q(TM)$ in $2n$-dimensional tangent space $T_q(TM)$. It is called **horizontal subspace**. Due to the equality $\pi_* \circ f = \mathrm{id}$ following maps

$$f\colon T_p(M) \to H_q(TM), \qquad\qquad \pi_*\colon H_q(TM) \to T_p(M) \tag{4.1}$$

are inverse to each other. Sum of vertical and horizontal subspaces is a direct sum:

$$H_q(TM) \oplus V_q(TM) = T_q(TM). \tag{4.2}$$

THEOREM 4.1. *Defining horizontal lift of vectors from M to TM is equivalent to defining direct complement $H_q(TM)$ for vertical subspace $V_q(TM)$ at each point q of tangent space TM.*

PROOF. If horizontal lift f is given, then horizontal subspace $H_q(TM)$ is defined as an image of the map $f\colon T_p(M) \to T_q(TM)$. Then (4.2) is a consequence of the equalities $\pi_* \circ f = \mathrm{id}$ and $V_q(TM) = \operatorname{Ker} \pi_*$.

And conversely, if at each point $q \in TM$ we have subspace $H_q(TM)$ satisfying the equality (4.2), then we can restrict canonical projection $\pi_*: T_q(TM) \to T_p(M)$ to $H_q(TM)$. Due to the equality (4.2) from $\operatorname{Ker} \pi_* = V_q(TM)$ we derive that restriction $\pi_*\colon H_q(TM) \to T_p(M)$ is one-to-one map. By inverting this map we can define the lift of vectors $\pi_*^{-1}\colon T_p(M) \to H_q(TM) \subset T_q(TM)$. One can easily check that this lift is horizontal in the sense of definition 2.5. □

Let's consider some horizontal lift f in local coordinates. Let's choose some local map $(U, x^1, \ldots, x^n)$ on the base manifold M and take coordinate vector fields:

$$\mathbf{E}_1 = \frac{\partial}{\partial x^1},\ \ldots,\ \mathbf{E}_n = \frac{\partial}{\partial x^n}. \tag{4.3}$$

Applying lift f to base vectors (4.3), we get the following expression:

$$f(\mathbf{E}_i) = \frac{\partial}{\partial x^i} - \sum_{k=1}^{n} \Gamma^k_i \frac{\partial}{\partial v^k}. \tag{4.4}$$

Such structure of $f(\mathbf{E}_k)$ is the consequence of $\pi_* \circ f = \mathrm{id}$ and obvious equalities

$$\pi_*\left(\frac{\partial}{\partial x^i}\right) = \mathbf{E}_i, \qquad\qquad \pi_*\left(\frac{\partial}{\partial v^k}\right) = 0. \tag{4.5}$$

Parameters $\Gamma^k_i = \Gamma^k_i(x^1, \ldots, x^n, v^1, \ldots, v^n)$ in thetag4.4 are called **components of lift** f in the map U. Under the change of maps $(U, x^1, \ldots, x^n)$ for $(\tilde{U}, \tilde{x}^1, \ldots, \tilde{x}^n)$ these components are transformed as follows:

$$\Gamma^k_i = \sum_{m=1}^{n} \sum_{a=1}^{n} S^k_m\, T^a_i\, \tilde{\Gamma}^m_a + \sum_{j=1}^{n} \sum_{m=1}^{n} v^j\, S^k_m\, \frac{\partial T^m_i}{\partial x^j}. \tag{4.6}$$

In order to prove the relationships (4.6) for Γ^k_i and $\tilde{\Gamma}^m_a$ let's remember that

$$\mathbf{E}_i = \sum_{a=1}^{n} T^a_k\, \tilde{\mathbf{E}}_a \ \text{ leads to } \ f(\mathbf{E}_i) = \sum_{a=1}^{n} T^a_i\, f(\tilde{\mathbf{E}}_a). \tag{4.7}$$

Components of transition matrix $T = S^{-1}$ in (4.7) are determined by formulas (4.6) in Chapter II. For $f(\tilde{\mathbf{E}}_a)$ in (4.7) we shall write formula similar to formula (4.4):

$$f(\tilde{\mathbf{E}}_a) = \frac{\partial}{\partial \tilde{x}^a} - \sum_{m=1}^{n} \tilde{\Gamma}^m_a \, \frac{\partial}{\partial \tilde{v}^m}. \tag{4.8}$$

In order to convert differential operators $\partial/\partial\tilde{x}^p$ and $\partial/\partial\tilde{v}^m$ in (4.8) to the variables $x^1, \ldots, x^n, v^1, \ldots, v^n$ we use the relationships (4.5) from Chapter II. Differentiating these relationships, we obtain

$$\begin{cases} \dfrac{\partial}{\partial \tilde{v}^m} = \displaystyle\sum_{p=1}^{n} S^p_m \, \frac{\partial}{\partial v^p}, \\[2ex] \dfrac{\partial}{\partial \tilde{x}^a} = \displaystyle\sum_{k=1}^{n} S^k_a \, \frac{\partial}{\partial x^k} + \sum_{k=1}^{n} \sum_{r=1}^{n} \tilde{v}^r \, \frac{\partial S^k_r}{\partial \tilde{x}^a} \, \frac{\partial}{\partial v^k}. \end{cases} \tag{4.9}$$

Now, substituting (4.4) and (4.8) into the relationship (4.7) and using (4.9), after some calculations we get (4.6).

Continuing the study of horizontal covariant differentiation ∇ in extended algebra of tensor fields $\mathbf{T}(M)$, consider the homomorphism (2.4) for it. Let q be some point of tangent bundle TM, and let $p = \pi(q)$ be its projection to M. To each vector $\mathbf{Y}$ at the point p homomorphism (2.4) puts into correspondence the differential operator acting upon extended tensor fields at the point q. Let's take some local map $(U, x^1, \ldots, x^n)$ in the neighborhood of the point p and, choosing one of base vector fields from (4.3), assume $\mathbf{Y} = \mathbf{E}_i$. Then we take another vector field $\mathbf{E}_j$ from (4.3) and apply the operator $\nabla_{\mathbf{Y}}$ to it. The resulting field $\nabla_{\mathbf{E}_i}\mathbf{E}_j$ can be expanded in the base of fields (4.3). We write it as follows:

$$\nabla_{\mathbf{E}_i}\mathbf{E}_j = \sum_{k=1}^{n} \Gamma^k_{ij} \, \mathbf{E}_k. \tag{4.10}$$

Vector field $\mathbf{E}_j$ is defined locally within the map U. But due to the lemma 1.2 this is not an obstacle for applying an operator $\nabla_{\mathbf{E}_i}$ to it.

Coefficients $\Gamma^k_{ij} = \Gamma^k_{ij}(x^1, \ldots, x^n, v^1, \ldots, v^n)$ in expansion (4.10) are called **components of connection** or **Christoffel symbols**. Under the change of local coordinates in the domain of intersection of two local maps these coefficients are transformed according to standard rule for connection components

$$\Gamma^k_{ij} = \sum_{m=1}^{n} \sum_{a=1}^{n} \sum_{c=1}^{n} S^k_m \, T^a_i \, T^c_j \, \tilde{\Gamma}^m_{ac} + \sum_{m=1}^{n} S^k_m \, \frac{\partial T^m_i}{\partial x^j}. \tag{4.11}$$

Quantities $\Gamma^k_{ij}(x^1, \ldots, x^n, v^1, \ldots, v^n)$, which are transformed according to (4.11), define special geometric structure on M. We call it **extended** or **dynamic** affine connection. It differs from ordinary affine connection by twice as more arguments in Christoffel symbols.

DEFINITION 4.1. Let M be some manifold. We say that an extended affine connection Γ on M is defined, if for each local map $(U, x^1, \ldots, x^2)$ on M we have the set of smooth functions

$$\Gamma^k_{ij} = \Gamma^k_{ij}(x^1, \ldots, x^n, v^1, \ldots, v^n)$$

which are transformed according to the rule (4.11) under the change of local coordinates in the domain of intersection of any two local maps.

THEOREM 4.2. *On any smooth paracompact manifold M at least one extended affine connection can be defined.*

Omitting details, we give only the scheme of proof for the theorem 3.2. Proof is based on the use of locally finite open covering $\{U_\alpha\}_{\alpha\in A}$ composed of local maps and subordinate smooth partition of unity

$$1 = \sum_{\alpha\in A} \eta_\alpha$$

(see details in [12], appendix 3). With each map U_α we associate some arbitrary set of smooth functions $\Gamma^k_{ij}(\alpha)$ in corresponding local coordinates $x^1, \ldots, x^n, v^1, \ldots, v^n$.

Suppose that p is some point on M. It has a neighborhood V that intersects with finite number of maps $\{U_\alpha\}_{\alpha\in A}$. Let's introduce some local coordinates $x^1, \ldots, x^n$ in V and transform each set of functions $\Gamma^k_{ij}(\alpha)$ to these local coordinates in V according to the rule (4.11). Then we set

$$\Gamma^k_{ij} = \sum_{\alpha\in A} \Gamma^k_{ij}(\alpha)\cdot\eta_\alpha. \tag{4.12}$$

Formula (4.12) determines the components of extended affine connection that we need in local map $(V, x^1, \ldots, x^n)$. Such maps cover the entire manifold M, therefore our connection is globally defined. Now we are only to do some formalities, testing transformation rules (4.11) for the connection components defined by (4.12).

DEFINITION 4.2. Horizontal covariant differentiation in extended algebra of tensor fields is called **space covariant differentiation** or **space gradient**, if the following condition is fulfilled:

$$\nabla_{\mathbf{Y}}\mathbf{v} = 0. \tag{4.13}$$

This means that covariant derivative $\nabla_{\mathbf{y}}$ applied to the velocity field $\mathbf{v}$ is identically zero for any vector field $\mathbf{Y}$ on M.

Let's express the condition (4.13) in local coordinates by choosing some map U on M. In order to do it we use the expansion

$$\mathbf{v} = \sum_{j=1}^{n} v^j\,\mathbf{E}_j \tag{4.14}$$

and we apply the relationship (4.10) by differentiating (4.14):

$$\begin{aligned}\nabla_{\mathbf{Y}}\mathbf{v}&=\sum_{j=1}^{n}(\nabla_{\mathbf{Y}}v^j)\,\mathbf{E}_j+\sum_{j=1}^{n}v^j\,\nabla_{\mathbf{Y}}\mathbf{E}_j=\\&=\sum_{j=1}^{n}f(\mathbf{Y})v^j\,\mathbf{E}_j+\sum_{i=1}^{n}\sum_{j=1}^{n}v^j\,Y^i\,\nabla_{\mathbf{E}_i}\mathbf{E}_j=\\&=\sum_{j=1}^{n}f(\mathbf{Y})v^j\,\mathbf{E}_j+\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}v^j\,Y^i\,\Gamma^k_{ij}\,\mathbf{E}_k.\end{aligned}$$

Here $Y^1, \ldots, Y^n$ are components of vector $\mathbf{Y}$. We calculate the result of applying $f(\mathbf{Y})$ to the function v^j by means of (4.4). Then

$$\nabla_{\mathbf{Y}}\mathbf{v}=\sum_{i=1}^{n}\sum_{k=1}^{n}Y^i\,\Gamma^k_i\,\mathbf{E}_k+\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}v^j\,Y^i\,\Gamma^k_{ij}\,\mathbf{E}_k. \tag{4.15}$$

Since components of the field $\mathbf{Y}$ in (4.15) are absolutely arbitrary, the condition (4.13) from definition 4.12 reduces to the relationships

$$\Gamma^k_i=\sum_{j=1}^{n}v^j\,\Gamma^k_{ij}. \tag{4.16}$$

The relationships (4.16) do not change their appearance under the change of local coordinates by transferring from one map to another. This can be seen by comparison the formulas (4.6) and (4.11). Thus, in case of space covariant differentiations the structure of horizontal lift of vectors is completely defined by the structure of extended affine connection. This result will be formulated as a theorem.

THEOREM 4.3. *Defining space covariant differentiation ∇ in extended algebra of tensor fields on the manifold M is equivalent to defining some extended affine connection on M.*

We shall terminate this section by writing an explicit formula for space covariant derivative $\hat{\mathbf{X}}=\nabla_{\mathbf{Y}}\mathbf{X}$ for extended tensor field $\hat{\mathbf{X}}=\mathbf{X}\in T^r_s(M)$ in local coordinates:

$$\begin{aligned}\hat{X}^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}&=\sum_{m=1}^{n}Y^m\,\frac{\partial X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}}{\partial x^m}-\\&-\sum_{m=1}^{n}\sum_{a=1}^{n}\sum_{b=1}^{n}Y^m\,v^a\,\Gamma^b_{ma}\,\frac{\partial X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}}{\partial v^b}+\\&+\sum_{m=1}^{n}\sum_{k=1}^{r}\sum_{a_k=1}^{n}Y^m\,\Gamma^{i_k}_{m\,a_k}\,X^{i_1\ldots\,a_k\ldots\,i_r}_{j_1\ldots\ldots\ldots\,j_s}-\\&-\sum_{m=1}^{n}\sum_{k=1}^{s}\sum_{b_k=1}^{n}Y^m\,\Gamma^{b_k}_{m\,j_k}\,X^{i_1\ldots\ldots\ldots\,i_r}_{j_1\ldots\,b_k\ldots\,j_s}\end{aligned} \tag{4.17}$$

Formula (4.17) is analogous to (3.8). It can be proved by induction in the number of indices r and s.

§ 5. Structural theorem for differentiations.

In sections 1–4 above we have defined and have completely described three types of differentiations in extended algebra of tensor fields $\mathbf{T}(M)$ for some manifold M. These are: **degenerate differentiation S**, which is given by some tensor field $\mathbf{S}$ of type $(1,1)$, and **two covariant differentiations** $\nabla_{\mathbf{X}}$ and $\tilde{\nabla}_{\mathbf{Y}}$. Here we shall prove that by means of $\mathbf{S}$, $\nabla_{\mathbf{X}}$, and $\tilde{\nabla}_{\mathbf{Y}}$ we can exhaust the whole variety of differentiations in extended algebra of tensor fields $\mathbf{T}(M)$.

THEOREM 5.1. *Let M be a real smooth manifold equipped with extended affine connection Γ. Then each differentiation D in extended algebra of tensor fields $\mathbf{T}(M)$ is expanded into a sum*

$$D = \nabla_{\mathbf{X}} + \tilde{\nabla}_{\mathbf{Y}} + \mathbf{S}, \tag{5.1}$$

where $\mathbf{X}$ and $\mathbf{Y}$ are some extended vector fields on M, and $\mathbf{S}$ is a degenerate differentiation defined by some extended tensor field $\mathbf{S}$ of type $(1,1)$.

PROOF. Take some arbitrary differentiation $D \in \mathfrak{D}(M)$. Its restriction to subspace $T^0_0(M) \subset \mathbf{T}(M)$, as we mentioned in § 1, is defined by some vector field $\mathbf{Z}$ on tangent bundle TM. Expanded affine connection Γ defines some horizontal lift of vectors f; its components in local coordinates are defined by relationships (4.16). Let's apply theorem 4.1 to the lift f. This yields the expansion of vector field $\mathbf{Z}$ into a sum of two components:

$$\mathbf{Z} = \mathbf{H} + \mathbf{V}. \tag{5.2}$$

Here $\mathbf{H}$ is horizontal vector field, and $\mathbf{V}$ is a vertical vector field on TM. Vector field $\mathbf{H}$ from (5.2) is projected into the expanded vector field $\mathbf{X}$ on M

$$\mathbf{X} = \pi_*(\mathbf{H}) = \pi_*(\mathbf{Z}), \qquad \mathbf{H} = f(\mathbf{X}). \tag{5.3}$$

Vertical component $\mathbf{V}$ in (5.3) is also related to some extended vector field on M. In order to get it we should remember the bijective linear map $w\colon T_p(M) \to V_q(TM)$. Then by inverting this map we get

$$\mathbf{Y} = w^{-1}(\mathbf{V}), \qquad \mathbf{V} = w(\mathbf{Y}), \tag{5.4}$$

where $\mathbf{Y}$ is an extended vector field on M. Taking into account (5.3) and (5.4) we rewrite (5.2) as follows:

$$\mathbf{Z} = f(\mathbf{X}) + w(\mathbf{Y}). \tag{5.5}$$

As a next step let's consider the differentiation $\hat{D} = \nabla_{\mathbf{X}} + \tilde{\nabla}_{\mathbf{Y}}$. Its restriction to the subspace $T^0_0(M)$ is defined by the vector field which coincides with (5.5). Therefore difference $D - \hat{D}$ is a degenerate differentiation in the extended algebra of tensor fields. According to the results of § 1 it is defined by some extended tensor field

$\mathbf{S}$ of the type $(1,1)$. Now from $D - \hat{D} = \mathbf{S}$ and $\hat{D} = \nabla_{\mathbf{X}} + \tilde{\nabla}_{\mathbf{Y}}$ for the differentiation D we get required expansion (5.1). Theorem is proved. □

§ 6. Commutational relationships and curvature tensors.

The set of differentiations $\mathfrak{D}(M)$ of extended algebra of tensor fields $\mathbf{T}(M)$ is an infinite-dimensional Lie algebra over the field of real numbers $\mathbb{R}$. Theorem 5.1 can give more detailed description of this Lie algebra. Let's begin with degenerate differentiations. Let $\mathbf{S}_1$ and $\mathbf{S}_2$ be some two degenerate differentiations defined by operator fields $\mathbf{S}_1$ and $\mathbf{S}_2$ from extended algebra. Then

$$[\mathbf{S}_1,\, \mathbf{S}_2] = C(\mathbf{S}_1 \otimes \mathbf{S}_2 - \mathbf{S}_2 \otimes \mathbf{S}_1). \tag{6.1}$$

Formula (6.1) means that commutator of two degenerate differentiations is degenerate differentiation defined by operator field being the pointwise commutator for operator fields $\mathbf{S}_1$ and $\mathbf{S}_2$.

Suppose that M is equipped with extended affine connection Γ. Let's commutate space covariant differentiation $\nabla_{\mathbf{X}}$ and velocity covariant differentiation $\tilde{\nabla}_{\mathbf{Y}}$ with degenerate differentiation $\mathbf{S}$:

$$[\nabla_{\mathbf{X}},\, \mathbf{S}] = \nabla_{\mathbf{X}}\mathbf{S}, \qquad\qquad [\tilde{\nabla}_{\mathbf{Y}},\, \mathbf{S}] = \tilde{\nabla}_{\mathbf{Y}}\mathbf{S}. \tag{6.2}$$

Formulas (6.2) show that such commutators are degenerate differentiations defined by operator fields $\nabla_{\mathbf{X}}\mathbf{S}$ and $\tilde{\nabla}_{\mathbf{Y}}\mathbf{S}$ respectively. As a consequence of (6.1) and (6.2) we get that the set of degenerate differentiation $\mathfrak{S}(M)$ is an ideal in $\mathfrak{D}(M)$.

Commutator of two velocity covariant differentiations $\tilde{\nabla}_{\mathbf{X}}$ and $\tilde{\nabla}_{\mathbf{Y}}$ is given by the following rather simple formula

$$[\tilde{\nabla}_{\mathbf{X}},\, \tilde{\nabla}_{\mathbf{Y}}] = \tilde{\nabla}_{\mathbf{U}}, \;\text{ where }\; \mathbf{U} = \tilde{\nabla}_{\mathbf{X}}\mathbf{Y} - \tilde{\nabla}_{\mathbf{Y}}\mathbf{X}. \tag{6.3}$$

In calculating the commutator $[\nabla_{\mathbf{X}},\, \tilde{\nabla}_{\mathbf{Y}}]$ we find some new extended tensor field $\mathbf{D}$ of the type $(1,3)$. Here we have formula

$$[\nabla_{\mathbf{X}},\, \tilde{\nabla}_{\mathbf{Y}}] = \tilde{\nabla}_{\mathbf{U}} - \nabla_{\mathbf{W}} + \mathbf{D}(\mathbf{X},\mathbf{Y}) \tag{6.4}$$

with $\mathbf{U} = \nabla_{\mathbf{X}}\mathbf{Y} - \mathbf{D}(\mathbf{X},\mathbf{Y})\,\mathbf{v}$ and $\mathbf{W} = \tilde{\nabla}_{\mathbf{Y}}\mathbf{X}$. Components of the tensor field $\mathbf{D}$ are expressed through the components of connection Γ by formula

$$D^k_{rij} = -\frac{\partial \Gamma^k_{ir}}{\partial v^j}. \tag{6.5}$$

By $\mathbf{S} = \mathbf{D}(\mathbf{X},\mathbf{Y})$ in thetag (6.4) we denote tensor field of type $(1,1)$ constructed by $\mathbf{D}$, $\mathbf{X}$, and $\mathbf{Y}$ as a result of contraction: $\mathbf{S} = C(\mathbf{D} \otimes \mathbf{X} \otimes \mathbf{Y})$. Being more precise, components of the field $\mathbf{S}$ are expressed through (6.5) as follows:

$$S^k_r = \sum_{i=1}^{n}\sum_{j=1}^{n} D^k_{rij}\, X^i\, Y^j. \tag{6.6}$$

Vector field $\mathbf{D}(\mathbf{X},\mathbf{Y})\,\mathbf{v}$ is obtained by contracting operator field $\mathbf{S}$ from (6.6) with vector field of velocity $\mathbf{v}$.

Tensor field $\mathbf{D}$ with components given by (6.6) is called the **dynamic curvature tensor**. In calculating commutator $[\nabla_{\mathbf{X}},\,\nabla_{\mathbf{Y}}]$ we find another curvature tensor $\mathbf{R}$:

$$[\nabla_{\mathbf{X}},\,\nabla_{\mathbf{Y}}]=\nabla_{\mathbf{U}}-\tilde{\nabla}_{\mathbf{W}}+\mathbf{R}(\mathbf{X},\mathbf{Y}),\tag{6.7}$$

Here $\mathbf{U}=\nabla_{\mathbf{X}}\mathbf{Y}-\nabla_{\mathbf{Y}}\mathbf{X}-\mathbf{T}(\mathbf{X},\mathbf{Y})$ and $\mathbf{W}=-\mathbf{R}(\mathbf{X},\mathbf{Y})\,\mathbf{v}$. Components of curvature tensor $\mathbf{R}$ is given by formula

$$\begin{aligned}R^k_{rij}&=\frac{\partial\Gamma^k_{jr}}{\partial x^i}-\frac{\partial\Gamma^k_{ir}}{\partial x^j}+\sum^n_{m=1}\Gamma^k_{im}\,\Gamma^m_{jr}-\sum^n_{m=1}\Gamma^k_{jm}\,\Gamma^m_{ir}-\\&-\sum^n_{m=1}\sum^n_{h=1}v^m\,\Gamma^h_{im}\,\frac{\partial\Gamma^k_{jr}}{\partial v^h}+\sum^n_{m=1}\sum^n_{h=1}v^m\,\Gamma^h_{jm}\,\frac{\partial\Gamma^k_{ir}}{\partial v^h}.\end{aligned}\tag{6.8}$$

Extended tensor field $\mathbf{T}$ of type $(1,2)$ is called the **torsion field**. Components of torsion field $\mathbf{T}$ are calculated by formula

$$T^k_{ij}=\Gamma^k_{ij}-\Gamma^k_{ji}.\tag{6.9}$$

They are skew symmetric in pair of lower indices. Components of $\mathbf{Z}=\mathbf{T}(\mathbf{X},\mathbf{Y})$ are expressed through (6.9) as follows:

$$Z^k=\sum^n_{i=1}\sum^n_{j=1}T^k_{ij}\,X^i\,Y^j.\tag{6.10}$$

Operator field $\mathbf{S}=\mathbf{R}(\mathbf{X},\mathbf{Y})$ in (6.7) is obtained by contracting $\mathbf{X}$ and $\mathbf{Y}$ with curvature tensor: $\mathbf{S}=C(\mathbf{R}\otimes\mathbf{X}\otimes\mathbf{Y})$. Its components are given by formula

$$S^k_r=\sum^n_{i=1}\sum^n_{j=1}R^k_{rij}\,X^i\,Y^j\tag{6.11}$$

similar to (6.6) and (6.10). Vector field $\mathbf{R}(\mathbf{X},\mathbf{Y})\,\mathbf{v}$ in turn is obtained by contracting (6.11) and the vector field of velocity $\mathbf{v}$.

§ 7. Coordinate form of covariant differentiations.

Keeping in mind theorem 5.1 further we shall use only two types of covariant differentiations in extended algebra of tensor fields. These are space covariant differentiation ∇ and velocity covariant differentiation $\tilde{\nabla}$. Let $\mathbf{X}$ be a tensor field of the type (r,s). Derivatives $\nabla_{\mathbf{Y}}\mathbf{X}$ and $\tilde{\nabla}_{\mathbf{Y}}\mathbf{X}$ are tensor fields of the same type. They are $\mathfrak{F}$-linear functions of $\mathbf{Y}$ (see definition 2.1 above). Therefore they can be expressed as the results of contracting $\mathbf{Y}$ with tensor fields of the type $(r,s+1)$:

$$\nabla_{\mathbf{Y}}\mathbf{X}=C(\mathbf{Y}\otimes\nabla\mathbf{X}),\qquad\qquad\tilde{\nabla}_{\mathbf{Y}}\mathbf{X}=C(\mathbf{Y}\otimes\tilde{\nabla}\mathbf{X}).\tag{7.1}$$

Fields $\nabla\mathbf{X}$ and $\tilde{\nabla}\mathbf{X}$ of type $(r, s+1)$ in (7.1) are called **covariant differentials** of the field $\mathbf{X}$: field $\nabla\mathbf{X}$ is a space covariant differential, and field $\tilde{\nabla}\mathbf{X}$ is a velocity covariant differential. According to what was said just above any covariant differentiation could be understood as a map from $T^r_s(M)$ to $T^r_{s+1}(M)$. However, definition 2.1 appears to be more convenient.

Let $X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}$ be the components of the field $\mathbf{X}$. For to numerate components of $\nabla\mathbf{X}$ and $\tilde{\nabla}\mathbf{X}$ we need an extra index, say, the index m. Components of the fields $\nabla\mathbf{X}$ and $\tilde{\nabla}\mathbf{X}$ are denoted as follows:

$$\nabla_m X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}, \qquad\qquad \tilde{\nabla}_m X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}. \tag{7.2}$$

In (7.2) signs ∇ and $\tilde{\nabla}$ are used to place an extra index m near them. In systematic work with tensors this trick is permanently used, and therefore signs ∇_m and $\tilde{\nabla}_m$ are understood as maps, that take the array of components of the field $\mathbf{X}$ to arrays of components of the fields $\nabla\mathbf{X}$ and $\tilde{\nabla}\mathbf{X}$:

$$\nabla_m\colon X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s} \to \nabla_m X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}, \qquad\qquad \tilde{\nabla}_m\colon X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s} \to \tilde{\nabla}_m X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}.$$

These maps are called covariant derivatives. From (4.17) we extract the following rule for calculating covariant derivative ∇_m:

$$\begin{aligned}\nabla_m X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s} &= \frac{\partial X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}}{\partial x^m} - \sum_{a=1}^{n}\sum_{b=1}^{n} v^a\,\Gamma^b_{ma}\,\frac{\partial X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}}{\partial v^b} + \\ &+ \sum_{k=1}^{r}\sum_{a_k=1}^{n}\Gamma^{i_k}_{m\,a_k}\,X^{i_1\ldots\,a_k\ldots\,i_r}_{j_1\ldots\ldots\ldots\,j_s} - \sum_{k=1}^{s}\sum_{b_k=1}^{n}\Gamma^{b_k}_{m\,j_k} X^{i_1\ldots\ldots\ldots\,i_r}_{j_1\ldots\,b_k\ldots\,j_s}.\end{aligned} \tag{7.3}$$

The rule for calculating $\tilde{\nabla}_m$ is extracted from (3.8). It is much more simple:

$$\tilde{\nabla}_m X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s} = \frac{\partial X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}}{\partial v^m}. \tag{7.4}$$

Due to (7.4) covariant derivative $\tilde{\nabla}_m$ is associated with $\partial/\partial v^m$. It is called covariant derivative with respect to m-th velocity variable. In similar way, due to (7.3) covariant derivative ∇_m is associated with $\partial/\partial x^m$. It is called covariant derivative with respect to m-th space variable x^m. Covariant derivatives $\tilde{\nabla}_m$ and ∇_m obey Leibniz rule. It is expressed by formulas

$$\begin{aligned}\nabla_m(A^{\cdots}_{\cdots}\cdot B^{\cdots}_{\cdots}) &= \nabla_m A^{\cdots}_{\cdots}\cdot B^{\cdots}_{\cdots} + A^{\cdots}_{\cdots}\cdot \nabla_m B^{\cdots}_{\cdots},\\ \tilde{\nabla}_m(A^{\cdots}_{\cdots}\cdot B^{\cdots}_{\cdots}) &= \tilde{\nabla}_m A^{\cdots}_{\cdots}\cdot B^{\cdots}_{\cdots} + A^{\cdots}_{\cdots}\cdot \tilde{\nabla}_m B^{\cdots}_{\cdots}.\end{aligned} \tag{7.5}$$

Moreover, they commutate with the operation of contraction:

$$\sum_{k=1}^{n}\nabla_m A^{\,..k...}_{...k..}=\nabla_m\left(\sum_{k=1}^{n}A^{\,..k...}_{...k..}\right),\qquad \sum_{k=1}^{n}\tilde{\nabla}_m A^{\,..k...}_{...k..}=\tilde{\nabla}_m\left(\sum_{k=1}^{n}A^{\,..k...}_{...k..}\right).\tag{7.6}$$

From (6.3) we extract commutator of covariant derivatives $\tilde{\nabla}_i$ and $\tilde{\nabla}_j$:

$$[\tilde{\nabla}_i,\,\tilde{\nabla}_j]=0.\tag{7.7}$$

The commutational relationship (6.4), when applied to scalar and vectorial fields φ and $\mathbf{X}$, is written as follows:

$$\begin{aligned}
&[\nabla_i,\,\tilde{\nabla}_j]\varphi=-\sum_{m=1}^{n}\sum_{k=1}^{n}v^m\,D^k_{mij}\,\tilde{\nabla}_k\varphi,\\
&[\nabla_i,\,\tilde{\nabla}_j]X^r=-\sum_{m=1}^{n}\sum_{k=1}^{n}v^m\,D^k_{mij}\,\tilde{\nabla}_kX^r+\sum_{m=1}^{n}D^r_{mij}\,X^m.
\end{aligned}\tag{7.8}$$

For the case of covectorial field $\mathbf{h}$ from (7.5), (7.6), and (7.8) we derive

$$[\nabla_i,\,\tilde{\nabla}_j]h_r=-\sum_{m=1}^{n}\sum_{k=1}^{n}v^m\,D^k_{mij}\,\tilde{\nabla}_kh_r-\sum_{m=1}^{n}D^m_{rij}\,h_m.\tag{7.9}$$

The relationships (7.8) and (7.9) can be expanded for the case of arbitrary tensor field of type (r,s), if we remember that $[\nabla_i,\,\tilde{\nabla}_j]$ acts as first order derivative.

From (6.7) we derive commutational relationships for two spatial covariant derivatives. When applied a scalar field φ they look like

$$[\nabla_i,\,\nabla_j]\varphi=-\sum_{m=1}^{n}\sum_{k=1}^{n}v^m\,R^k_{mij}\,\tilde{\nabla}_k\varphi-\sum_{k=1}^{n}T^k_{ij}\,\nabla_k\varphi.\tag{7.10}$$

For the case of vectorial field $\mathbf{X}$ and covectorial field $\mathbf{h}$ we have

$$\begin{aligned}
[\nabla_i,\,\nabla_j]X^r=&-\sum_{m=1}^{n}\sum_{k=1}^{n}v^m\,R^k_{mij}\,\tilde{\nabla}_kX^r-\\
&-\sum_{k=1}^{n}T^k_{ij}\,\nabla_kX^r+\sum_{m=1}^{n}R^r_{mij}\,X^m,
\end{aligned}\tag{7.11}$$

$$\begin{aligned}
[\nabla_i,\,\nabla_j]h_r=&-\sum_{m=1}^{n}\sum_{k=1}^{n}v^m\,R^k_{mij}\,\tilde{\nabla}_kh_r-\\
&-\sum_{k=1}^{n}T^k_{ij}\,\nabla_kh_r-\sum_{m=1}^{n}R^m_{rij}\,h_m.
\end{aligned}\tag{7.12}$$

Further the relationships (7.10), (7.11), and (7.12) can be expanded for the case of arbitrary tensor field of type (r,s). But we shall not write this formula.

CHAPTER IV

TENSOR FIELDS ON CURVES AND THEIR DIFFERENTIATION.

§ 1. Vectorial and tensorial field on curves.

Let M be real smooth manifold. By assigning to each point $p \in M$ some tensor at this point we obtain a tensor field, i. e. tensor field is a tensor-valued function on M. By extending the domain of such function up to a tangent bundle we come to the concept of extended tensor field (see definition 5.3 in Chapter II). However, sometimes we need to consider tensor-valued functions with very narrow domains. In the theory of dynamical systems admitting the normal shift we consider tensor fields on curves in M. Curves here arise in a natural way as trajectories of Newtonian dynamical systems.

Suppose that $p(t)$ is smooth parametric curves on the manifold M. This is smooth map from some open interval (a, b) on real axis $\mathbb{R}$ to the manifold M. In geometry, considering parametric curves, one often reparameterizes them, changing initial parameter t for some new parameter $t' = f(t)$, where f is some smooth monotonic function. Here we shall not yet consider the reparametrization of curves, since in theory of Newtonian dynamical systems we have canonical parameter on all trajectories. This is time variable t.

Let $p(t)$ be smooth parametric curve on the manifold M. Upon choosing some local map $(U, x^1, \dots, x^n)$ in M it is represented by numeric functions

$$x^1(t),\ \dots,\ x^n(t). \tag{1.1}$$

Let's differentiate these functions with respect to t. Their time derivatives

$$\dot{x}^1(t),\ \dots,\ \dot{x}^n(t). \tag{1.2}$$

are the components of tangent vector to this parametric curve at the point $p(t)$. We denote it by $\mathbf{v}(t) = \dot{p}(t)$. Vector $\mathbf{v}(t)$ is called **velocity** vector of the point moving along the curve.

In geometry the points, where $\mathbf{v}(t) = 0$, are called **singular** point of the curve. At these points smoothness of the curve $p(t)$ could be broken. In describing the motion of the point along the curve this is not crucial, if the functions (1.1) are smooth. For instance, if we consider a circle rolling along the straight line on the plane, each point of the circle is moving along a cycloid, which has singular points.

Velocity vector $\mathbf{v}(t)$ is a simplest example of vector field on the curve. It is defined only at the points of curve[1]. Let $p(t)$ be a parametric curve on the manifold M.

DEFINITION 1.1. If each value of parameter t is associated with some vector $\mathbf{X}(t)$ from the tangent space $T_p(M)$ to M at the point $p = p(t)$, then we say that the *vector field* $\mathbf{X}$ *on the curve* $p(t)$ is defined.

DEFINITION 1.2. If each value of parameter t is associated with some tensor $\mathbf{X}(t)$ from tensor space $T^r_s(p, M)$ at the point $p = p(t)$, then we say that the *tensor field* $\mathbf{X}$ *on the curve* $p(t)$ is defined.

Note that if curve $p(t)$ has self intersection, i. e. if $p(t_1) = p(t_2)$, then the values of field $\mathbf{X}(t_1)$ and $\mathbf{X}(t_2)$ shouldn't coincide. This means that curve here is treated not only as geometric set of points, but as an object with fixed parametrization, where parameter is a time variable t.

Vectorial and tensorial fields on curves can arise in many ways. Let's consider one of them. Suppose that tensor field $\mathbf{X}$ on the manifold M is given, i. e. at each point $p \in M$ we have a tensor $\mathbf{X}(p)$. Then for the points of curve we take $\mathbf{X}(t) = \mathbf{X}(p(t))$. This yields a tensor field on the curve $p(t)$. Such field is called the **restriction** of tensor field $\mathbf{X}$ from M to the curve.

DEFINITION 1.3. Tensor field $\mathbf{X}(t)$ of type (r, s) on the curve $p(t) \in M$ is called **smooth field**, if for any local map U on M its components $X^{i_1 \ldots i_r}_{j_1 \ldots j_s}(t)$ in this map are smooth functions of parameter t.

Its evident, that by restricting smooth tensor field $\mathbf{X}$ from M to the curve we get smooth tensor field on this curve. This fact is easily proved by considering the components of $\mathbf{X}$ in local maps on M.

§ 2. Lift of curves to tangent bundle.

Suppose that $\mathbf{X}(t)$ is some smooth vector field on the curve $p(t) \in M$. Pair $q(t) = (p(t), \mathbf{X}(t))$ is a point of tangent bundle depending on t. Such pairs define smooth parametric curve $q(t)$ in tangent bundle TM. This curve is called the **lift** of curve $p(t)$ from M to TM. Canonical projection $\pi\colon TM \to M$ maps it back to the initial curve $p(t)$. Curve $p(t)$ has infinitely many lifts to tangent bundle, but one of them is especially important.

DEFINITION 2.1. Lift $q(t) = (p(t), \mathbf{v}(t))$ of curve $p(t)$ defined by vector field of velocity $\mathbf{v}(t) = \dot{p}(t)$ is called **natural lift** of this curve from M to TM.

Consider some tensor field $\mathbf{Y}$ from extended algebra $\mathbf{T}(M)$. It can't be restricted to the curve $p(t) \in M$ immediately, since tensor $\mathbf{Y}$ at the point p for such field is defined not by p, but by the point q in the fiber $\pi^{-1}(p)$ of tangent bundle over the point p (see definition 5.3 in Chapter II). However, if we have the lift of curve

[1]It may be defined on TM, when treated as an extended vector field, but it is not a vector field on the manifold M itself in the usual sense.

$q(t) = (p(t), \mathbf{X}(t))$, then we can restrict $\mathbf{Y}$ to such lift:

$$\mathbf{Y}(t) = \mathbf{Y}(q(t)) = \mathbf{Y}(p(t), \mathbf{X}(t)). \tag{2.1}$$

Tensor field $\mathbf{Y}(t)$ on the curve $p(t)$ defined by (2.1) is called the **restriction** of extended tensor field $\mathbf{Y}$ to the curve $p(t)$ **due to its lift** $q(t) = (p(t), \mathbf{X}(t))$.

Definition 2.2. If in restriction (2.1) we use natural lift of curve $p(t)$ given by vector field of velocity $\mathbf{v}(t)$, then such restriction is called **natural restriction** of extended tensor field $\mathbf{Y}$ to the curve $p(t)$ on M.

Velocity vector $\mathbf{v}$ can be treated as extended vector field on M. If we take $\mathbf{Y} = \mathbf{v}$, then natural restriction of $\mathbf{Y}$ to the curve $p(t)$ on M coincides with vector field of velocity $\mathbf{v}(t) = \dot{p}(t)$.

§ 3. Differentiation of tensor fields with respect to the parameter along the curves.

Let $p(t)$ be smooth parametric curve on real smooth manifold M. Denote by $T^r_s(t)$ the set of smooth tensor fields of type (r, s) on this curve and take

$$\mathbf{T}(t) = \bigoplus_{r=0}^{\infty} \bigoplus_{s=0}^{\infty} T^r_s(t). \tag{3.1}$$

The set $T^0_0(t)$ is a ring of smooth functions with numeric argument t. Denote it by $\mathfrak{F}(t)$. Each set $T^r_s(t)$ possess the structure of module over the ring $\mathfrak{F}(t)$. Their direct sum $\mathbf{T}(t)$ in (3.1) is an algebra over this ring. It is called the **algebra of tensor fields** on the curve $p(t)$.

Definition 3.1. The map $D : \mathbf{T}(t) \to \mathbf{T}(t)$ is called the **differentiation** in algebra of tensor fields on the curve $p(t)$, if the following conditions are fulfilled:

(1) concordance with grading: $D(T^r_s(t)) \subset T^r_s(t)$;
(2) $\mathbb{R}$-linearity: $D(\mathbf{X} + \mathbf{Y}) = D(\mathbf{X}) + D(\mathbf{Y})$ and $D(\lambda\mathbf{X}) = \lambda D(\mathbf{X})$ for $\lambda \in \mathbb{R}$;
(3) commutation with contractions: $D(C(\mathbf{X})) = C(D(\mathbf{X}))$;
(4) Leibniz rule: $D(\mathbf{X} \otimes \mathbf{Y}) = D(\mathbf{X}) \otimes \mathbf{Y} + \mathbf{X} \otimes D(\mathbf{Y})$.

Theory of differentiations in algebra $\mathbf{T}(t)$ is analogous to the theory of differentiations for extended tensor fields, though it is substantially more simple. Suppose that D is a differentiation in algebra $\mathbf{T}(t)$. Denote by δ and ξ the restrictions of D to $T^0_0(t)$ and $T^1_0(t)$ respectively:

$$\delta\colon T^0_0(t) \to T^0_0(t), \qquad\qquad \xi\colon T^1_0(t) \to T^1_0(t). \tag{3.2}$$

The map δ is a differentiation in the ring $\mathfrak{F}(t)$:

$$\delta(\varphi \cdot \psi) = \delta(\varphi) \cdot \psi + \varphi \cdot \delta(\psi). \tag{3.3}$$

Another map ξ is bound with δ by the following relationship:

$$\xi(\varphi\cdot\mathbf{X})=\delta(\varphi)\cdot\mathbf{X}+\varphi\cdot\xi(\mathbf{X}). \tag{3.4}$$

any differentiation (3.3) in the ring of smooth functions $\mathfrak{F}(t)$ is given by formula

$$\delta=f(t)\,\frac{d}{dt}, \tag{3.5}$$

ı. e., in main, it reduces to the operator of differentiation with respect to time variable t. For non-restricted operator D in whole we have the following propositions similar to lemma 1.2 and theorem 1.3 in Chapter III.

Lemma 3.1. *If two tensor fields $\mathbf{X}(t)$ and $\mathbf{Y}(t)$ on the curve $p(t)$ coincide with each other in some neighborhood of some point $t=t_0$, then tensor fields $D(\mathbf{X})$ and $D(\mathbf{Y})$ have equal values for $t=t_0$.*

Theorem 3.1. *Defining the differentiation D in the algebra of tensor fields $\mathbf{T}(t)$ on the curve is equivalent to defining the function $f(t)$ in* (3.5) *and the $\mathbb{R}$-map $\xi\colon T^1_0(t)\to T^1_0(t)$ satisfying the relationships* (3.4).

If we take $f(t)=0$ in (3.5), then we get the definition degenerate differentiation.

Definition 3.2. Differentiation D in the algebra of tensor fields $\mathbf{T}(t)$ is called **degenerate differentiation**, if its restriction δ to subspace $T^0_0(t)$ is identically zero.

Any degenerate differentiation D in $\mathbf{T}(t)$ is defined by some tensor field $\mathbf{S}(t)$ of the type $(1,1)$. This can be done my means of formulas similar to formulas (1.8) and (1.9) in Chapter III.

Definition 3.3. Differentiation D in the algebra of tensor fields $\mathbf{T}(t)$ is called **covariant differentiation** with respect to parameter t along the curve $p(t)\in M$, if function $f(t)$ in (3.5) for it is identically equal to unity: $f(t)=1$.

Covariant differentiation with respect to parameter t along the curve is denoted by ∇_t. Condition $f(t)=1$ determines the restriction of ∇_t to $T^0_0(t)$. In order to describe restriction of ∇_t to $T^1_0(t)$ let's choose some local map $(U,x^1,\ldots,x^n)$ and consider coordinate vector fields $\mathbf{E}_1,\ \ldots\ ,\ \mathbf{E}_n$ in this map (see (4.3) in Chapter III). Upon restricting them to the curve we get the fields $\mathbf{E}_1(t),\ \ldots\ ,\ \mathbf{E}_n(t)$. Let's apply ∇_t to $\mathbf{E}_j(t)$ and expand resulting field in the base of vector fields $\mathbf{E}_1(t),\ \ldots\ ,\ \mathbf{E}_n(t)$:

$$\nabla_t\mathbf{E}_j=\sum_{k=1}^{n}\Gamma^k_j(t)\,\mathbf{E}_k. \tag{3.6}$$

Coefficients $\Gamma^k_j(t)$ in (3.6) play the same role as the components of affine connection in formula (4.10) from Chapter III). In transferring from one local map to another they are transformed according to the following rule:

$$\Gamma^k_j=\sum_{m=1}^{n}\sum_{a=1}^{n}S^k_m\,T^a_j\,\tilde\Gamma^m_a+\sum_{m=1}^{n}S^k_m\,\frac{dT^m_j}{dt}. \tag{3.7}$$

Components of transition matrices S and T here are function of t, since their natural arguments $x^1, \ldots, x^n$ and $\tilde{x}^1, \ldots, \tilde{x}^n$ depend on t due to (1.1). Therefore

$$\frac{dT^m_j}{dt} = \sum_{i=1}^{n} \frac{\partial T^m_j}{\partial x^i}\,\dot{x}^i = \sum_{i=1}^{n} \frac{\partial T^m_i}{\partial x^j}\,\dot{x}^i. \tag{3.8}$$

Time derivatives $\dot{x}^1, \ldots, \dot{x}^n$ in (3.8) are the components of velocity vector $\mathbf{v}(t)$ on the curve. Substituting (3.8) into (3.7), now we obtain

$$\Gamma^k_j = \sum_{m=1}^{n}\sum_{a=1}^{n} S^k_m\,T^a_j\,\tilde{\Gamma}^m_a + \sum_{m=1}^{n}\sum_{i=1}^{n} v^i\,S^k_m\,\frac{\partial T^m_i}{\partial x^j}. \tag{3.9}$$

Let's compare (3.9) with formula (4.11) in Chapter III). As a result of this comparison we find that quantities Γ^k_j in (3.6) could be defined by formula

$$\Gamma^k_j = \sum_{i=1}^{n} v^i\,\Gamma^k_{ij}(x^1, \ldots, x^n, v^1, \ldots, v^n), \tag{3.10}$$

where Γ^k_{ij} are components of some extended connection in the manifold M, and their arguments $x^1, \ldots, x^n, v^1, \ldots, v^n$ depend on t due to the natural lift of curve $p(t)$ from M to TM.

Theorem 3.2. *Any extended affine connection Γ on the manifold M defines some covariant differentiation ∇_t of tensor fields with respect to parameter t on curves.*

Proof. In the set of scalar fields we define differentiation ∇_t to be coincident with ordinary differentiation with respect to parameter t:

$$\nabla_t \varphi = \frac{d\varphi}{dt}. \tag{3.11}$$

This is in accordance with definition 3.3. Then we calculate parameters Γ^k_i by formula (3.10), and for vector field $\mathbf{X}$ with components $X^1, \ldots, X^n$ in the base of coordinate vectors $\mathbf{E}_1(t), \ldots, \mathbf{E}_n(t)$ we set

$$\nabla_t \mathbf{X} = \sum_{k=1}^{n} \left(\frac{dX^k}{dt} + \sum_{i=1}^{n} \Gamma^k_i\,X^i \right) \mathbf{E}_k. \tag{3.12}$$

Formulas (3.11) and (3.12) define two maps δ and ξ of the form (3.2). Due to the theorem 3.1 these maps completely define the differentiation ∇_t. As for the relationships (3.3) and (3.4), one can easily check them to be fulfilled for the maps δ and ξ defined by (3.11) and (3.12). $\square$

Let $\mathbf{X}(t)$ be a tensor field with components $X^{i_1 \ldots i_r}_{j_1 \ldots j_s}$ on the curve $p(t)$. Let's apply the differentiation ∇_t, which was defined just above, to the field $\mathbf{X}(t)$. And denote by $\nabla_t X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}$ components of resulting tensor field $\nabla_t\mathbf{X}(t)$. Relying on (3.11) and (3.12) we can derive the following formula for these components

$$\begin{aligned}\nabla_t X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s} &= \frac{dX^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}}{dt}+\\ &+\sum_{k=1}^{r}\sum_{m=1}^{n}\sum_{a_k=1}^{n}\dot{x}^m\,\Gamma^{i_k}_{m\,a_k}\,X^{i_1\ldots\,a_k\ldots\,i_r}_{j_1\ldots\ldots\ldots\,j_s}-\\ &-\sum_{k=1}^{s}\sum_{m=1}^{n}\sum_{b_k=1}^{n}\dot{x}^m\,\Gamma^{b_k}_{m\,j_k}\,X^{i_1\ldots\ldots\ldots\,i_r}_{j_1\ldots\,b_k\ldots\,j_s}.\end{aligned}\tag{3.13}$$

Formula (3.13) is an analog of formula (4.17) in Chapter III. The sign ∇_t further will be used in two cases. For the tensor field $\mathbf{X}(t)$ by $\nabla_t\mathbf{X}(t)$ we shall denote the result of differentiating it with respect to parameter t along the curve. In local coordinates by $\nabla_t X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}$ we shall denote the array of components for the field $\nabla_t\mathbf{X}(t)$. Therefore in second case ∇_t is understood as an operation transforming one array into another array. For this operation we have the following relationships:

$$\nabla_t(A^{\ldots}_{\ldots}\cdot B^{\ldots}_{\ldots}) = \nabla_t A^{\ldots}_{\ldots}\cdot B^{\ldots}_{\ldots} + A^{\ldots}_{\ldots}\cdot\nabla_t B^{\ldots}_{\ldots},\tag{3.14}$$

$$\sum_{k=1}^{n}\nabla_t A^{..k\ldots}_{\ldots k..} = \nabla_t\left(\sum_{k=1}^{n} A^{..k\ldots}_{\ldots k..}\right).\tag{3.15}$$

These relationships (3.14) and (3.15) are the analogs of the relationships (7.5) and (7.6) in Chapter III.

As a final result in this section we formulate the structural theorem for differentiations in the algebra of tensor fields $\mathbf{T}(t)$.

THEOREM 3.3. *Let M be real smooth manifold equipped with extended affine connection Γ, and let $p(t)$ be smooth parametric curve on M. Then any differentiation in the algebra of tensor fields on this curve can be expressed as a sum*

$$D = f(t)\cdot\nabla_t + \mathbf{S},\tag{3.16}$$

where $f(t)$ is some smooth function on the curve $p(t)$, and $\mathbf{S}$ is some degenerate differentiation defined by tensor field $\mathbf{S}(t)$ of the type $(1,1)$.

Theorem 3.3 is an analog of theorem 5.1 in Chapter III, and the expansion (3.16) is analogous to the expansion (5.1) in Chapter III.

§ 4. Vector of acceleration. Differentiation of extended tensor fields along the curves.

Suppose that M is a real smooth manifold with extended affine connection Γ. In this situation we have two covariant differentiations ∇ and $\tilde{\nabla}$ in extended algebra of tensor fields $\mathbf{T}(M)$ and differentiation $\tilde{\nabla}$ with respect to parameter t on curves.

Let $p(t)$ be smooth parametric curve on M with parameter t understood as a time variable. Then we can apply ∇_t to the vector field of velocity $\mathbf{v}(t) = \dot{p}(t)$ on this curve. As a result we get vector field $\mathbf{a}(t) = \nabla_t \mathbf{v}$ with the following components:

$$a^k = \ddot{x}^k + \sum_{i=1}^{n} \sum_{j=1}^{n} \Gamma^k_{ij}\, \dot{x}^i\, \dot{x}^j, \quad \text{where} \quad k = 1, \ldots, n. \tag{4.1}$$

Vector $\mathbf{a}(t)$ with components (4.1) is called the **acceleration vector** for the point moving on M along the curve $p(t)$. This terminology is in concordance with the results of Chapter II, where we considered constrained mechanical systems.

Now let $\mathbf{X}$ be some extended tensor field on the manifold M. It has natural restriction $\mathbf{X}(t)$ to the curve $p(t)$ due to the natural lift of this curve from M to TM (see definition 2.1 and definition 2.2 in § 2 above). Let's apply ∇_t to the field $\mathbf{X}(t)$ and let's calculate components of resulting field $\nabla_t \mathbf{X}(t)$ by means of formula (3.13). To find time derivative of $X^{i_1 \ldots\, i_r}_{j_1 \ldots\, j_s}$ in formula (3.13) we should differentiate composite function of the following form:

$$X^{i_1 \ldots\, i_r}_{j_1 \ldots\, j_s} = X^{i_1 \ldots\, i_r}_{j_1 \ldots\, j_s}(x^1(t), \ldots, x^n(t), \dot{x}^1(t), \ldots, \dot{x}^n(t)).$$

Upon differentiating this composite function let's substitute the resulting expression into (3.13) and rearrange terms in this formula

$$\begin{gathered}
\nabla_t X^{i_1 \ldots\, i_r}_{j_1 \ldots\, j_s} = \sum_{m=1}^{n} \dot{x}^m \left(\frac{\partial X^{i_1 \ldots\, i_r}_{j_1 \ldots\, j_s}}{\partial x^m} + \sum_{k=1}^{r} \sum_{a_k=1}^{n} \Gamma^{i_k}_{m\, a_k}\, X^{i_1 \ldots\, a_k \ldots\, i_r}_{j_1 \ldots \ldots \ldots\, j_s} - \right.\\
\left. - \sum_{k=1}^{s} \sum_{b_k=1}^{n} \Gamma^{b_k}_{m\, j_k}\, X^{i_1 \ldots \ldots \ldots\, i_r}_{j_1 \ldots\, b_k \ldots\, j_s}\right) + \sum_{m=1}^{n} \ddot{x}^m \frac{\partial X^{i_1 \ldots\, i_r}_{j_1 \ldots\, j_s}}{\partial v^m}.
\end{gathered}$$

Now compare the expression in round brackets here with the formula (7.3) from Chapter III. The difference is only in one term

$$-\sum_{a=1}^{n} \sum_{b=1}^{n} \dot{x}^a\, \Gamma^b_{ma} \frac{\partial X^{i_1 \ldots\, i_r}_{j_1 \ldots\, j_s}}{\partial v^b}.$$

Let's add and then subtract this term. After opening brackets and redesignating some summation indices we get

$$\begin{gathered}
\nabla_t X^{i_1 \ldots\, i_r}_{j_1 \ldots\, j_s} = \sum_{m=1}^{n} \dot{x}^m\, \nabla_m X^{i_1 \ldots\, i_r}_{j_1 \ldots\, j_s} + \\
+ \sum_{m=1}^{n} \left(\ddot{x}^m + \sum_{a=1}^{n} \sum_{b=1}^{n} \Gamma^m_{ab}\, \dot{x}^a\, \dot{x}^b\right) \frac{\partial X^{i_1 \ldots\, i_r}_{j_1 \ldots\, j_s}}{\partial v^m}.
\end{gathered}$$

Let's compare this expression with formula (4.1) for the components of acceleration vector, and with formula (7.4) in Chapter III. This yields

$$\nabla_t X^{i_1\dots\,i_r}_{j_1\dots\,j_s} = \sum_{m=1}^{n} v^m\,\nabla_m X^{i_1\dots\,i_r}_{j_1\dots\,j_s} + \sum_{m=1}^{n} a^m\,\tilde\nabla_m X^{i_1\dots\,i_r}_{j_1\dots\,j_s}. \tag{4.2}$$

Formula (4.2) is the rule for differentiating natural restriction of extended tensor field $\mathbf{X}$ with respect to parameter t along the curve. In coordinate-free form this rule is written as follows:

$$\nabla_t\mathbf{X} = C(\mathbf{v}(t)\otimes\nabla\mathbf{X}) + C(\mathbf{a}(t)\otimes\tilde\nabla\mathbf{X}). \tag{4.3}$$

If we take into account the relationships (7.1) from Chapter III, then formula (4.3) can be brought to the following equality:

$$\nabla_t\mathbf{X} = \nabla_{\mathbf{v}}\mathbf{X} + \tilde\nabla_{\mathbf{a}}\mathbf{X}. \tag{4.4}$$

The relationships (4.2), (4.3), and (4.4) can be understood as particular subcase of the rule of differentiating composite functions written in covariant form and applied to tensor fields.

§ 5. Deformation of curves on manifold. Vector of variation.

Let $p(t)$ be smooth parametric curve on real smooth manifold M. In order to describe the deformation of this curve lets consider the function of two numeric arguments $p(t,u)$ with values being point of manifold M. Let's denote

$$p(t) = p(t,u)\Big|_{u=0}. \tag{5.1}$$

If condition (5.1) is fulfilled, we say that $p(t,u)$ defines the **deformation** of curve $p(t)$ on the manifold M. Parameter t is main parameter. Further it will be understood as time variable. Parameter u is auxiliary parameter. It is called the **parameter of deformation**.

Let's choose some local map $(U, x^1, \dots, x^n)$ on the manifold M. Deformations of curve $p(t)$ in local map U is expressed by the functions

$$x^1(t,u),\ \dots,\ x^n(t,u). \tag{5.2}$$

We say that $p(t,u)$ is a smooth deformation of the curve $p(t)$ if it is expressed by smooth functions (5.2) in any local map.

In case of smooth deformation $p(t,u)$ we can differentiate functions (5.2) with respect to the auxiliary parameter u. Derivatives

$$\tau^i(t) = \frac{\partial x^i(t,u)}{\partial u}\bigg|_{u=0},\ \text{ where }\ i = 1,\,\dots,\,n, \tag{5.3}$$

are the components of some vector $\boldsymbol{\tau}(t)$ on the curve $p(t)$. This vector is called the **vector of variation**. Its components define first order terms in Taylor series expansions for the functions (5.2) in auxiliary variable u about the point $u = 0$:

$$\begin{aligned} &x^1(t,u) = x^1(t) + \tau^1(t)\cdot u + O(u^2),\\ &\dots\dots\dots\dots\dots\dots\dots\dots\\ &x^n(t,u) = x^n(t) + \tau^n(t)\cdot u + O(u^2). \end{aligned} \tag{5.4}$$

Usually in considering deformations of curves only starting terms in the series expansions (5.4) appear to be of importance. In this case all results are formulated in terms of variation vector $\boldsymbol{\tau}(t)$.

Now suppose that manifold M is equipped with extended affine connection Γ, and suppose that smooth deformation of curve $p(t,u)$ on M is given. This means that for each fixed value of u we have some smooth curve with parameter t. Let's calculate velocity vector $\mathbf{v}(t,u) = \dot{p}(t,u)$ on each such curve and define its natural lift $q(t,u) = (p(t,u), \mathbf{v}(t,u))$ to TM. This construction is called **natural lift for deformation of curves** $p(t,u)$. Some asymmetry between main and auxiliary parameters t and u is prebuilt into this construction.

Suppose that for each fixed u on each curve $p(t,u)$ the tensor field $\mathbf{X}(t,u)$ of type (r,s) is defined. If $p(t) = p(t,u)$ and $\mathbf{X}(t) = \mathbf{X}(t,0)$, then we say that the **deformation** of tensor field $\mathbf{X}(t)$ is given. This deformation is called **smooth**, if it is represented by smooth tensor-valued function $\mathbf{X}(t)$ of two numeric arguments.

Let $\mathbf{X}(t,u)$ be smooth deformation of tensor field corresponding to the smooth deformation of the curve $p(t,u)$. Having fixed the parameter u, we can apply covariant differentiation ∇_t defined in § 3 to $\mathbf{X}(t,u)$. However, this is not enough. In what follows we sometimes have to differentiate $\mathbf{X}(t,u)$ with respect to its second parameter u for fixed t. Therefore we define covariant differentiation with respect to parameter of deformation. For the deformation of scalar field $\varphi(t,u)$ we set

$$\nabla_u\varphi = \frac{d\varphi}{du}. \tag{5.5}$$

According to the results of § 3, in order to expand (5.5) up to a covariant differentiation with respect to parameter u we should associate with each local map on M the set of functions $\Gamma^k_j = \Gamma^k_j(t,u)$. Under the change of local maps on M these function should obey the transformation rules

$$\Gamma^k_j = \sum_{m=1}^{n}\sum_{a=1}^{n} S^k_m\,T^a_j\,\tilde{\Gamma}^m_a + \sum_{m=1}^{n} S^k_m\,\frac{dT^m_j}{du} \tag{5.6}$$

similar to (3.7). Let's define them by the following formula:

$$\Gamma^k_j = \sum_{i=1}^{n}\tau^i\,\Gamma^k_{ij}(x^1,\dots,x^n,v^1,\dots,v^n). \tag{5.7}$$

Here $\tau^1, \dots, \tau^n$ are the components of variation vector $\boldsymbol{\tau}(t,u)$, and $v^1, \dots, v^n$ are the components of velocity vector $\mathbf{v}(t,u)$. It is not difficult to check up that the relationships (5.6) for the quantities (5.7) are fulfilled. Note that the quantities (5.7) are intentionally chosen in a slightly different way than the quantities (3.10). This is due to different roles of the variables t and u.

Differentiation ∇_u defined by extended affine connection Γ due to (5.5) and (5.7) is called **covariant differentiation with respect to the parameter of deformation**. For the components of the tensor field $\nabla_u\mathbf{X}(t,u)$ we have formula analogous to formula (3.13) for $\nabla_t\mathbf{X}(t,u)$:

$$\begin{aligned} \nabla_u X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s} &= \frac{dX^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}}{du}+\\ &+\sum_{k=1}^{r}\sum_{m=1}^{n}\sum_{a_k=1}^{n}\tau^m\,\Gamma^{i_k}_{m\,a_k}\,X^{i_1\ldots\,a_k\ldots\,i_r}_{j_1\ldots\ldots\ldots\,j_s}-\\ &-\sum_{k=1}^{s}\sum_{m=1}^{n}\sum_{b_k=1}^{n}\tau^m\,\Gamma^{b_k}_{m\,j_k}\,X^{i_1\ldots\ldots\ldots\,i_r}_{j_1\ldots\,b_k\ldots\,j_s}. \end{aligned} \tag{5.8}$$

For the fields corresponding to deformations of curves the derivative in u is calculated for the fixed value of t, and time derivative in (3.13) is calculated for the fixed value of u. These are partial derivatives. However, in (5.8) we used the symbol of ordinary derivative in order to keep similarity of formulas (3.13) and (5.8).

The map $\nabla_u\colon \mathbf{T}(t,u)\to\mathbf{T}(t,u)$ is the differentiation of tensor fields on deformations of curves (in the sense of definition analogous to the definition 3.1). Therefore in local coordinates we have the relationships

$$\nabla_u(A^{\ldots}_{\ldots}\cdot B^{\ldots}_{\ldots})=\nabla_u A^{\ldots}_{\ldots}\cdot B^{\ldots}_{\ldots}+A^{\ldots}_{\ldots}\cdot\nabla_u B^{\ldots}_{\ldots}, \tag{5.9}$$

$$\sum_{k=1}^{n}\nabla_u A^{..k\ldots}_{\ldots k..}=\nabla_u\left(\sum_{k=1}^{n}A^{..k\ldots}_{\ldots k..}\right) \tag{5.10}$$

similar to formulas (3.14) and (3.15). For the components of vectors $\mathbf{v}(t,u)$ and $\boldsymbol{\tau}(t,u)$ we have the obvious equality

$$\frac{\partial\tau^k}{\partial t}=\frac{\partial^2x^k}{\partial t\,\partial u}=\frac{\partial^2x^k}{\partial u\,\partial t}=\frac{\partial v^k}{\partial u}, \tag{5.11}$$

which is the consequence of permutability of partial derivatives in u and t. Written in covariant derivatives ∇_u and and ∇_t, the equality (5.11) takes less trivial appearance:

$$\nabla_t\tau^k-\nabla_u v^k=-\sum_{i=1}^{n}\sum_{j=1}^{n}T^k_{ij}\,\tau^i\,v^j. \tag{5.12}$$

And finally, in coordinate-free form (5.12) is written as follows:

$$\nabla_t\boldsymbol{\tau}-\nabla_u\mathbf{v}=-\mathbf{T}(\boldsymbol{\tau},\mathbf{v}). \tag{5.13}$$

Tensor field $\mathbf{T}$ in (5.13) is a tensor field of torsion, its components T^k_{ij} are calculated by formula (6.8) from Chapter III.

Now let's find formula for commutator of covariant derivatives ∇_u and ∇_t. It is very important for further development of the theory. For $[\nabla_u, \nabla_t]$ we get

$$[\nabla_u,\, \nabla_t] = \mathbf{S}, \tag{5.14}$$

where $\mathbf{S}$ is degenerate differentiation defined by tensor field $\mathbf{S}$ of type $(1, 1)$:

$$\mathbf{S} = \mathbf{R}(\boldsymbol{\tau}, \mathbf{v}) + \mathbf{D}(\boldsymbol{\tau}, \mathbf{a}) - \mathbf{D}(\mathbf{v}, \nabla_t \boldsymbol{\tau}) - \mathbf{D}(\mathbf{v}, \mathbf{T}(\boldsymbol{\tau}, \mathbf{v})). \tag{5.15}$$

Here $\mathbf{T}$ is a tensor field of torsion, while $\mathbf{D}$ and $\mathbf{R}$ are tensor fields of curvature. Components of $\mathbf{D}$ and $\mathbf{R}$ are calculated by formulas (6.5) and (6.8) from Chapter III. Vector $\mathbf{a} = \nabla_t \mathbf{v}$ in formula (5.15) is a vector of acceleration with components calculated by formula (4.1).

Degeneracy of differentiation $\mathbf{S}$ in right hand side of (5.14) means that commutator of derivatives ∇_u and ∇_t is zero when applied to a scalar field $\varphi(t, u)$:

$$[\nabla_u,\, \nabla_t]\varphi = 0 \tag{5.16}$$

Let's apply commutator (5.14) to vector field $\mathbf{X}(t, u)$ and let's write the result in local coordinates. Taking into account (5.15), we get

$$\begin{aligned} &[\nabla_u,\, \nabla_t]X^k = -\sum_{r=1}^{n}\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{m=0}^{n}\sum_{h=0}^{n} v^m\, T^h_{im}\, D^k_{rjh}\, \tau^i\, v^j\, +\right.\\ &\left. + \sum_{i=1}^{n}\sum_{j=1}^{n}\left(D^k_{rij}\, v^i\, \nabla_t \tau^j - R^k_{rij}\, \tau^i\, v^j - D^k_{rij}\, \tau^i\, a^j\right)\right) X^r. \end{aligned} \tag{5.17}$$

Now let's apply commutator (5.14) to covector field $\mathbf{h}(t, u)$. The result is written as

$$\begin{aligned} &[\nabla_u,\, \nabla_t]h_k = \sum_{r=1}^{n}\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{m=0}^{n}\sum_{h=0}^{n} v^m\, T^h_{im}\, D^r_{kjh}\, \tau^i\, v^j\, +\right.\\ &\left. + \sum_{i=1}^{n}\sum_{j=1}^{n}\left(D^r_{kij}\, v^i\, \nabla_t \tau^j - R^r_{kij}\, \tau^i\, v^j - D^r_{kij}\, \tau^i\, a^j\right)\right) h_r. \end{aligned} \tag{5.18}$$

Formulas (5.16), (5.17), and (5.18) are analogs of formulas (7.10), (7.11), and (7.12) from Chapter III.

Let $\mathbf{X}$ be extended tensor field on the manifold M, and let $p(t, u)$ be the deformation of curves on this manifolds. For each fixed u we have parametric curve $p(t, u)$ with parameter t. Natural lift of such curve, as we mentioned above, define the deformation of curves $q(u, t)$ on tangent bundle TM. This construction, which is called **natural lift** for $p(t, u)$, can be used for restricting extended tensor fields to the deformation of curves on M. For the field $\mathbf{X}(t,u)$ obtained as a result of such **natural restriction** we have the formula

$$\nabla_u \mathbf{X} = C(\boldsymbol{\tau} \otimes \nabla \mathbf{X}) + C(\nabla_t \boldsymbol{\tau} \otimes \tilde{\nabla} \mathbf{X}) + C(\mathbf{T}(\boldsymbol{\tau}, \mathbf{v}) \otimes \tilde{\nabla} \mathbf{X}), \tag{5.19}$$

being analog of formula (4.3). By denoting $\mathbf{b} = \nabla_t \boldsymbol{\tau} + \mathbf{T}(\boldsymbol{\tau}, \mathbf{v})$ we can write formula (5.19) in the form similar to (4.4):

$$\nabla_u \mathbf{X} = \nabla_{\boldsymbol{\tau}} \mathbf{X} + \tilde{\nabla}_{\mathbf{b}} \mathbf{X}. \tag{5.20}$$

Being written in local coordinates, formula (5.20) looks like

$$\nabla_u X^{i_1 \ldots\, i_r}_{j_1 \ldots\, j_s} = \sum_{m=1}^{n} \tau^m \, \nabla_m X^{i_1 \ldots\, i_r}_{j_1 \ldots\, j_s} + \sum_{m=1}^{n} b^m \, \tilde{\nabla}_m X^{i_1 \ldots\, i_r}_{j_1 \ldots\, j_s}. \tag{5.21}$$

It's easier first to prove formula (5.21) by direct calculations in local coordinates, then back step to (5.20) is obvious.

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http://arXiv.org/ps/math/0002202v1

CHAPTER V

NORMAL SHIFT IN RIEMANNIAN MANIFOLDS.

§ 1. Geodesic normal shift.

In this Chapter we return to the case of Riemannian manifolds and derive main results of thesis. They consist in developing the theory of **Newtonian dynamical systems admitting the normal shift** on such manifolds. As a starting point for this theory we choose well-known classical construction of **geodesic normal shift**.

Let M be Riemannian manifold with metric tensor $\mathbf{g}$. Tensor field $\mathbf{g}$ defines scalar product of tangent vectors on the manifold M. Let's denote it as

$$\mathbf{g}(\mathbf{X},\mathbf{Y}) = (\mathbf{X}\,|\,\mathbf{Y}). \tag{1.1}$$

Here $\mathbf{X}$ and $\mathbf{Y}$ are two vectors at one point $p \in M$. Metric $\mathbf{g}$ defines metric connection Γ with identically zero torsion field $\mathbf{T} = 0$. Its components in local map $(U, x^1, \ldots, x^n)$ are determined by components of metric tensor $\mathbf{g}$ according to the following standard formula (see [12], [32], [76], or [77]):

$$\Gamma^k_{ij} = \frac{1}{2}\sum_{s=1}^{n} g^{ks}\left(\frac{\partial g_{sj}}{\partial x^i} + \frac{\partial g_{is}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^s}\right), \tag{1.2}$$

Quantities g_{ij} and Γ^k_{ij} in (1.2) do not depend on the components of velocity vector $v^1, \ldots, v^n$, i. e. we have ordinary (not extended) tensor field $\mathbf{g}$ and ordinary (not extended) connection Γ. Applying velocity gradient $\tilde{\nabla}$ to the field $\mathbf{g}$, we get zero, while space gradient ∇ applied to $\mathbf{g}$ coincides with traditional covariant differential of this field. Therefore we have the equalities

$$\nabla_{\mathbf{g}} = 0, \qquad\qquad \tilde{\nabla}\mathbf{g} = 0. \tag{1.3}$$

First equality in (1.2) is standard (see [12], [32], [76], or [77]). It expresses the **concordance of** metric and connection).

Geodesic lines play important role in geometry. In local coordinates the are defined by ordinary differential equations

$$\ddot{x}^k + \sum_{i=1}^{n}\sum_{j=1}^{n}\Gamma^k_{ij}\,\dot{x}^i\,\dot{x}^j = 0, \quad \text{where} \quad k = 1, \ldots, n, \tag{1.4}$$

provided special parametrization is used, where t is a length of the curve measured from some fixed reference point on it. Let's write the equations (1.4) in terms of covariant differentiation with respect to parameter t along the curve (such covariant differentiations were considered in Chapter IV):

$$\dot x^k = v^k, \qquad\qquad \nabla_t v^k = 0. \tag{1.5}$$

The equations (1.5) form the system of $2n$ ordinary differential equations of the first order. In setting up Cauchy problem for these equations initial values of local coordinates $x^1, \ldots, x^n$ and components of velocity vector $v^1, \ldots, v^n$ are given:

$$x^k(t)\Bigr|_{t=0} = x^k(0), \qquad\qquad v^k(t)\Bigr|_{t=0} = v^k(0). \tag{1.6}$$

This means that we fix initial point $p = p(0)$ and initial vector of velocity $\mathbf{v}(0)$ at the point $p(0)$. Note that equations (1.4) are invariant under the following transformation of independent variable: $t \to \mathrm{const}\cdot t$. Therefore without loss of generality we can assume the initial vector $\mathbf{v}(0)$ in (1.6) to be of unit length: $|\mathbf{v}(0)| = 1$. Once fulfilled for $t = 0$ this condition remains fulfilled for nonzero t as well:

$$|\mathbf{v}(t)| = 1. \tag{1.7}$$

This is the very case when parameter t is a length of curve measured from some fixed reference point, it is called **natural parameter** on geodesic line.

Let S be some hypersurface in M. For the sake of simplicity S assumed to be connected and simply connected smooth submanifold of codimension 1 without boundary such that the closure of S is compact. Moreover, we assume that S is compactly imbedded into some other open hypersurface, i. e. $S \Subset S'$. This strong requirement eliminates most difficulties that could arise on the boundary ∂S. Under the above assumptions hypersurface S is orientable, and one can choose smooth field of unitary normal vectors on it, which has smooth continuation to any point on the boundary ∂S. This field defines unitary normal vector $\mathbf{n}(p)$ at each point $p \in S$. Let's choose components of $\mathbf{n}(p)$ as the components of initial velocity in setting up initial data for Cauchy problem (1.6). Then the condition $|\mathbf{v}(0)| = 1$ is fulfilled due to $|\mathbf{n}(p)| = 1$. Let's write (1.6) as

$$x^k(t)\Bigr|_{t=0} = x^k(p), \qquad\qquad v^k(t)\Bigr|_{t=0} = n^k(p). \tag{1.8}$$

Solution of the equations (1.5) satisfying initial data (1.8) defines the family of geodesic lines beginning at the points of hypersurface S and being perpendicular to S at that points (See Fig. 1.1). The value of parameter t on these lines coincides with their length measured from starting points on S.

Let p_0 be a pint on the hypersurface S, and let U be some map on S covering this point. Suppose that $u^1, \dots, u^{n-1}$ are local coordinates of the point p in the map U. Then geodesic lines obtained as the solution of Cauchy problem (1.8) for the system of equations (1.5) are represented by the set of n functions

$$\begin{aligned} x^1 &= x^1(u^1, \dots, u^{n-1}, t), \\ &\dots\dots\dots\dots\dots\dots \\ x^n &= x^n(u^1, \dots, u^{n-1}, t), \end{aligned} \tag{1.9}$$

where $x^1, \dots, x^n$ are local coordinates in M. According to well-known "existence and uniqueness" theorem (see [82] or [83]) the domain of functions (1.9) is such that they define the map $U' \times I_\delta \to M$, where U' is an open subset of local map U containing the point p_0, and $I_\delta = (-\delta, +\delta)$ is open δ-neighborhood of zero on real axis $\mathbb{R}$. Due to the compactness of the closure of S, or, being more exact, due to $S \Subset S'$ local maps $U' \times I_\delta \to M$ can be glued into one global map $S \times I_\varepsilon \to M$.

Let's differentiate the functions (1.9) with respect to parameters $u^1, \dots, u^{n-1}$, and let's form the following vectors by the obtained derivatives:

$$\mathbf{K}_i = \sum_{j=1}^{n} \frac{\partial x^j}{\partial u^i} \frac{\partial}{\partial x^j}, \text{ where } i = 1, \dots, n-1. \tag{1.10}$$

Differentiating the functions (1.9) with respect to their last argument t, we form one more vector $\mathbf{N}$. It is defined by the formula similar to (1.10):

$$\mathbf{N} = \sum_{j=1}^{n} \frac{\partial x^j}{\partial t} \frac{\partial}{\partial x^j}. \tag{1.11}$$

For $t = 0$ vectors (1.10) coincide with coordinate tangent vectors to S, while vector (1.11) coincides with normal vector of S. Therefore vectors $\mathbf{K}_1, \dots, \mathbf{K}_{n-1}, \mathbf{N}$ are linearly independent for $t = 0$. If we denote $t = u^n$, and if we build Jacoby matrix

$$J = \begin{Vmatrix} \dfrac{\partial x^1}{\partial u^1} & \dots & \dfrac{\partial x^1}{\partial u^n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial x^n}{\partial u^1} & \dots & \dfrac{\partial x^n}{\partial u^n} \end{Vmatrix},$$

it will be non-degenerate at the point p_0. This means, that the variables $u^1, \dots, u^n$ (where $u^n = t$) could be used as local coordinates on M in some neighborhood of the point p_0. These coordinates are called **semigeodesic coordinates** (see [**]). Hypersurface S in these coordinates is described by the equation $u^n = 0$.

Let's consider the equation $u^n = t = \text{const}$ in semigeodesic coordinates. It determines a piece of some hypersurface. Due to $S \Subset S'$ we can find $\varepsilon > 0$ such that for $t \in I_\varepsilon$ various pieces of hypersurfaces $u^n = t$ are glued into an open hypersurface S_t diffeomorphic to S. At the expense of choosing sufficiently small value of ε we can reach the situation, when for $t \in I_\varepsilon$ none of hypersurfaces S_t will intersect another, nor will it have self-intersection. Therefore S has some neighborhood in M diffeomorphic to $S \times I_\varepsilon$, and for each $t \in I_\varepsilon$ there is a diffeomorphism $f_t\colon\, S \to S_t$. Such diffeomorphism built by geodesic lines as described above is called **geodesic shift** of hypersurface S to the distance t. It's important to emphasize that any hypersurface S_t has compact closure, and there exists some open hypersurface S'_t such that $S_t \Subset S'_t$.

Let's study the geodesic shift of hypersurface S in semigeodesic coordinates $x^1 = u^1, \ldots, x^n = u^n$ associated with it. In this case $x^n = t$ is a parameter of shift, and vectors (1.10) are coordinate tangent vectors to hypersurfaces S_t described by the equations $x^n = t = \text{const}$. Due to $x^j = u^j$ they are the followings:

$$\mathbf{K}_1 = \frac{\partial}{\partial x^1},\ \ldots,\ \mathbf{K}_{n-1} = \frac{\partial}{\partial x^{n-1}}. \tag{1.12}$$

Geodesic lines implementing geodesic shift of hypersurface $x^n = 0$ in semigeodesic coordinates are described by functions

$$x^1(t) = \text{const},\ \ldots,\ x^{n-1}(t) = \text{const},\ x^n(t) = t. \tag{1.13}$$

Let's substitute (1.13) into the equations of geodesic lines (1.4). This yields

$$\Gamma^k_{nn} = 0 \ \text{ for all } \ k = 1, \ldots, n. \tag{1.14}$$

Having calculated connection components Γ^k_{nn} by formula (1.2), due to non-degeneracy of the matrix of metric tensor g^{ks} from (1.14) we derive the relationships

$$2\,\frac{\partial g_{sn}}{\partial x^n} - \frac{\partial g_{nn}}{\partial x^s} = 0 \ \text{ for all } \ s = 1, \ldots, n. \tag{1.15}$$

Particularly for $s = n$ from (1.15) we extract the following equation:

$$\frac{\partial g_{nn}}{\partial x^n} = 0. \tag{1.16}$$

Thus, diagonal component g_{nn} in metric tensor doesn't depend on shift parameter $x^n = t$; and g_{nn} is the square of the length of n-th coordinate vector, which in semigeodesic coordinates coincides with vector (1.11). The latter one plays the role of velocity vector on geodesic lines used to shift S:

$$\mathbf{K}_n = \mathbf{N} = \mathbf{v}(t) = \frac{\partial}{\partial x^n}. \tag{1.17}$$

For $x^n = t = 0$ velocity vector $\mathbf{v}(t)$ coincides with unitary normal vector to S (see (1.8)). Hence $g_{nn} = 1$ for $x^n = 0$. From this equality combined with (1.16) we derive

$$g_{nn} \equiv 1. \tag{1.18}$$

In particular, (1.18) could be the proof for the equality (1.7) for the velocity vector on geodesic lines. Substituting (1.18) into (1.15) for $s \neq n$, we immediately get

$$\frac{\partial g_{sn}}{\partial x^n} = 0 \quad \text{for} \quad s = 1, \ldots, n-1. \tag{1.19}$$

Components g_{sn} in metric tensor are the scalar products of vectors (1.12) and (1.17):

$$g_{sn} = (\mathbf{K}_s \,|\, \mathbf{K}_n). \tag{1.20}$$

At the initial instant, when $x^n = t = 0$, scalar products (1.20) are equal to zero, since $\mathbf{K}_n$ coincides with unitary normal vector on S, while vectors (1.12) are tangent to S. From this fact combined with the equations (1.19) we get

$$g_{sn} \equiv 0 \quad \text{for all} \quad s = 1, \ldots, n-1. \tag{1.21}$$

The equalities (1.21) reflect very important geometric property of geodesic shift. They mean that not only the initial hypersurface S is perpendicular to geodesic lines used to shift it, but all other hypersurfaces S_t generated by shift procedure are perpendicular as well. Therefore geodesic shift is a **normal shift** of hypersurfaces.

§ 2. Normal shift along trajectories of dynamical system.

Geodesic normal shift described in § 1 is a well-known classical construction. The **main idea** that brought about the advent of the theory of **dynamical systems admitting the normal shift** consists in replacing geodesic lines by **wider class of parametric curves**. As a first pretender to this role we considered the trajectories of Newtonian dynamical systems, since

(1) they possess prescribed parametrization with the parameter having physical meaning of time;
(2) they are associated with positively definite quadratic form of kinetic energy that equips configuration space with the structure of Riemannian manifold;
(3) in local coordinates they are described by systems of differential equations compatible with Cauchy problems of the form (1.6).

Let's consider Newtonian dynamical system with force field $\mathbf{F}$, configuration space of which is a Riemannian manifold M with metric $\mathbf{g}$. Let's choose some local map $(U, x^1, \ldots, x^n)$ on M and canonically associated local coordinates on tangent bundle TM (see § 6 in Chapter II). Then trajectories of Newtonian dynamical system with the field $\mathbf{F}$ can be constructed by solving the system of differential equations (4.1) from Chapter II. By analogy with (1.5) these equations could be written in terms of covariant derivative with respect to parameter t:

$$\dot{x}^k = v^k, \qquad \nabla_t v^k = F^k. \tag{2.1}$$

Definition 2.1. We say that hypersurface S in M belongs to **transformation class**, if it is connected and simply connected open hypersurface with compact closure such that $S \Subset S'$ for some other open hypersurface S'.

Note that closed hypersurface without boundary belongs to transformation class, if it is connected, simply connected, and compact, since for closed hypersurface $\overline{S} = S$. Here we can take $S' = S$.

Suppose that $S \subset M$ is a hypersurface from transformation class. Let's fix the field of unitary normal vectors $\mathbf{n}(p)$ on S, and let's set up the following initial data for the system of differential equations (2.1) at the points of S:

$$x^k(t)\Big|_{t=0} = x^k(p), \qquad v^k(t)\Big|_{t=0} = \nu(p)\cdot n^k(p). \tag{2.2}$$

Solution of Cauchy problem with initial data (2.2) determines the family of trajectories of Newtonian dynamical system (2.1) beginning at the points of S and outgoing from S along normal vector of this hypersurface (see Fig. 1.1). The value of $\nu(p)$ determines modulus of velocity vector at the initial point p on such trajectory. In the case of geodesic normal shift the value of $\nu(p)$ is chosen to be equal to unity everywhere on S. Here we shall not keep this restriction, we shall require only that $\nu(p) \neq 0$ for all points $p \in S$.

Definition 2.2. Let $S \subset M$ be a hypersurface from transformation class. We say that smooth function $\nu(p)$ on S belongs to **transformation class**, if it has smooth expansion to some bigger open hypersurface S' such that $S \Subset S'$, and if it is nonzero everywhere on the closure $\overline{S}$.

Suppose that on the hypersurface S belonging to the transformation class on M we defined some function $\nu(p)$ belonging to the transformation class on S. For each point $p = p_0$ on S we draw the trajectory of dynamical system (2.1) passing through p_0 and satisfying initial data (2.2) at this point. Then consider the point $p(t) = p(p_0, t)$ on such trajectory corresponding to the value t of time variable, and associate it with p_0. So we have constructed a transformation of the hypersurface S that maps p_0 to $p(t)$. It is called the **shift along trajectories of dynamical system** for the time t.

In the case of shift along trajectories of Newtonian dynamical system, as well as in the case of geodesic shift, we can point out some interval I_ε for time variable t such that the map $S \times I_\varepsilon \to M$ is injective and has no singular points. This map stratifies some full-dimensional neighborhood of S into a family of hypersurfaces S_t diffeomorphic to S: for any $t \in I_\varepsilon$ we have diffeomorphism $f_t\colon S \to S_t$. For $t \in I_\varepsilon$ hypersurfaces S_t all are in transformation class, they have no self-intersection points and they do not intersect each other.

Definition 2.3. Shift of hypersurface S along trajectories of Newtonian dynamical system (2.1) initiated by the function $\nu(p)$ in Cauchy problem (2.2) is called **normal shift**, if for parameter t in some interval $I_\varepsilon = (-\varepsilon, +\varepsilon)$ all hypersurfaces S_t are orthogonal to the trajectories used to shift S.

The definition 2.3 is quite necessary, since, in contrast to geodesic normal shift, here the orthogonality of S_t and trajectories is not an unconditional property of shift. Presence (or absence) of this property essentially depends on how we deal with arbitrariness in the choice of the function $\nu(p)$ on S and in the choice of force field $\mathbf{F}$ for dynamical system.

§ 3. Differential equations for the vector of variation.

Let's choose some local map $(U, x^1, \ldots, x^n)$ on M and consider the shift of some hypersurface S along trajectories of Newtonian dynamical system (2.1). If $u^1, \ldots, u^{n-1}$ are local coordinates on S, then trajectories of dynamical system (2.1) beginning at the points of this hypersurface are expressed by functions

$$
\begin{aligned}
x^1 &= x^1(u^1, \ldots, u^{n-1}, t),\\
&\ldots\ldots\ldots\ldots\ldots\\
x^n &= x^n(u^1, \ldots, u^{n-1}, t).
\end{aligned} \tag{3.1}
$$

Parameter t in (3.1) is the main parameter, it has physical meaning of time, other parameters $u^1, \ldots, u^{n-1}$ are auxiliary ones. If we choose one of auxiliary parameters u^i and fix all others, then we get the situation considered in § 5 of Chapter IV. In this situation partial derivatives

$$
\frac{\partial x^1}{\partial u^i}, \ \ldots, \ \frac{\partial x^n}{\partial u^i} \tag{3.2}
$$

are the components of the vector of variation. We denote it by $\boldsymbol{\tau}_i$. On the other hand, these derivatives are the components of the vector $\mathbf{K}_i$ from (1.10) as well. Thus, on the trajectories of dynamical system (2.1) used in the construction of shift we have several vectors of variation simultaneously:

$$
\boldsymbol{\tau}_1 = \mathbf{K}_1, \ \ldots, \ \boldsymbol{\tau}_{n-1} = \mathbf{K}_{n-1}. \tag{3.3}
$$

From geometrical point of view vectors (3.3) are interpreted as coordinate tangent vectors on hypersurface S_t in local coordinates $u^1, \ldots, u^{n-1}$ induced to S_t from S by means of shift diffeomorphism $f_t\colon S \to S_t$.

Let's study time dependence of the vectors (3.3), i. e. their evolution as we move along trajectories of the system (2.1). Components of any of them appear to satisfy the same differential equations, which can be obtained by linearization of the equations (2.1). In deriving these equations we denote $u^i = u$ and $\mathbf{K}_i = \boldsymbol{\tau}_i = \boldsymbol{\tau}$ for to simplify further calculations. The equations (2.1) determine not only the functions $x^i(u^1, \ldots, u^{n-1}, t)$ in (3.1), but the functions

$$\begin{aligned} v^1 &= v^1(u^1, \ldots, u^{n-1}, t), \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ v^n &= v^n(u^1, \ldots, u^{n-1}, t) \end{aligned} \tag{3.4}$$

as well. Therefore we have natural lift of trajectories to the tangent bundle TM (see definition 2.1 in Chapter IV). Let's apply the operator ∇_u of covariant differentiation with respect to parameter u (see § 5 in Chapter IV) to both sides of second part of the equations (2.1). The result can be written as a vectorial equality

$$\nabla_u \nabla_t \mathbf{v} = \nabla_u(\mathbf{F}). \tag{3.5}$$

In order to transform left hand side of (3.5) we transpose covariant differentiations ∇_u and ∇_t by means of formula (5.14) from Chapter IV:

$$\nabla_u \nabla_t \mathbf{v} = [\nabla_u, \nabla_t]\mathbf{v} + \nabla_t \nabla_u \mathbf{v} = \mathbf{S}\mathbf{v} + \nabla_t \nabla_u \mathbf{v}. \tag{3.6}$$

Operator field $\mathbf{S}$ in (3.6) is determined by formula (5.1) from Chapter IV. Applying this formula, we take into account that dynamic curvature field $\mathbf{D}$ and torsion field T for metric connection (1.2) are zero. Therefore

$$\nabla_u \nabla_t \mathbf{v} = \nabla_t \nabla_u \mathbf{v} + \mathbf{R}(\boldsymbol{\tau}, \mathbf{v})\mathbf{v}. \tag{3.7}$$

In order to calculate $\nabla_u \mathbf{v}$ we use formula (5.13) from Chapter IV and remember that torsion is zero. Then from (3.7) we derive

$$\nabla_u \nabla_t \mathbf{v} = \nabla_{tt} \boldsymbol{\tau} + \mathbf{R}(\boldsymbol{\tau}, \mathbf{v})\mathbf{v}. \tag{3.8}$$

In order to transform right hand side of equality (3.5) we use formula (5.19) or formula (5.20) from Chapter IV and remember that $\mathbf{T} = 0$:

$$\nabla_u(\mathbf{F}) = C(\boldsymbol{\tau} \otimes \nabla \mathbf{F}) + C(\nabla_t \boldsymbol{\tau} \otimes \tilde{\nabla} \mathbf{F}). \tag{3.9}$$

Now, equating the expressions (3.8) and (3.9), we obtain vectorial differential equation for the vector of variation $\boldsymbol{\tau}$:

$$\nabla_{tt} \boldsymbol{\tau} = C(\nabla_t \boldsymbol{\tau} \otimes \tilde{\nabla} \mathbf{F}) + C(\boldsymbol{\tau} \otimes \nabla \mathbf{F}) - \mathbf{R}(\boldsymbol{\tau}, \mathbf{v})\mathbf{v}. \tag{3.10}$$

Written in terms of components in some local map, (3.10) looks like

$$\nabla_{tt}\tau^k = -\sum_{m=1}^{n}\sum_{i=1}^{n}\sum_{j=1}^{n} R^k_{mij}\,\tau^i\,v^j\,v^m + \sum_{m=1}^{n}\nabla_t\tau^m\,\tilde{\nabla}_m F^k + \sum_{m=1}^{n}\tau^m\,\nabla_m F^k. \tag{3.11}$$

From (3.11) we conclude that components of variation vector $\boldsymbol{\tau}$ satisfy the system of n linear ordinary equation of second order respective to time variable t. Coefficients in these linear equations are depending on t. They are determined by the force field $\mathbf{F}$ of dynamical system, and they depend on the choice of its particular trajectory, i. e. they depend on the functions (3.1) and (3.4).

§ 4. Function of deviation and its derivatives.

As a vector of variation $\boldsymbol{\tau}$ in the equations (3.10) we can choose any one of vectors $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_{n-1}$ from (3.3). In the case, when $f_t\colon\, S \to S_t$ is a normal shift, they all are perpendicular to the vector $\mathbf{v}(t)$, which is tangent to the trajectory. Therefore as a measure of deviation from normality we use the following scalar products:

$$\varphi_i = (\boldsymbol{\tau}_i\,|\,\mathbf{v}), \quad \text{where} \quad i = 1, \ldots, n-1. \tag{4.1}$$

In the situation of normal shift all these functions are identically zero for all trajectories of shift.

Definition 4.1. Scalar product of velocity vector $\mathbf{v}(t)$ with the vector of variation $\boldsymbol{\tau}(t)$ is called the function of **deviation** on the trajectory of Newtonian dynamical system.

Suppose that $\varphi(t)$ is a function of deviation on some particular trajectory of Newtonian dynamical system:

$$\varphi = (\mathbf{v}\,|\,\boldsymbol{\tau}). \tag{4.2}$$

Let's calculate two time derivatives of this function $\dot{\varphi}(t)$ and $\ddot{\varphi}(t)$. For the first derivative of the function (4.2) we have

$$\dot{\varphi} = \nabla_t\varphi = \nabla_t(\mathbf{v}\,|\,\boldsymbol{\tau}) = (\nabla_t\mathbf{v}\,|\,\boldsymbol{\tau}) + (\mathbf{v}\,|\,\nabla_t\boldsymbol{\tau}) \tag{4.3}$$

Differentiating scalar product $(\mathbf{v}\,|\,\boldsymbol{\tau})$ in (4.3), we used Leibniz rule. This is correct, since we can do the following more detailed calculations

$$\begin{aligned} \nabla_t(\mathbf{v}\,|\,\boldsymbol{\tau}) &= \nabla_t C(\mathbf{g}\otimes\mathbf{v}\otimes\boldsymbol{\tau}) = C(\nabla_t\mathbf{g}\otimes\mathbf{v}\otimes\boldsymbol{\tau}) + C(\mathbf{g}\otimes\nabla_t\mathbf{v}\otimes\boldsymbol{\tau}) + \\ &+ C(\mathbf{g}\otimes\mathbf{v}\otimes\nabla_t\boldsymbol{\tau}) = (\nabla_t\mathbf{v}\,|\,\boldsymbol{\tau}) + (\mathbf{v}\,|\,\nabla_t\boldsymbol{\tau}) + C(\nabla_t\mathbf{g}\otimes\mathbf{v}\otimes\boldsymbol{\tau}). \end{aligned} \tag{4.4}$$

Covariant derivative $\nabla_t\mathbf{g}$ in (4.4) is equal to zero. This follows from the formula

(4.3) in Chapter IV and the relationships (1.3) that express concordance of metric and connection. Indeed, we have

$$\nabla_t \mathbf{g} = C(\mathbf{v} \otimes \nabla \mathbf{g}) + C(\nabla_t \mathbf{v} \otimes \tilde{\nabla} \mathbf{g}) = 0. \tag{4.5}$$

For the further transformation of (4.3) we use the equations of dynamics (2.1), second part of which can be written in vectorial form:

$$\nabla_t \mathbf{v} = \mathbf{F}. \tag{4.6}$$

Substituting (4.6) into the formula (4.3), for the derivative $\dot{\varphi}$ we get

$$\dot{\varphi} = (\mathbf{F} \,|\, \boldsymbol{\tau}) + (\mathbf{v} \,|\, \nabla_t \boldsymbol{\tau}). \tag{4.7}$$

Now let's calculate second derivative $\ddot{\varphi}$. In order to do it let's differentiate (4.7), taking into account the equation (4.6) thereby:

$$\ddot{\varphi} = \nabla_t \dot{\varphi} = (\nabla_t \mathbf{F} \,|\, \boldsymbol{\tau}) + 2\,(\mathbf{F} \,|\, \nabla_t \boldsymbol{\tau}) + (\mathbf{v} \,|\, \nabla_{tt} \boldsymbol{\tau}). \tag{4.8}$$

For to calculate $\nabla_t \mathbf{F}$ we apply formula (4.3) from Chapter IV:

$$\nabla_t \mathbf{F} = C(\mathbf{v} \otimes \nabla \mathbf{F}) + C(\mathbf{F} \otimes \tilde{\nabla} \mathbf{F}). \tag{4.9}$$

Second derivative of variation vector $\nabla_{tt} \boldsymbol{\tau}$ is expressed through $\boldsymbol{\tau}$ and through first derivative $\nabla_t \boldsymbol{\tau}$ by means of the equation (3.10). If additionally we take into account (4.9), then for the derivative $\ddot{\varphi}$ we get

$$\begin{aligned} \ddot{\varphi} = {} & (C(\mathbf{v} \otimes \nabla \mathbf{F}) \,|\, \boldsymbol{\tau}) + (C(\mathbf{F} \otimes \tilde{\nabla} \mathbf{F}) \,|\, \boldsymbol{\tau}) + 2\,(\mathbf{F} \,|\, \nabla_t \boldsymbol{\tau}) + \\ & + (\mathbf{v} \,|\, C(\boldsymbol{\tau} \otimes \nabla \mathbf{F})) + (\mathbf{v} \,|\, C(\nabla_t \boldsymbol{\tau} \otimes \tilde{\nabla} \mathbf{F})) - (\mathbf{v} \,|\, \mathbf{R}(\boldsymbol{\tau}, \mathbf{v}) \mathbf{v}). \end{aligned} \tag{4.10}$$

Last term in (4.10) is equal to zero. This follows from some properties of curvature tensor of metric connection (1.2). Indeed, let's write the expression $(\mathbf{v} \,|\, \mathbf{R}(\boldsymbol{\tau}, \mathbf{v}) \mathbf{v})$ in local coordinates. Here we have

$$(\mathbf{v} \,|\, \mathbf{R}(\boldsymbol{\tau}, \mathbf{v}) \mathbf{v}) = \sum_{k=1}^{n} \sum_{m=1}^{n} \sum_{i=1}^{n} \sum_{j=1}^{n} R_{kmij}\, v^k\, \tau^i\, v^j\, v^m = 0. \tag{4.11}$$

Sum in (4.11) vanishes due to skew symmetry of R_{kmij} with respect to first two indices k and m (see [12], [32], [76], or [77]). Therefore

$$\begin{aligned} \ddot{\varphi} = {} & (C(\mathbf{v} \otimes \nabla \mathbf{F}) \,|\, \boldsymbol{\tau}) + (C(\mathbf{F} \otimes \tilde{\nabla} \mathbf{F}) \,|\, \boldsymbol{\tau}) + \\ & + 2\,(\mathbf{F} \,|\, \nabla_t \boldsymbol{\tau}) + (\mathbf{v} \,|\, C(\boldsymbol{\tau} \otimes \nabla \mathbf{F})) + (\mathbf{v} \,|\, C(\nabla_t \boldsymbol{\tau} \otimes \tilde{\nabla} \mathbf{F})). \end{aligned} \tag{4.12}$$

Let's write the equalities (4.2), (4.7), and (4.12) in local coordinates. For the function of deviation φ and for its first derivative $\dot{\varphi}$ we have

$$\varphi = \sum_{i=1}^{n} v_i\,\tau^i, \tag{4.13}$$

$$\dot{\varphi} = \sum_{i=1}^{n} F_i\,\tau^i + \sum_{i=1}^{n} v_i\,\nabla_t\tau^i \tag{4.14}$$

For the sake of brevity in (4.13) and (4.14) we used covariant components of vectors $\mathbf{v}$ and $\mathbf{F}$, which are obtained by lowering the indices:

$$v_i = \sum_{j=1}^{n} g_{ij}\,v^j, \qquad\qquad F_i = \sum_{j=1}^{n} g_{ij}\,F^j. \tag{4.15}$$

Procedure of lowering indices (4.15) is commutating with various covariant differentiations due to the concordance of metric and connection expressed by (1.3) (see more details in [77]). This procedure is invertible, therefore we can make almost no difference between covariant and contravariant components of tensorial objects in Riemannian geometry.

Now let's write in local coordinates the expression (4.12) for the second derivative of the function of variation:

$$\begin{aligned}\ddot{\varphi} = \sum_{i=1}^{n}\left(2\,F_i + \sum_{j=1}^{n} v^j\,\tilde{\nabla}_i F_j\right)\nabla_t\tau^i\,+\\ +\sum_{i=1}^{n}\left(\sum_{j=1}^{n} v^j\,(\nabla_j F_i + \nabla_i F_j) + \sum_{j=1}^{n} F^j\,\tilde{\nabla}_j F_i\right)\tau^i.\end{aligned} \tag{4.16}$$

In this formula (4.16) we use covariant and contravariant components of force field $\mathbf{F}$ simultaneously.

§ 5. Projectors defined by the vector of velocity.

Function of deviation φ depends only on orthogonal projection of variation vector $\boldsymbol{\tau}$ to the direction of velocity vector. This follows from (4.13) or from the initial formula (4.2) for the function φ. Similarly, first derivative $\dot{\varphi}$ also depends only on projection of variation vector $\boldsymbol{\tau}$ to the direction of velocity vector. In what follows it will be important to subdivide all entries of $\boldsymbol{\tau}$ and $\nabla_t\boldsymbol{\tau}$ in formulas for φ, $\dot{\varphi}$, and $\ddot{\varphi}$ into two parts, one directed along the velocity vector $\mathbf{v}$, and the second directed perpendicular to $\mathbf{v}$. With this aim in mind we define a few auxiliary tensor fields from extended algebra $\mathbf{T}(M)$.

Scalar field of modulus of velocity vector $v = |\mathbf{v}|$ is an extended tensor field of type $(0, 0)$. Its domain is the whole tangent bundle TM, but it is smooth only at that points $q=(p,\mathbf{v})$, where $\mathbf{v}\neq 0$. In local coordinates for the field v we have

$$v=\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}g_{ij}\,v^i\,v^j}. \tag{5.1}$$

Denote by $\mathbf{N}$ the field of unit vectors directed along the velocity vector $\mathbf{v}$. Then

$$\mathbf{N}=\frac{\mathbf{v}}{v}. \tag{5.2}$$

From (5.2) we see that $\mathbf{N}$ is an extended vector field, the domain of which is the whole tangent bundle TM except for those points $q=(p,\mathbf{v})$, where $\mathbf{v}=0$. The same condition $\mathbf{v}\neq 0$ restricts domain of the following two extended operator fields: $\mathbf{Q}$ and $\mathbf{P}$. $\mathbf{Q}$ is an operator of orthogonal projection to the direction of the vector $\mathbf{v}$. Operator $\mathbf{P}$ is a projector onto the hyperplane perpendicular to $\mathbf{v}$. Then

$$\mathbf{P}+\mathbf{Q}=\mathbf{1}. \tag{5.3}$$

Here $\mathbf{1}$ is the field of identical operators. In local coordinates operator fields $\mathbf{P}$ and $\mathbf{Q}$ have the following components:

$$P^i_j=\delta^i_j-N^i\,N_j, \qquad\qquad Q^i_j=N^i\,N_j. \tag{5.4}$$

Here N^i and N_j are the covariant and contravariant components of the field (5.2).

From (5.1), (5.2), and (5.4) one can easily derive the formulas for differentiating all the above defined auxiliary fields. Let's write these formulas in local coordinates. In case of vector field of velocity $\mathbf{v}$ and scalar field $v=|\mathbf{v}|$ we have

$$\nabla_k v^i=0, \qquad\qquad \tilde{\nabla}_k v^i=\delta^i_k, \tag{5.5}$$

$$\nabla_k v=0, \qquad\qquad \tilde{\nabla}_k v=N_k. \tag{5.6}$$

In case of vector field $\mathbf{N}$ from the relationships (5.5) and (5.6) we derive

$$\nabla_k N^i=0, \qquad\qquad \tilde{\nabla}_k N^i=\frac{1}{v}\,P^k_i. \tag{5.7}$$

Spatial gradients of projector fields $\mathbf{P}$ and $\mathbf{Q}$ are identically zero:

$$\nabla_k P^i_j=0, \qquad\qquad \nabla_k Q^i_j=0. \tag{5.8}$$

For velocity gradients of these two fields we have the formulas:

$$\begin{aligned}\tilde{\nabla}_k P^i_j&=-\frac{N_j\,P^i_k}{v}-\sum_{r=1}^{n}\frac{g_{jr}\,P^r_k\,N^i}{v},\\ \tilde{\nabla}_k Q^i_j&=\frac{N_j\,P^i_k}{v}+\sum_{r=1}^{n}\frac{g_{jr}\,P^r_k\,N^i}{v}.\end{aligned} \tag{5.9}$$

Spatial gradients are identically zero not only for the fields $\mathbf{P}$ and $\mathbf{Q}$; for the fields v and $\mathbf{N}$ they vanish as well. This reflects the fact that they all are produced by the velocity field $\mathbf{v}$ and Riemannian metric field $\mathbf{g}$, which satisfy the relationships (1.3). Derivation of formulas (5.5), (5.6), (5.7), (5.8), and (5.9) is rather simple. It is based on formulas (7.3) and (7.4) from Chapter III. We shall not give it here.

Let's use the relationship (5.3) in order to transform first summand in formula (4.14) for the derivative $\dot{\varphi}$. In local coordinates (5.3) is written as $\delta^i_j = P^i_j + Q^i_j$. We can insert δ^i_j into (4.14) and add one more summation there:

$$\dot{\varphi} = \sum_{i=1}^{n}\sum_{j=1}^{n} F_i\,\delta^i_j\tau^j + \sum_{i=1}^{n} v_i\,\nabla_t\tau^i \tag{5.10}$$

Now, substituting $P^i_j + Q^i_j$ for δ^i_j in (5.10), we get the formula

$$\dot{\varphi} = \sum_{i=1}^{n}\sum_{j=1}^{n} F_i\,P^i_j\tau^j + \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{F_i\,N^i}{v}\,v_j\tau^j + \sum_{i=1}^{n} v_i\,\nabla_t\tau^i. \tag{5.11}$$

Formula (5.11) can be written in coordinate-free form as follows:

$$\dot{\varphi} = (\mathbf{F}\,|\,\mathbf{P}\,\boldsymbol{\tau}) + \frac{(\mathbf{F}\,|\,\mathbf{N})}{v}\,(\mathbf{v}\,|\,\boldsymbol{\tau}) + (\mathbf{v}\,|\,\nabla_t\boldsymbol{\tau}). \tag{5.12}$$

Last two terms in formula (5.12) contain scalar products of $\boldsymbol{\tau}$ and $\nabla_t\boldsymbol{\tau}$ with the vector of velocity, while the first term does not depend on the projection of these two vectors to the direction of $\mathbf{v}$.

Now let's perform some similar transformations with the formula (4.16) for the second derivative $\ddot{\varphi}$. Here it's convenient to introduce auxiliary covectorial fields $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ with the following components

$$\begin{aligned} \alpha_i &= 2\,F_i + \sum_{j=1}^{n} v^j\,\tilde{\nabla}_i F_j,\\ \beta_i &= \sum_{j=1}^{n} v^j\,(\nabla_j F_i + \nabla_i F_j) + \sum_{j=1}^{n} F^j\,\tilde{\nabla}_j F_i. \end{aligned} \tag{5.13}$$

Then for the second derivative of the function of deviation we get

$$\ddot{\varphi} = \boldsymbol{\alpha}(\mathbf{P}\nabla_t\boldsymbol{\tau}) + \boldsymbol{\beta}(\mathbf{P}\boldsymbol{\tau}) + \frac{\boldsymbol{\alpha}(\mathbf{N})}{v}\,(\mathbf{v}\,|\,\nabla_t\boldsymbol{\tau}) + \frac{\boldsymbol{\beta}(\mathbf{N})}{v}\,(\mathbf{v}\,|\,\boldsymbol{\tau}). \tag{5.14}$$

Last two terms in (5.14) contain scalar product of $\boldsymbol{\tau}$ and $\nabla_t\boldsymbol{\tau}$ with the vector of velocity $\mathbf{v}$. First two terms do not depend oh the projections of $\boldsymbol{\tau}$ and $\nabla_t\boldsymbol{\tau}$ to the direction of the vector $\mathbf{v}$.

§ 6. The condition of weak normality.

Let's look on the equations (3.11) and on the formulas (4.13), (4.14), and (4.16) from the point of view of differential equations theorist. When we fix some trajectory of dynamical system (2.1), the equations (3.11) form the system of linear ordinary differential equations, overall order of which is equal to $2n$. Denote by $\mathfrak{T}$ the space of solutions of this system of equations. Then $\dim\mathfrak{T} = 2n$. As the coordinates in this space we can use components of two vectors $\boldsymbol{\tau}(t)$ and $\nabla_t\boldsymbol{\tau}(t)$ taken for some fixed value of parameter $t = t_0$. Comparing formulas (4.13), (4.14), and (4.16), we see that the function of deviation φ and its derivatives $\dot\varphi$ and $\ddot\varphi$ depend linearly on $\boldsymbol{\tau}$ and $\nabla_t\boldsymbol{\tau}$. This is true for all time derivatives of the function of deviation, it is proved by applying repeatedly ∇_t to the equality (4.12). So the value of φ by itself, and values of all its derivatives (corresponding to some fixed value of $t = t_0$)

$$\varphi(t_0),\ \dot\varphi(t_0),\ \ddot\varphi(t_0),\ \varphi^{(3)}(t_0),\ \ldots,\ \varphi^{(2n)}(t_0), \tag{6.1}$$

can be considered as **linear functionals** on the space $\mathfrak{T}$. The number of functionals (6.1) is equal to $2n+1$, it is one as more than the dimension of dual space $\mathfrak{T}^*$. Therefore these functionals are linearly dependent. Hence

$$\sum_{i=0}^{2n} C_i(t_0)\,\varphi^{(i)}(t_0) = 0. \tag{6.2}$$

Coefficients of linear combination (6.2) certainly depend on t_0. For each particular n they can be calculated explicitly. These considerations lead to the following theorem.

Theorem 6.1. *Suppose that Newtonian dynamical system on n-dimensional Riemannian manifold M is given. For any its trajectory and for any choice of variation vector $\boldsymbol{\tau}$ on this trajectory corresponding function of deviation $\varphi(t)$ satisfies some linear homogeneous ordinary differential equation of the order not greater than $2n$.*

Definition 6.1. We say that Newtonian dynamical system on Riemannian manifold M of the dimension $n \geqslant 2$ satisfies **weak normality** condition, if for each its trajectory there exists some ordinary differential equation

$$\ddot\varphi = \mathcal{A}(t)\,\dot\varphi + \mathcal{B}(t)\,\varphi \tag{6.3}$$

such that any function of deviation $\varphi(t)$ corresponding to any choice of variation vector $\boldsymbol{\tau}$ on that trajectory is the solution of this equation.

Definition 6.1 is one of the main definitions in the theory of dynamical systems admitting the normal shift. The definition of this kind was first stated in [38] (see also preprint [34]). It was found that weak normality condition leads to the system of partial differential equations for the force field $\mathbf{F}$ of Newtonian dynamical system. Derivation of these equations is our main purpose in this section.

Thus suppose, that Newtonian dynamical system with force field $\mathbf{F}$ satisfies weak normality condition. Let's fix some trajectory $p(t)$ and write corresponding differential equation (6.2) in the following form:

$$\ddot\varphi - \mathcal{A}\,\dot\varphi - \mathcal{B}\,\varphi = 0. \tag{6.4}$$

Let $p = p(t_0)$ be a point on the fixed trajectory, where velocity vector doesn't vanish, i. e. $\mathbf{v}(t_0) \neq 0$. In left hand side of (6.3) we substitute the expression (4.2) for the function of deviation φ, and substitute the above expressions (5.12) and (5.14) for its derivatives. After some reductions we get

$$\begin{gathered}\left(\frac{\boldsymbol{\alpha}(\mathbf{N})}{v} - \mathcal{A}\right)(\mathbf{v}\,|\,\nabla_t\boldsymbol{\tau}) + \boldsymbol{\alpha}(\mathbf{P}\nabla_t\boldsymbol{\tau}) + \boldsymbol{\beta}(\mathbf{P}\boldsymbol{\tau}) - \\ - \mathcal{A}\,(\mathbf{F}\,|\,\mathbf{P}\,\boldsymbol{\tau}) + \left(\frac{\boldsymbol{\beta}(\mathbf{N})}{v} - \mathcal{A}\,\frac{(\mathbf{F}\,|\,\mathbf{N})}{v} - \mathcal{B}\right)(\mathbf{v}\,|\,\boldsymbol{\tau}) = 0.\end{gathered} \tag{6.5}$$

Let's consider the value of left hand side of (6.5) for $t = t_0$. After fixing $t = t_0$ we have only one arbitrariness rest, it consists in the choice of solution of the equation (3.11). This choice determines the value of $\boldsymbol{\tau}(t_0)$ and the value of its derivative $\nabla_t\boldsymbol{\tau}(t_0)$. Left hand side of (6.5) is linear in $\boldsymbol{\tau}$ and $\nabla_t\boldsymbol{\tau}$. Therefore, considered for some fixed $t = t_0$, (6.5) is an equality of linear functionals in $\mathfrak{T}^*$. The space of solutions of the equations (3.11) on a given trajectory $p(t)$ is isomorphic to direct sum of two copies of tangent space $T_p(M)$ at the point $p = p(t_0)$:

$$\mathfrak{T} \cong T_p(M) \oplus T_p(M). \tag{6.6}$$

This is true, since $\mathfrak{T}$ is parameterized by components of two vectors $\boldsymbol{\tau}(t_0) \in T_p(M)$ and $\nabla_t\boldsymbol{\tau}(t_0) \in T_p(M)$. Velocity vector $\mathbf{v}(t_0) \in T_p(M)$ subdivides each summand in formula (6.6) into direct sum of two subspaces:

$$T_p(M) = \langle\mathbf{v}\rangle \oplus \langle\mathbf{v}\rangle_{\perp}. \tag{6.7}$$

Here subspace $\langle\mathbf{v}\rangle$ is a linear span of velocity vector $\mathbf{v}(t_0) \neq 0$, and $\langle\mathbf{v}\rangle_{\perp}$ is an orthogonal complement of $\langle\mathbf{v}\rangle$ to a whole space $T_p(M)$ in the metric g. From (6.6) and (6.7) we derive the expansion of $\mathfrak{T}$ into a direct sum of four subspaces:

$$\mathfrak{T} \cong \langle\mathbf{v}\rangle \oplus \langle\mathbf{v}\rangle_{\perp} \oplus \langle\mathbf{v}\rangle \oplus \langle\mathbf{v}\rangle_{\perp}. \tag{6.8}$$

The expansion (6.8) generates dual expansion in dual space $\mathfrak{T}^*$:

$$\mathfrak{T}^* \cong \langle\mathbf{v}\rangle^* \oplus \langle\mathbf{v}\rangle^*_{\perp} \oplus \langle\mathbf{v}\rangle^* \oplus \langle\mathbf{v}\rangle^*_{\perp}. \tag{6.9}$$

Functionals $(\mathbf{v}\,|\,\nabla_t\boldsymbol{\tau})$, $\boldsymbol{\alpha}(\mathbf{P}\nabla_t\boldsymbol{\tau})$, $\boldsymbol{\beta}(\mathbf{P}\boldsymbol{\tau}) - \mathcal{A}\,(\mathbf{F}\,|\,\mathbf{P}\boldsymbol{\tau})$ and $(\mathbf{v}\,|\,\boldsymbol{\tau})$ in (6.5) belong to four different subspaces in the expansion (6.9). Therefore the equation (6.5) is reduced to four separate equations. Two of them determine $\mathcal{A}$ and $\mathcal{B}$ in (6.4):

$$\mathcal{A} = \frac{\boldsymbol{\alpha}(\mathbf{N})}{v}, \qquad\qquad \mathcal{B} = \frac{\boldsymbol{\beta}(\mathbf{N})}{v} - \mathcal{A}\,\frac{(\mathbf{F}\,|\,\mathbf{N})}{v}. \tag{6.10}$$

Second pair of equation derived from (6.5) has the following form:

$$\boldsymbol{\alpha}(\mathbf{P}\nabla_t\boldsymbol{\tau}) = 0, \qquad\qquad \boldsymbol{\beta}(\mathbf{P}\boldsymbol{\tau}) = \mathcal{A}\,(\mathbf{F}\,|\,\mathbf{P}\,\boldsymbol{\tau}). \tag{6.11}$$

Let's calculate coefficients $\mathcal{A}$ and $\mathcal{B}$ in the equation (6.3) by means of above formulas (6.10) and by means of formulas (5.13) for the components of convectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$:

$$\mathcal{A}=\sum_{i=1}^{n}\frac{2\,F_i\,N^i}{v}+\sum_{i=1}^{n}\sum_{j=1}^{n}N^i\,N^j\,\tilde{\nabla}_iF_j, \tag{6.12}$$

$$\begin{aligned}\mathcal{B}=&\sum_{i=1}^{n}\sum_{j=1}^{n}\left(N^j\,(\nabla_jF_i+\nabla_iF_j)+\frac{F^j\,\tilde{\nabla}_jF_i}{v}\right)N^i-\\&-\sum_{i=1}^{n}\sum_{k=1}^{n}\left(\frac{2\,F_i\,N^i}{v}+\sum_{j=1}^{n}N^i\,N^j\,\tilde{\nabla}_iF_j\right)\frac{F_k\,N^k}{v}.\end{aligned} \tag{6.13}$$

Now consider the equations (6.11). Using (5.13), one can write them in local coordinates. Afterwards one should take into account arbitrariness of vectors $\boldsymbol{\tau}$ and $\nabla_t\boldsymbol{\tau}$. As a result first equation (6.11) is written in the following form:

$$\sum_{i=1}^{n}\left(2\,F_i+\sum_{j=1}^{n}v^j\,\tilde{\nabla}_iF_j\right)P^i_k=0. \tag{6.14}$$

In order to transform another equation (6.11) we use formula (6.12) for the coefficient $\mathcal{A}$ in the equation (6.3). This yields

$$\begin{aligned}&\sum_{i=1}^{n}\left(\sum_{j=1}^{n}v^j\,(\nabla_jF_i+\nabla_iF_j)+\sum_{j=1}^{n}F^j\,\tilde{\nabla}_jF_i\right)P^i_k=\\&=\sum_{r=1}^{n}\left(\sum_{i=1}^{n}\frac{2\,F_i\,N^i}{v}+\sum_{i=1}^{n}\sum_{j=1}^{n}N^i\,N^j\,\tilde{\nabla}_iF_j\right)F_r\,P^r_k.\end{aligned} \tag{6.15}$$

All done. We are only to do some slight cosmetic transformations in the equation (6.14) and (6.15). The equation (6.14) is written as

$$\sum_{i=1}^{n}\left(v^{-1}\,F_i+\sum_{j=1}^{n}\tilde{\nabla}_i\left(N^j\,F_j\right)\right)P^i_k=0. \tag{6.16}$$

In deriving (6.16) we took into account $v^j=v\,N^j$, and used second relationship (5.7) together with well-known property $\mathbf{P}^2=\mathbf{P}$ of projector $\mathbf{P}$. In equation (6.15) we shall only transpose some terms:

$$\begin{aligned}&\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\nabla_iF_j+\nabla_jF_i-2\,v^{-1}\,F_i\,F_j\right)N^j\,P^i_k+\\&+\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{F^j\,\tilde{\nabla}_jF_i}{v}-\sum_{r=1}^{n}\frac{N^r\,N^j\,\tilde{\nabla}_jF_r}{v}\,F_i\right)P^i_k=0.\end{aligned} \tag{6.17}$$

The equations (6.16) and (6.17), which we have derived just now, are called **weak normality** equations. In this form they were first obtained in [58] (see also [75]). The above derivation of these equations proves the following theorem.

THEOREM 6.2. *Newtonian dynamical system on Riemannian manifold of the dimension* $n \geqslant 2$ *satisfies weak normality condition if and only if its force field satisfies the differential equations* (6.16) *and* (6.17) *at all points* $q = (p, \mathbf{v})$ *of tangent bundle* TM*, except for those, where* $\mathbf{v} = 0$*.*

The equations of weak normality (6.16) and (6.17) form the system of partial differential equations for the force field $\mathbf{F}$. The number of these equations is $2n$. However, projector matrix P^i_k is degenerate, its rank is $n - 1$. Therefore actual number of independent equations in (6.16) and (6.17) is $2n - 2$. The problem of their compatibility is studied in next two Chapters, where we construct great many of their solutions. Here we note only that identically zero force field $\mathbf{F}$ is the solution of the equations (6.16) and (6.17). This proves the following theorem.

THEOREM 6.3. *Geodesic flow on Riemannian manifold* M *of the dimension* $n \geqslant 2$ *is a Newtonian dynamical system satisfying weak normality condition.*

§ 7. Cauchy problem for the function of deviation.

Suppose that M is a Riemannian manifold with Newtonian dynamical system satisfying weak normality condition on it. Let's use this system to define the shift $f_t\colon S \to S_t$ for some hypersurface S from transformation class on M (see definition 2.1). Functions of deviation on the trajectories of shift satisfy the differential equations (6.3). These are linear homogeneous differential equations of the second order:

$$\ddot{\varphi} = \mathcal{A}(t)\,\dot{\varphi} + \mathcal{B}(t)\,\varphi. \tag{7.1}$$

Coefficients $\mathcal{A}(t)$ and $\mathcal{B}(t)$ can be different for different trajectories of shift. Upon choosing local coordinates $u^1, \ldots, u^{n-1}$ on S and local coordinates $x^1, \ldots, x^n$ in M, we can define shift functions (3.1). Their derivatives determine vectors of variation (3.3), which coincide with coordinate tangent vectors to hypersurfaces S_t:

$$\boldsymbol{\tau}_i = \sum_{j=1}^{n} \frac{\partial x^j}{\partial u^i}\,\frac{\partial}{\partial x^j}, \text{ where } i = 1, \ldots, n-1. \tag{7.2}$$

By differentiating shift functions (3.1) with respect to time variable t for the fixed values of $u^1, \ldots, u^{n-1}$ we obtain the velocity vector on the trajectories of shift:

$$\mathbf{v} = \sum_{j=1}^{n} \dot{x}^j\,\frac{\partial}{\partial x^j}. \tag{7.3}$$

Scalar products of velocity vector $\mathbf{v}$ from (7.3) with variation vectors $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_{n-1}$ from (7.2) define the set functions of deviations

$$\varphi_1, \ \ldots, \ \varphi_{n-1}. \tag{7.4}$$

This is in complete agreement with formulas (4.1). Each function (7.4) satisfies the equations (7.1) for fixed values of variables $u^1, \ldots, u^{n-1}$. Normality condition for the shift $f_t\colon\, S \to S_t$ consists in identical vanishing of all functions (7.4) (see definition 2.3). In order to satisfy this condition it is sufficient to provide vanishing of functions φ_k and their first derivatives $\dot\varphi_k$ at the initial instant of time $t = 0$, i. e. at the points of initial hypersurface S. Thus, we are to set up the following Cauchy problem for differential equations (7.1) on trajectories of shift:

$$\varphi_k(t)\Bigr|_{t=0} = 0, \qquad\qquad \dot\varphi_k(t)\Bigr|_{t=0} = 0. \tag{7.5}$$

First part of initial conditions (7.5) appears to be fulfilled unconditionally due to the initial data (2.2) that define trajectories of shift $f_t\colon\, S \to S_t$. Indeed, according to (2.2) vector of initial velocity $\mathbf{v}(0)$ is directed along the normal vector of S. Therefore it is perpendicular to the vectors $\boldsymbol\tau_1(0), \ldots, \boldsymbol\tau_{n-1}(0)$, which are tangent to S.

Let's study in more details the second pert of initial conditions (7.5). In order to do it we use formula (4.7). Here this formula is written as

$$\dot\varphi_k = (\mathbf{F}\,|\,\boldsymbol\tau_k) + (\mathbf{v}\,|\,\nabla_t\boldsymbol\tau_k). \tag{7.6}$$

In order to calculate vector $\nabla_t\boldsymbol\tau_k$ in (7.6) we apply formula (5.13) from Chapter IV and take unto account that here $u = u^k$. Tensor field of torsion $\mathbf{T}$ for Riemannian connection (1.2) is equal to zero, therefore

$$\nabla_t\boldsymbol\tau_k = \nabla_{u^k}\mathbf{v}. \tag{7.7}$$

Let's write the equality (7.7) for $t = 0$, taking into account initial data (2.2). In order to calculate covariant derivative $\nabla_{u^k}\mathbf{u}$ in right hand side of (7.7) one should only know the values of velocity vector on initial hypersurface S. This is clear, for instance, due to formula (5.8) from Chapter IV. Then

$$\nabla_t\boldsymbol\tau_k\Bigr|_{t=0} = \nabla_{u^k}(\nu\cdot\mathbf{n}) = \frac{\partial\nu}{\partial u^k}\cdot\mathbf{n} + \nu\cdot\nabla_{u^k}\mathbf{n}. \tag{7.8}$$

Covariant derivatives $\nabla_{u^k}\mathbf{n}$ are natural in the theory of hypersurfaces embedded into Riemannian manifold. They are encountered in derivational formulas of Weingarten (see [16], [32], [84], or [77]). In our case for the hypersurface S in M derivational formulas of Weingarten are written as follows:

$$\nabla_{u^k}\boldsymbol\tau_r = \sum_{m=1}^{n-1}\theta^m_{kr}\,\boldsymbol\tau_m + b_{kr}\,\mathbf{n}, \tag{7.9}$$

$$\nabla_{u^k}\mathbf{n} = -\sum_{m=1}^{n-1} b^m_k\,\boldsymbol\tau_m. \tag{7.10}$$

If we denote by ρ_{ij} components of metric tensor $\boldsymbol\rho$ for induced Riemannian metric $\boldsymbol\rho$ in local coordinates $u^1, \ldots, u^{n-1}$ on S, then quantities θ^m_{kr} in (7.9) are the components of corresponding metric connection for the metric $\boldsymbol{\rho}$. They are calculated by formula analogous to formula (1.2) for Γ^k_{ij}:

$$\theta^k_{ij} = \frac{1}{2}\sum_{s=1}^{n-1}\rho^{ks}\left(\frac{\partial\rho_{sj}}{\partial u^i} + \frac{\partial\rho_{is}}{\partial u^j} - \frac{\partial\rho_{ij}}{\partial u^s}\right). \tag{7.11}$$

Quantities b_{kr} in (7.9) are the components of inner tensor field $\mathbf{b}$ of type $(0, 2)$ in S which is called **second fundamental form** of hypersurface S. They are symmetric, i. e. they do not change in transposing indices:

$$b_{kr} = b_{rk}. \tag{7.12}$$

Quantities b^m_k in formula (7.10) are derived from b_{kr} by index raising procedure:

$$b^m_k = \sum_{r=1}^{n-1} b_{kr}\,\rho^{rm}. \tag{7.13}$$

They are the components of inner tensor field of type $(1, 1)$ in submanifold S. Such field is an operator version of second fundamental form $\mathbf{b}$ in S.

Let's apply derivational formula of Weingarten (7.10) in order to express covariant derivative $\nabla_{u^k}\mathbf{n}$ in formula (7.8). This yields

$$\nabla_t\boldsymbol{\tau}_k\Big|_{t=0} = \frac{\partial\nu}{\partial u^k}\cdot\mathbf{n} - \nu\cdot\sum_{m=1}^{n-1} b^m_k\,\boldsymbol{\tau}_m. \tag{7.14}$$

Now we substitute (7.14) into (7.6) and calculate derivative $\dot\varphi_k$ for initial instant of time $t = t_0$ on initial hypersurface S:

$$\dot\varphi_k\Big|_{t=0} = (\mathbf{F}\,|\,\boldsymbol{\tau}_k) + \nu\,\frac{\partial\nu}{\partial u^k}. \tag{7.15}$$

In deriving (7.15) we took into account second part of the relationships (2.2) that determine initial value of velocity vector on S.

Let's substitute the above formula (7.15) expressing initial value of derivative $\dot\varphi$ into the equations (7.5). This yields the equations

$$\frac{\partial\nu}{\partial u^k} = -\nu^{-1}\,(\mathbf{F}\,|\,\boldsymbol{\tau}_k) \tag{7.16}$$

for the function $\nu(u^1, \ldots, u^{n-1})$. If $\nu(u^1, \ldots, u^{n-1})$ is the solution of equations (7.16), then for all function of deviation (7.4) initial conditions (7.5) are fulfilled. In the case of dynamical system satisfying weak normality condition (see definition 6.1) it is sufficient to assure the normality of shift $f_t\colon S \to S_t$.

§ 8. Pfaff equations for the modulus of initial velocity.

The equations (7.16) determine two essential different cases: **two dimensional case**, when $\dim M = n = 2$, when we have only one equation (7.16), and **multidimensional case**, when $\dim M = n \geqslant 3$, when the number of equations is greater than 1. We intentionally omit here two dimensional case, since it is considered in **other thesis** [36].

Let's consider the equations (7.16) in multidimensional case. All quantities in right hand side of thetag7.16 are localized on initial hypersurface S. Force vector $\mathbf{F}$ depends on the point of tangent bundle $q = (p, \mathbf{v})$. Substituting $\mathbf{F}$ into (7.16) we choose $p \in S$ and $\mathbf{v} = \nu \cdot \mathbf{n}(p)$ in accordance with initial data (2.2). In local coordinates $u^1, \ldots, u^{n-1}$ the point $p \in S$ and coordinate tangent vectors $\boldsymbol{\tau}_k$ are parameterized by these coordinates: $p = p(u^1, \ldots, u^{n-1})$ and $\boldsymbol{\tau}_k = \boldsymbol{\tau}_k(u^1, \ldots, u^{n-1})$. Therefore the equations (7.16) has the following structure:

$$\frac{\partial \nu}{\partial u^k} = \psi_k(\nu, u^1, \ldots, u^{n-1}), \text{ where } k = 1, \ldots, n-1. \tag{8.1}$$

For $n \geqslant 3$ the equations (8.1) form **complete system of Pfaff equations**. Central point of the theory of Pfaff equations is the **problem of compatibility**. Suppose that $\nu = \nu(u^1, \ldots, u^{n-1})$ is a solution of the equations (8.1). Let's calculate second order partial derivative ν with respect to variables u^k and u^r by virtue of the equations (8.1). This can be done in two ways:

$$\frac{\partial^2 \nu}{\partial u^k\,\partial u^r} = \frac{\partial \psi_k}{\partial u^r} + \frac{\partial \psi_k}{\partial \nu}\,\psi_r = \vartheta_{kr}(\nu, u^1, \ldots, u^{n-1}), \tag{8.2}$$

$$\frac{\partial^2 \nu}{\partial u^r\,\partial u^k} = \frac{\partial \psi_r}{\partial u^k} + \frac{\partial \psi_r}{\partial \nu}\,\psi_k = \vartheta_{rk}(\nu, u^1, \ldots, u^{n-1}). \tag{8.3}$$

Functions $\vartheta_{kr}(\nu, u^1, \ldots, u^{n-1})$ and $\vartheta_{kr}(\nu, u^1, \ldots, u^{n-1})$ in right hand sides of (8.2) and (8.3) can be calculated regardless to whether has the system of Pfaff equations thetag8.1 some solution or it hasn't. In the case, when it has, the derivatives (8.2) and (8.3) are equal to each other. Equating (8.2) and (8.3), we get

$$\begin{aligned} &\vartheta_{kr}(\nu(u^1, \ldots, u^{n-1}), u^1, \ldots, u^{n-1}) = \\ &= \vartheta_{rk}(\nu(u^1, \ldots, u^{n-1}), u^1, \ldots, u^{n-1}). \end{aligned} \tag{8.4}$$

In general, the equality (8.4) doesn't provide identical coincidence of the functions ϑ_{kr} and ϑ_{rk}. In order to test whether it is fulfilled or not we are to have some solution $\nu(u^1, \ldots, u^{n-1})$ of the equations (8.1). But there is some situation when we can escape this difficulty.

Definition 8.1. System of Pfaff equations (8.1) is called **compatible**, if for any fixed point $(\nu_0,\, u_0^1,\, \ldots,\, u_0^{n-1})$ from the domain of the functions ψ_k there is a solution $\nu = \nu(u^1, \ldots, u^{n-1})$ of the equations (8.1) defined in some neighborhood of the point $(u_0^1, \ldots, u_0^{n-1})$ and normalized by the condition

$$\nu(u_0^1, \ldots, u_0^{n-1}) = \nu_0. \tag{8.5}$$

If the system of Pfaff equations (8.1) is compatible, then from (8.5) and (8.4) we get that functions ϑ_{kr} and ϑ_{rk} do identically coincide

$$\vartheta_{kr}(\nu, u^1, \ldots, u^{n-1}) = \vartheta_{rk}(\nu, u^1, \ldots, u^{n-1}). \tag{8.6}$$

The relationships (8.6) are called **compatibility conditions** for complete system of Pfaff equations (8.1). Let's write them in more explicit form:

$$\frac{\partial \psi_k}{\partial u^r} + \frac{\partial \psi_k}{\partial \nu}\,\psi_r = \frac{\partial \psi_r}{\partial u^k} + \frac{\partial \psi_r}{\partial \nu}\,\psi_k. \tag{8.7}$$

THEOREM 8.1. *Compatibility conditions expressed by relationships* (8.7)*, where* $k = 1, \ldots, n-1$ *and* $r = 1, \ldots, n-1$*, are necessary and sufficient for the system of Pfaff equations* (8.1) *to be compatible in the sense of definition 8.1.*

PROOF. The necessity was proved by derivations of relationships (8.7). We are to prove sufficiency. Let's do it by induction in n. For $n = 2$ we have one equation (8.1) which is ordinary differential equation with independent variable u^1. Normalization (8.5) is a Cauchy problem for this equation. Compatibility conditions (8.7) in this case are identically fulfilled, since $k = r = 1$. Thus, for $n = 2$ the proposition of theorem follows from local solvability of Cauchy problem for ordinary differential equation (see "existence and uniqueness" theorem in [82] or [83]). This is the base of induction.

Let's do the step of induction from $n-1$ to n. If we exclude the last equation from the system (8.1) and if we fix the last variable $u^{n-1} = u_0^{n-1}$, then we get into the situation when inductive hypothesis is applicable. This yields the function

$$\nu = \nu(u^1, \ldots, u^{n-2}) \tag{8.8}$$

that satisfies the system of first $(n-2)$ Pfaff equations in (8.1), and that satisfies normalizing condition (8.5) on the hyperplane $u^{n-1} = u_0^{n-1}$. Now let's consider again the last equation in (8.1), treating it as an ordinary differential equation with independent variable u^{n-1}, and let's set up the Cauchy problem

$$\nu\Big|_{u^{n-1}=u_0^{n-1}} = \nu(u^1, \ldots, u^{n-2}), \tag{8.9}$$

using the function (8.8) as initial data. We get the required function $\nu(u^1, \ldots, u^{n-1})$ as a solution of the above Cauchy problem with initial data (8.9).

Function $\nu(u^1, \ldots, u^{n-1})$ satisfies the last equation in the system (8.1) by construction. Moreover, it satisfies other equations (8.1) on the hyperplane $u^{n-1} = u_0^{n-1}$ by inductive hypothesis. Suppose that $1 \leqslant k \leqslant n-2$. Let's show that k-th equation

(8.1) is fulfilled for all values of arguments in $\nu(u^1,\ldots,u^{n-1})$ where this function is defined. Consider the following functions:

$$x_k(u^1,\ldots,u^{n-1})=\frac{\partial\nu}{\partial u^k},\tag{8.10}$$

$$y_k(u^1,\ldots,u^{n-1})=\psi_k(\nu,u^1,\ldots,u^{n-1}).\tag{8.11}$$

Let's calculate the derivatives of (8.10) and (8.11) with respect to the variable u^{n-1}. For the first of these two functions we have

$$\frac{\partial x_k}{\partial u^{n-1}}=\frac{\partial^2\nu}{\partial u^k\,\partial u^{n-1}}=\frac{\partial}{\partial u^k}\left(\psi_{n-1}(\nu,u^1,\ldots,u^{n-1})\right).$$

By differentiating composite function in the above expression we get

$$\frac{\partial x_k}{\partial u^{n-1}}=\frac{\partial\psi_{n-1}}{\partial u^k}+\frac{\partial\psi_{n-1}}{\partial\nu}\,x^k.\tag{8.12}$$

By means of analogous differentiation for the function (8.11) we obtain

$$\frac{\partial y_k}{\partial u^{n-1}}=\frac{\partial\psi_k}{\partial u^{n-1}}+\frac{\partial\psi_k}{\partial\nu}\,\psi_{n-1}.\tag{8.13}$$

Let's transform (8.13) with the help of compatibility condition (8.7), taking $r=n-1$ in it. This yields the relationship

$$\frac{\partial y_k}{\partial u^{n-1}}=\frac{\partial\psi_{n-1}}{\partial u^k}+\frac{\partial\psi_{n-1}}{\partial\nu}\,y^k.\tag{8.14}$$

Comparing (8.12) and (8.14), we see that functions x_k and y_k satisfy identical linear ordinary differential equations of the first order with to independent variable u^{n-1}. For $u^{n-1}=u^{n-1}_0$ their values are equal to each other by inductive hypothesis. Therefore $x_k=y_k$ for all $u^1,\ \ldots,\ u^{n-1}$. This means that function $\nu(u^1,\ldots,u^{n-1})$ constructed above is the solution of whole system of Pfaff equations (8.1). And it satisfies normalizing condition (8.5) as well. Theorem is proved. $\square$

Let's apply the theorem 8.1 to the equations (7.16), and let's find compatibility condition for them. First calculate partial derivatives (8.2) and (8.3) for the function $\nu(u^1,\ldots,u^{n-1})$ by virtue of the equations (7.16):

$$\begin{aligned}\frac{\partial^2\nu}{\partial u^k\,\partial u^r}&=-\nabla_{u^r}\left(\nu^{-1}\,(\mathbf{F}\,|\,\boldsymbol{\tau}_k)\right)=\frac{(\mathbf{F}\,|\,\boldsymbol{\tau}_k)}{\nu^2}\,\frac{\partial\nu}{\partial u^r}-\frac{(\nabla_{u^r}\mathbf{F}\,|\,\boldsymbol{\tau}_k)}{v}-\\&-\frac{(\mathbf{F}\,|\,\nabla_{u^r}\boldsymbol{\tau}_k)}{v}=-\frac{(\mathbf{F}\,|\,\boldsymbol{\tau}_k)\,(\mathbf{F}\,|\,\boldsymbol{\tau}_r)}{\nu^3}-\frac{(\nabla_{u^r}\mathbf{F}\,|\,\boldsymbol{\tau}_k)}{\nu}-\frac{(\mathbf{F}\,|\,\nabla_{u^r}\boldsymbol{\tau}_k)}{\nu},\end{aligned}\tag{8.15}$$

$$\begin{aligned}\frac{\partial^2\nu}{\partial u^r\,\partial u^k}&=-\nabla_{u^k}\left(\nu^{-1}\,(\mathbf{F}\,|\,\boldsymbol{\tau}_r)\right)=\frac{(\mathbf{F}\,|\,\boldsymbol{\tau}_r)}{\nu^2}\,\frac{\partial\nu}{\partial u^k}-\frac{(\nabla_{u^k}\mathbf{F}\,|\,\boldsymbol{\tau}_r)}{v}-\\&-\frac{(\mathbf{F}\,|\,\nabla_{u^k}\boldsymbol{\tau}_r)}{v}=-\frac{(\mathbf{F}\,|\,\boldsymbol{\tau}_r)\,(\mathbf{F}\,|\,\boldsymbol{\tau}_k)}{\nu^3}-\frac{(\nabla_{u^k}\mathbf{F}\,|\,\boldsymbol{\tau}_r)}{\nu}-\frac{(\mathbf{F}\,|\,\nabla_{u^k}\boldsymbol{\tau}_r)}{\nu}.\end{aligned}\tag{8.16}$$

Let's subtract (8.16) from (8.15), taking into account that $\nabla_{u^r}\boldsymbol{\tau}_k=\nabla_{u^k}\boldsymbol{\tau}_r$. The latter equality is the consequence of derivational formulas (7.9) due to symmetry of components of metric connection (7.11), and due to the relationship (7.12) expressing symmetry of second fundamental form. Taking into account all above facts, we get

$$(\nabla_{u^r}\mathbf{F}\,|\,\boldsymbol{\tau}_k)=(\nabla_{u^k}\mathbf{F}\,|\,\boldsymbol{\tau}_r).\tag{8.17}$$

In order to calculate covariant derivative $\nabla_{u^r}\mathbf{F}$ in the equality (8.17) we apply the formula (5.19) from Chapter IV:

$$\nabla_{u^r}\mathbf{F}=C(\nabla_{u^r}\mathbf{v}\otimes\tilde{\nabla}\mathbf{F})+C(\boldsymbol{\tau}_r\otimes\nabla\mathbf{F}).\tag{8.18}$$

In calculating $\nabla_{u^r}\mathbf{v}$ we take into account the equality $\mathbf{v}=\nu\cdot\mathbf{n}$ for the initial velocity on the initial hypersurface S. Then we get

$$\nabla_{u^r}\mathbf{v}=\nabla_{u^r}(\nu\cdot\mathbf{n})=-\frac{(\mathbf{F}\,|\,\boldsymbol{\tau}_r)}{\nu}\cdot\mathbf{n}-\nu\cdot\sum_{m=1}^{n-1}b^m_r\,\boldsymbol{\tau}_m.\tag{8.19}$$

In obtaining (8.19) we used derivational formulas (7.10) and the equation (7.16) for ν by itself. Now let's substitute (8.19) into (8.18), and let's use the resulting expression for to transform the equation (8.17):

$$\begin{aligned}&\frac{(\mathbf{F}\,|\,\boldsymbol{\tau}_k)(C(\mathbf{n}\otimes\tilde{\nabla}\mathbf{F})\,|\,\boldsymbol{\tau}_r)}{\nu}-\frac{(\mathbf{F}\,|\,\boldsymbol{\tau}_r)(C(\mathbf{n}\otimes\tilde{\nabla}\mathbf{F})\,|\,\boldsymbol{\tau}_k)}{\nu}+\\&+(C(\boldsymbol{\tau}_r\otimes\nabla\mathbf{F})\,|\,\boldsymbol{\tau}_k)-(C(\boldsymbol{\tau}_k\otimes\nabla\mathbf{F})\,|\,\boldsymbol{\tau}_r)=\\&=\sum_{m=1}^{n-1}\nu\left(b^m_r\,(C(\boldsymbol{\tau}_m\otimes\tilde{\nabla}\mathbf{F})\,|\,\boldsymbol{\tau}_k)-b^m_k\,(C(\boldsymbol{\tau}_m\otimes\tilde{\nabla}\mathbf{F})\,|\,\boldsymbol{\tau}_r)\right).\end{aligned}\tag{8.20}$$

The relationship (8.20) is the compatibility condition (8.7) applied to the Pfaff equations (7.16). Let's write it in local coordinates on M:

$$\begin{aligned}&\sum_{i=1}^{n}\sum_{j=1}^{n}\tau^i_k\,\tau^j_r\left(\sum_{m=1}^{n}n^m\,\frac{F_i\,\tilde{\nabla}_mF_j-F_j\,\tilde{\nabla}_mF_i}{\nu}+\nabla_jF_i-\nabla_iF_j\right)=\\&=\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{m=1}^{n-1}\nu\left(b^m_r\,\tau^j_m\tilde{\nabla}_jF_i\,\tau^i_k-b^m_k\,\tau^j_m\tilde{\nabla}_jF_i\,\tau^i_r\right).\end{aligned}\tag{8.21}$$

Here n^m are the components of unitary normal vector to S, and τ^i_k are components of the vector $\boldsymbol{\tau}_k$ in local map $(U, x^1, \ldots, x^n)$ on M:

$$\boldsymbol{\tau}_k = \sum_{i=1}^{n} \tau^i_k \, \frac{\partial}{\partial x^i}, \text{ where } k = 1, \ldots, n-1. \tag{8.22}$$

Variation vectors (8.22) tangent to S and the components b^m_k of inner operator field $\mathbf{b}$ from (7.10) are determined by the choice of local coordinates $u^1, \ldots, u^{n-1}$ on the hypersurface S. The equations (8.21) expressing compatibility condition (8.20) in local coordinates appear to be more convenient for the further analysis.

§ 9. The additional normality condition.

Normalization condition (8.5) in the theory of Pfaff equations is similar to initial data in cauchy problem for ordinary differential equations. Theorem 8.1 is analog of the theorem on local solvability of Cauchy problem for ordinary differential equations. The condition (8.5) prompt us normalization condition for the function $\nu(p)$ that determines modulus of initial velocity in (2.2). Let p_0 be some fixed point on hypersurface S in M, and let $\nu_0 \neq 0$ be some fixed real number. Then we set

$$\nu(p)\Big|_{p=p_0} = \nu_0 \tag{9.1}$$

This equality (9.1) is called **normalization condition** for the function $\nu(p)$ on S.

Definition 9.1. Suppose that Riemannian manifold M of the dimension $n \geqslant 2$ is equipped with some Newtonian dynamical system, which is used to arrange the shift of hypersurfaces along its trajectories according to the initial data (2.2). We say that this dynamical system satisfies **additional normality** condition, if for any hypersurface S in M, for any point p_0 on S, and for any real number $\nu_0 \neq 0$ there exists some smaller piece S' of hypersurface S belonging to transformation class and containing the point p_0, and there exists some function $\nu(p)$ from transformation class on S' normalized by the condition (9.1) and such that, when substituted to (2.2), it defines the shift $f_t\colon\, S' \to S'_t$ such that

$$\dot\varphi_k(t)\Big|_{t=0} = 0, \quad \text{where} \quad k = 1, \ldots, n-1. \tag{9.2}$$

Here $\varphi_k = (\boldsymbol{\tau}_k \,|\, \mathbf{v})$ are the functions of deviation defined by some choice of local coordinates $u^1, \ldots, u^{n-1}$ on S and corresponding vectors of variation $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_{n-1}$.

Note that (9.2) is a second part of the equalities (7.5). If additional normality condition is fulfilled, then equalities (7.5) are also fulfilled, since first part of these equalities holds unconditionally due to the initial data (2.2).

Functions of deviations φ_k in (9.2) depend on the choice of local coordinates $u^1, \ldots, u^{n-1}$ on S. But additional normality condition from definition 9.1 is fulfilled or not fulfilled regardless to any particular choice of local coordinates. Indeed, if we change local coordinates $u^1, \ldots, u^{n-1}$ on S for other local coordinates

$\tilde{u}^1, \ldots, \tilde{u}^{n-1}$, then corresponding vectors of variation are transformed as follows

$$\tilde{\boldsymbol{\tau}}_i = \sum_{k=1}^{n-1} \sigma_i^k \, \boldsymbol{\tau}_k, \tag{9.3}$$

Here σ_i^k are components of transition matrix given by

$$\sigma_i^k = \frac{\partial u^k}{\partial \tilde{u}^i}.$$

They do not depend on t. The matter is that local coordinates $u^1, \ldots, u^{n-1}$ and $\tilde{u}^1, \ldots, \tilde{u}^{n-1}$ are transferred from S' to S'_t by bijective map of shift $f_t\colon\, S' \to S'_t$. Transition functions

$$\begin{aligned}
&\tilde{u}^1 = \tilde{u}^1(u^1, \ldots, u^{n-1}), && u^1 = u^1(\tilde{u}^1, \ldots, \tilde{u}^{n-1}),\\
&\ldots\ldots\ldots\ldots\ldots\ldots && \ldots\ldots\ldots\ldots\ldots\ldots\\
&\tilde{u}^{n-1} = \tilde{u}^{n-1}(u^1, \ldots, u^{n-1}), && u^{n-1} = u^{n-1}(\tilde{u}^1, \ldots, \tilde{u}^{n-1})
\end{aligned}$$

thereby remain same as on initial hypersurface S'. Transformation rule (9.3) then is the direct consequence of formula (7.2), which, in turn, is the definition of variation vectors $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_{n-1}$.

Let's substitute (9.3) into (4.1). As a result we obtain the transformation rule for the functions of deviation relating $\varphi_1, \ldots, \varphi_{n-1}$ and $\tilde{\varphi}_1, \ldots, \tilde{\varphi}_{n-1}$:

$$\tilde{\varphi}_i = \sum_{k=1}^{n-1} \sigma_i^k \, \varphi_k. \tag{9.4}$$

Differentiating (9.4) with respect to t and taking into account $\dot{\sigma}_i^k = 0$, we get

$$\dot{\tilde{\varphi}}_i = \sum_{k=1}^{n-1} \sigma_i^k \, \dot{\varphi}_k. \tag{9.5}$$

The relationships (9.5) shows that, once fulfilled for some particular choice of local coordinates on S, the relationships (9.2) will remain true for any other choice of local coordinates on S.

The statement of additional normality condition in definition 9.1 exploits some hypersurface S and some local coordinates on it. But, as we have seen above, these objects are auxiliary tools only. Additional normality condition, when it is fulfilled, is a property of dynamical system by itself. Similar to weak normality condition from definition 6.1, it can be written in form of partial differential equations for the force field $\mathbf{F}$ of Newtonian dynamical system. Exception is two-dimensional case $\dim M = n = 2$.

Theorem 9.1. *In two-dimensional case* $\dim M = 2$ *additional normality condition is fulfilled unconditionally for arbitrary Newtonian dynamical system on* M.

We shall not prove and comment this theorem, since two-dimensional case is considered in **other thesis** [36].

Suppose that $\dim M \geqslant 3$. Let's consider the Newtonian dynamical system with force field $\mathbf{F}$ on M. Suppose that this system satisfies additional normality condition. Let's choose some fixed point p_0 in M and some nonzero vector $\mathbf{v}_0$ at this point. Denote by α the hyperplane perpendicular to vector $\mathbf{v}_0$, and denote by $\mathbf{P}$ orthogonal projector to this hyperplane (see Fig. 9.1). Then $\mathbf{P}$ is a value of projector field $\mathbf{P}$ from § 5 at the point $q = (p_0, \mathbf{v}_0)$ of tangent bundle TM. One can draw infinitely many surfaces passing through the point p_0 and being perpendicular to $\mathbf{v}_0$ at this point. Let's choose one of them belonging to transformation class and denote it by S. Then choose local coordinates $u^1, \ldots, u^{n-1}$ in the neighborhood of the point p_0 and use this point to write normalization condition (9.1). According to the results of § 7 for $n \geqslant 3$ weak the equalities (9.2) lead to the system of Pfaff equations (7.16) for the function $\nu(p)$, while arbitrariness of $\nu_0 = |\mathbf{v}_0|$ in thetag9.1 means that these Pfaff equations are compatible in the sense of definition 8.1. Hence the relationships (8.20) are fulfilled. In local coordinates they are written as (8.21). We shall consider them at the point p_0 on S.

THEOREM 9.2. *Let S be some hypersurface in Riemannian manifold M and let $\mathbf{P}$ be operator of orthogonal projection onto the hyperplane tangent to S at the point p_0 on S. Then for the matrix components of operator $\mathbf{P}$ in local coordinates $x^1, \ldots, x^n$ on M there exists the formula*

$$P^i_\varepsilon = \sum_{s=1}^{n} \sum_{a=1}^{n-1} \sum_{k=1}^{n-1} g_{\varepsilon s}\, \tau^s_a\, \rho^{ak}\, \tau^i_k, \tag{9.6}$$

where τ^s_a and τ^i_k are components of coordinate tangent vectors $\boldsymbol{\tau}_a$ and $\boldsymbol{\tau}_k$ that are defined by some local coordinates $u^1, \ldots, u^{n-1}$ on S; ρ^{ak} are components of dual metric tensor for induced Riemannian metric ρ in coordinates $u^1, \ldots, u^{n-1}$ on S; and finally, $g_{\varepsilon s}$ are components of metric $\mathbf{g}$ in coordinates $x^1, \ldots, x^n$ on M.

PROOF. Let $\mathbf{y}$ be some arbitrary vector at the point p_0 on M. Its projection $\mathbf{Py}$ can be expanded in the base of vectors $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_{n-1}$:

$$\mathbf{Py} = \sum_{k=1}^{n-1} \beta^k\, \boldsymbol{\tau}_k. \tag{9.7}$$

In order to find numeric coefficients β^k in (9.7) we consider scalar products of the vector $\boldsymbol{\tau}_a$ with both sides of the equality (9.7). This yields

$$(\boldsymbol{\tau}_a\,|\,\mathbf{Py})=\sum_{k=1}^{n-1}\beta^k\,(\boldsymbol{\tau}_a\,|\,\boldsymbol{\tau}_k). \tag{9.8}$$

In left hand side of (9.8) we obtain $(\boldsymbol{\tau}_a\,|\,\mathbf{Py})=(\mathbf{P}\boldsymbol{\tau}_a\,|\,\mathbf{y})=(\boldsymbol{\tau}_a\,|\,\mathbf{y})$, while in right hand side the quantities $(\boldsymbol{\tau}_a\,|\,\boldsymbol{\tau}_k)$ coincide with components of metric tensor for induced Riemannian metric on S: $\rho_{ak}=(\boldsymbol{\tau}_a\,|\,\boldsymbol{\tau}_k)$. Therefore we can bring (9.8) to

$$(\boldsymbol{\tau}_a\,|\,\mathbf{y})=\sum_{k=1}^{n-1}\beta^k\,\rho_{ak},$$

and we can calculate β^k with the use of components ρ^{ak} for dual metric tensor:

$$\beta^k=\sum_{a=1}^{n-1}(\boldsymbol{\tau}_a\,|\,\mathbf{y})\,\rho^{ak}. \tag{9.9}$$

Let's substitute (9.9) into the formula (9.7) for the vector $\mathbf{Py}$. As a result we get

$$\mathbf{Py}=\sum_{a=1}^{n-1}\sum_{k=1}^{n-1}(\boldsymbol{\tau}_a\,|\,\mathbf{y})\,\rho^{ak}\,\boldsymbol{\tau}_k. \tag{9.10}$$

Now we are only to write vectorial equality (9.10) in local coordinates $x^1,\,\ldots\,,\,x^n$ on the manifold M. This yields the following relationship:

$$\sum_{\varepsilon=1}^{n}P^i_\varepsilon\,y^\varepsilon=\sum_{\varepsilon=1}^{n}\sum_{s=1}^{n}\sum_{a=1}^{n-1}\sum_{k=1}^{n-1}g_{\varepsilon s}\,\tau^s_a\,y^\varepsilon\;\;\rho^{ak}\,\tau^i_k.$$

Components of vector $\mathbf{y}$ in both sides of the obtained equality are arbitrary real numbers. Therefore it is reduced to the equality (9.6) that was to be proved. $\square$

We shall use theorem 9.2 to make further changes in (8.21). In order to do it let's take two copies of the equality (9.6), redesignating indices in the second copy:

$$P^i_\varepsilon=\sum_{s=1}^{n}\sum_{a=1}^{n-1}\sum_{k=1}^{n-1}g_{\varepsilon s}\,\tau^s_a\,\rho^{ak}\,\tau^i_k,$$
$$P^j_\sigma=\sum_{w=1}^{n}\sum_{b=1}^{n-1}\sum_{r=1}^{n-1}g_{\sigma w}\,\tau^w_b\,\rho^{br}\,\tau^j_r.$$

By multiplying these two equalities, we get the relationship

$$P^i_\varepsilon\,P^j_\sigma=\sum_{s=1}^{n}\sum_{a=1}^{n-1}\sum_{k=1}^{n-1}\sum_{w=1}^{n}\sum_{b=1}^{n-1}\sum_{r=1}^{n-1}g_{\varepsilon s}\,\tau^s_a\,\rho^{ak}\,g_{\sigma w}\,\tau^w_b\,\rho^{br}\,(\tau^i_k\,\tau^j_r).$$

Here we intentionally enclosed into brackets last two terms in right hand side of this relationships. Exactly the same terms $\tau^i_k\,\tau^j_r$ are in left hand side of (8.21). Let's multiply (8.21) by $g_{\varepsilon s}\,\tau^s_a\,\rho^{ak}\,g_{\sigma w}\,\tau^w_b\,\rho^{br}$ and let's sum the resulting equality in s, a, k, w, b, and r. Then left hand side of (8.21) takes the form

$$\sum_{i=1}^{n}\sum_{j=1}^{n}P^i_\varepsilon\,P^j_\sigma\left(\sum_{m=1}^{n}n^m\,\frac{F_i\,\tilde\nabla_mF_j-F_j\,\tilde\nabla_mF_i}{\nu}+\nabla_jF_i-\nabla_iF_j\right).$$

Now let's introduce the following notations

$$H^j_\varepsilon=\sum_{s=1}^{n}\sum_{a=1}^{n-1}\sum_{k=1}^{n-1}\sum_{m=1}^{n-1}g_{\varepsilon s}\,\tau^s_a\,\rho^{ak}\,b^m_k\,\tau^j_m. \tag{9.11}$$

Then in right hand side of transformed equality (8.21) we can detect the quantities H^j_ε and H^j_σ that were formed as a result of the above transformations. I terms of notations (9.11) right hand side of (8.21) is written as

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\nu\,(P^i_\varepsilon\,H^j_\sigma-P^i_\sigma\,H^j_\varepsilon)\,\tilde\nabla_jF_i.$$

Now, equating transformed expressions for left and right hand sides of (8.21), we complete first step in transforming this equality:

$$\begin{aligned}&\sum_{i=1}^{n}\sum_{j=1}^{n}P^i_\varepsilon\,P^j_\sigma\left(\sum_{m=1}^{n}n^m\,\frac{F_i\,\tilde\nabla_mF_j-F_j\,\tilde\nabla_mF_i}{\nu^2}\;+\right.\\&\left.+\;\frac{\nabla_jF_i-\nabla_iF_j}{\nu}\right)=\sum_{i=1}^{n}\sum_{j=1}^{n}(P^i_\varepsilon\,H^j_\sigma-P^i_\sigma\,H^j_\varepsilon)\,\tilde\nabla_jF_i.\end{aligned} \tag{9.12}$$

Let's postpone for a while further transformation of (9.12). Instead, let's study the quantities H^j_ε introduced by (9.11). They have transparent geometric interpretation. In (9.11) we find the quantities b^m_k that were derived from b_{kr} by index raising procedure (see formula (7.13)). They are the components of inner tensor field $\mathbf{B}$ of type $(1,1)$ in S. The value of such field at the point $p_0\in S$ can be interpreted as linear operator in the tangent space $T_{p_0}(S)$ to S. Tangent space $T_{p_0}(S)$ is naturally embedded into the tangent space $T_{p_0}(M)$ as $(n-1)$-dimensional hyperplane α (see Fig. 9.1). We have orthogonal projector $\mathbf{P}$ onto this hyperplane. Therefore composition $\mathbf{H}=\mathbf{B}\circ\mathbf{P}$ can be interpreted as linear operator in $T_{p_0}(M)$. Let's calculate the matrix of this operator in local coordinates $x^1,\,\ldots\,,\,x^n$. Suppose that $\mathbf{y}$ is some arbitrary tangent vector at the point p_0 on M. Then

$$\mathbf{H}(\mathbf{y})=\mathbf{B}\circ\mathbf{P}(\mathbf{y})=\mathbf{B}(\mathbf{P}\mathbf{y}). \tag{9.13}$$

In order to calculate projection $\mathbf{Py}$ in (9.13) we use formula (9.10). The result of applying $\mathbf{B}$ to base vectors $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_{n-1}$ in hyperplane α is defined by matrix of $\mathbf{B}$:

$$\mathbf{B}(\boldsymbol{\tau}_k)=\sum_{m=1}^{n-1} b^m_k\,\boldsymbol{\tau}_m. \tag{9.14}$$

Combining (9.13), (9.14), and (9.10), for the vector $\mathbf{H}(\mathbf{y})$ in (9.13) we get

$$\mathbf{H}(\mathbf{y})=\sum_{m=1}^{n-1}\sum_{a=1}^{n-1}\sum_{k=1}^{n-1}(\boldsymbol{\tau}_a\,|\,\mathbf{y})\,\rho^{ak}\,b^m_k\,\boldsymbol{\tau}_m. \tag{9.15}$$

Now let's bring vectorial equality (9.15) to coordinate form:

$$\sum_{\varepsilon=1}^{n} H^j_\varepsilon\,y^\varepsilon=\sum_{\varepsilon=1}^{n}\sum_{s=1}^{n}\sum_{m=1}^{n-1}\sum_{a=1}^{n-1}\sum_{k=1}^{n-1} g_{\varepsilon s}\,\tau^s_a\,y^\varepsilon\,\rho^{ak}\,b^m_k\,\tau^j_m.$$

Components of vector $\mathbf{y}$ are arbitrary real numbers. Taking into account this fact, we find that components of operator $\mathbf{H}$ in local coordinates $x^1, \ldots, x^n$ are given exactly by formula (9.11).

Theorem 9.3. *Operator $\mathbf{H}=\mathbf{B}\circ\mathbf{P}$ in tangent space $T_{p_0}(M)$ defined by second fundamental form of hypersurface S according to the formula* (9.11) *for its components is symmetric in Riemannian metric $\mathbf{g}$ of the manifold M, i. e.*

$$(\mathbf{x}\,|\,\mathbf{Hy})=(\mathbf{Hx}\,|\,\mathbf{y}), \tag{9.16}$$

where $\mathbf{x}$ and $\mathbf{y}$ are arbitrary vectors from tangent space $T_{p_0}(M)$ at the point $p_0\in S$.

Proof. In order to prove the relationship (9.16) we use the above formula (9.15) for the vector $\mathbf{Hy}=\mathbf{H}(\mathbf{y})$. This yields

$$(\mathbf{x}\,|\,\mathbf{Hy})=\sum_{m=1}^{n-1}\sum_{a=1}^{n-1}\sum_{k=1}^{n-1}(\boldsymbol{\tau}_a\,|\,\mathbf{y})\,\rho^{ak}\,b^m_k\,(\boldsymbol{\tau}_m\,|\,\mathbf{x}).$$

Taking into account (7.13), this relationship can be transformed as follows:

$$(\mathbf{x}\,|\,\mathbf{Hy})=\sum_{m=1}^{n-1}\sum_{a=1}^{n-1}\sum_{k=1}^{n-1}\sum_{r=1}^{n-1}(\boldsymbol{\tau}_a\,|\,\mathbf{y})\,\rho^{ak}\,b_{kr}\,\rho^{rm}\,(\boldsymbol{\tau}_m\,|\,\mathbf{x}). \tag{9.17}$$

Right hand side of the equality (9.17) is invariant under the transposition of vectors $\mathbf{x}$ and $\mathbf{y}$. This is due to the symmetry in components of second fundamental form of S (see equality (7.12) in § 7). Therefore we get the required relationship (9.16): $(\mathbf{x}\,|\,\mathbf{Hy})=(\mathbf{y}\,|\,\mathbf{Hx})=(\mathbf{Hx}\,|\,\mathbf{y})$. $\square$

Symmetric in metric $\mathbf{g}$ operator $\mathbf{H}$ commutate with operator $\mathbf{P}$ of orthogonal projection to hyperplane $\alpha = T_{p_0}(S)$, i. e. we have:

$$\mathbf{P}\circ\mathbf{H} = \mathbf{H}\circ\mathbf{P} = \mathbf{H}. \tag{9.18}$$

Let $\mathbf{x} \in T_{p_0}(M)$. Operator $\mathbf{H}$ maps vector $\mathbf{x}$ to the vector $\mathbf{Hx} \in \alpha$. Therefore, applying $\mathbf{P}$ to $\mathbf{Hx}$, we get the same vector $\mathbf{Hx}$, i. e. $\mathbf{P}(\mathbf{Hx}) = \mathbf{Hx}$. This means that $\mathbf{P}\circ\mathbf{H} = \mathbf{H}$. On the other hand $\mathbf{H}\circ\mathbf{P} = \mathbf{B}\circ\mathbf{P}^2$. If we recall that $\mathbf{P}^2 = \mathbf{P}$, then we get $\mathbf{H}\circ\mathbf{P} = \mathbf{B}\circ\mathbf{P} = \mathbf{H}$. This proves the equality (9.18).

Theorem 9.4 (**on the second fundamental form).** *Let $\mathbf{v}_0$ be nonzero vector at some point p_0 on Riemannian manifold M, and let $\mathbf{P}$ be the operator of orthogonal projection onto the hyperplane α perpendicular to $\mathbf{v}_0$. Then for any linear operator $\mathbf{H}$ in $T_{p_0}(M)$ symmetric in metric $\mathbf{g}$ of Riemannian manifold M and satisfying the relationships* (9.18) *one can find hypersurface S passing through the point p_0 and perpendicular to $\mathbf{v}_0$ such that matrix elements of the operator $\mathbf{H}$ are determined by second fundamental form of S according to the formula* (9.11).

We shall prove this theorem in § 10 (see below). Now we use it to analyze the equations (9.12). Identically zero operator $\mathbf{H} = 0$ satisfy the hypothesis of theorem 9.4. Therefore its matrix $H^i_j = 0$ can be present in right hand side of (9.12). Substituting $H^j_\sigma = 0$ and $H^j_\varepsilon = 0$ into (9.12) we obtain

$$\sum_{i=1}^{n}\sum_{j=1}^{n} P^i_\varepsilon\,P^j_\sigma\left(\nabla_j F_i - \nabla_i F_j + \sum_{m=1}^{n} n^m\,\frac{F_i\,\tilde\nabla_m F_j - F_j\,\tilde\nabla_m F_i}{\nu}\right) = 0.$$

Left hand side of (9.12) doesn't depend on second fundamental form of S, hence it doesn't depend on $\mathbf{H}$. The above equality means that left hand side of (9.12) is equal to zero. Therefore right hand side of (9.12) is zero too:

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(P^i_\varepsilon\,H^j_\sigma - P^i_\sigma\,H^j_\varepsilon)\,\tilde\nabla_j F_i = 0. \tag{9.19}$$

First of the above two relationships can be written in the form that doesn't depend on S at all. In order to do it let's recall that normal vector to S at the point p_0 is directed along the vector of velocity, while function ν iz normalized at this point by $\nu = \nu_0 = |\mathbf{v}_0|$. Hence $\mathbf{n}$ can be replaced by $\mathbf{N}$, and ν can be replaced by $v = |\mathbf{v}|$:

$$\sum_{i=1}^{n}\sum_{j=1}^{n} P^i_\varepsilon\,P^j_\sigma\left(\nabla_j F_i - \nabla_i F_j + \sum_{m=1}^{n} N^m\,\frac{F_i\,\tilde\nabla_m F_j - F_j\,\tilde\nabla_m F_i}{v}\right) = 0.$$

Arbitrariness in the choice of point p_0 (see definition 9.1) means that this equality holds at all points $q = (p, \mathbf{v})$ of tangent bundle TM, where $\mathbf{v} \neq 0$. It is the differential equation for the force field of dynamical system on M that follows from additional normality condition for it. For the sake of more symmetry we shall write it as

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{n} P^i_\varepsilon\,P^j_\sigma\left(\sum_{m=1}^{n} N^m\,\frac{F_i\,\tilde\nabla_m F_j}{v} - \nabla_i F_j\right) &=\\ =\sum_{i=1}^{n}\sum_{j=1}^{n} P^i_\varepsilon\,P^j_\sigma\left(\sum_{m=1}^{n} N^m\,\frac{F_j\,\tilde\nabla_m F_i}{v} - \nabla_j F_i\right)&.\end{aligned}\tag{9.20}$$

The equation (9.19), unlike to (9.20), require further transformation, since it contain the quantities H^j_σ and H^j_ε, which depend on the choice of hypersurface S. Let's consider a linear operator in $T_{p_0}(M)$ with the following components:

$$K^\sigma_\varepsilon = \sum_{m=1}^{n}\sum_{i=1}^{n}\sum_{j=1}^{n} g^{\sigma m}\,P^j_m\,\tilde\nabla_j F_i\,P^i_\varepsilon.\tag{9.21}$$

Operator $\mathbf{K}$ with components (9.21) satisfies the relationships

$$\mathbf{K}\circ\mathbf{P} = \mathbf{P}\circ\mathbf{K} = \mathbf{K}.\tag{9.22}$$

The relationships (9.22) are analogous to the relationships (9.18) for the operator H. They can be verified by direct calculations, if we take into account that $\mathbf{P}^2 = \mathbf{P}$ and take into account the equalities

$$\sum_{m=1}^{n} g^{\sigma m}\,P^j_m = \sum_{m=1}^{n} P^\sigma_m\,g^{mj}\tag{9.23}$$

that reflect the symmetry of projector $\mathbf{P}$ with respect to Riemannian metric $\mathbf{g}$ on M. Further let's consider the operator $\mathbf{M} = \mathbf{H}\circ\mathbf{K}$, and let's calculate its components in local coordinates. For the components of $\mathbf{M}$ we obtain

$$M^\gamma_\varepsilon = \sum_{\sigma=1}^{n} H^\gamma_\sigma\,K^\sigma_\varepsilon = \sum_{\sigma=1}^{n}\sum_{i=1}^{n}\sum_{j=1}^{n} g^{\gamma\sigma}\,H^j_\sigma\,\tilde\nabla_j F_i\,P^i_\varepsilon.$$

In deriving this formula we used (9.23), and used one of the relationships (9.18). By means of $\mathbf{M}$ we define bilinear form

$$\mathbf{\Theta}(\mathbf{x},\mathbf{y}) = (\mathbf{x}\,|\,\mathbf{M}\mathbf{y}).\tag{9.24}$$

Form (9.24) is associated with tensor $\mathbf{\Theta}$ of type $(0,2)$. Here are its components:

$$\theta_{\sigma\varepsilon} = \sum_{i=1}^{n}\sum_{j=1}^{n} H^j_\sigma\,\tilde\nabla_j F_i\,P^i_\varepsilon.\tag{9.25}$$

Let's compare formulas (9.25) and (9.19). It's easy to note that (9.19) is exactly the condition of symmetry for bilinear form $\mathbf{\Theta}$, i. e. $\theta_{\sigma\varepsilon} = \theta_{\varepsilon\sigma}$ or $\mathbf{\Theta}(\mathbf{x},\mathbf{y}) = \mathbf{\Theta}(\mathbf{y},\mathbf{x})$.

Hence $\mathbf{M}$ is a symmetric operator in Riemannian metric $\mathbf{g}$ of the manifold M:

$$(\mathbf{x}\,|\,\mathbf{M}\mathbf{y}) = (\mathbf{M}\mathbf{x}\,|\,\mathbf{y}), \tag{9.26}$$

Let's state this result as the following lemma.

LEMMA 9.1. *The equations* (9.19) *are equivalent to the symmetry of operator* $\mathbf{M} = \mathbf{H} \circ \mathbf{K}$ *in the metric* $\mathbf{g}$ *of* M*, operators* $\mathbf{H}$ *and* $\mathbf{K}$ *being defined by their components* (9.11) *and* (9.21).

First let's take $\mathbf{H} = \mathbf{P}$ and let's apply theorem 9.4 to such operator $\mathbf{H}$. This is correct, since operator $\mathbf{H} = \mathbf{P}$ is symmetric, and it satisfies the relationships (9.18). From theorem 9.4 we get that operator $\mathbf{H} = \mathbf{P}$ can be defined by second fundamental form of some hypersurface S passing through the point p_0. For $\mathbf{H} = \mathbf{P}$ we obtain $\mathbf{M} = \mathbf{K}$. Therefore due to lemma 9.1 we conclude that operator $\mathbf{K}$ with components (9.21) is symmetric in Riemannian metric $\mathbf{g}$ of the manifold M.

The equality $\mathbf{H} = \mathbf{P}$, which holds for some particular choice of hypersurface S, is only some special case, when theorem 9.4 is applicable. In general, operator $\mathbf{H}$ is an arbitrary symmetric operator satisfying the relationships (9.18) and such that composition of two symmetric operators $\mathbf{H}$ and $\mathbf{K}$ is symmetric operator $\mathbf{M} = \mathbf{H} \circ \mathbf{K}$. This is possible if and only if operators $\mathbf{H}$ and $\mathbf{K}$ are commutating. Indeed, from the equality (9.26) we derive

$$(\mathbf{x}\,|\,\mathbf{H}(\mathbf{K}\mathbf{y})) = (\mathbf{H}(\mathbf{K}\mathbf{x})\,|\,\mathbf{y}). \tag{9.27}$$

On the other hand symmetry of operators $\mathbf{H}$ and $\mathbf{K}$ yields

$$(\mathbf{x}\,|\,\mathbf{H}(\mathbf{K}\mathbf{y})) = (\mathbf{H}\mathbf{x}\,|\,\mathbf{K}\mathbf{y}) = (\mathbf{K}(\mathbf{H}\mathbf{x})\,|\,\mathbf{y}). \tag{9.28}$$

Comparing (9.27) and (9.28), and taking into account the arbitrariness of $\mathbf{x}$ and $\mathbf{y}$ in these relationships, we obtain the operator equality

$$\mathbf{H} \circ \mathbf{K} = \mathbf{K} \circ \mathbf{H}. \tag{9.29}$$

This equality (9.29) means that $\mathbf{H}$ and $\mathbf{K}$ are commutating. We state this result in form of the following lemma.

LEMMA 9.2. *Suppose that the equations* (9.19) *are fulfilled for any hypersurface* S *passing through the point* p_0 *and perpendicular to the vector* $\mathbf{v}_0$ *at this point. Then operator* $\mathbf{K}$ *commutates with any symmetric operator* $\mathbf{H}$ *that satisfies the relationships* (9.18).

Let α be a hyperplane perpendicular to the vector $\mathbf{v}_0$. Tangent space to the manifold M at the point p_0 is represented by a sum of two subspaces

$$T_{p_0}(M) = \alpha \oplus \langle \mathbf{v}_0 \rangle. \tag{9.30}$$

Here $\langle \mathbf{v}_0 \rangle$ is a linear span of vector $\mathbf{v}_0$. Each subspace in the expansion (9.30) is invariant under the action of operators $\mathbf{H}$ and $\mathbf{K}$. This is easily derived from (9.18) and (9.22). Moreover, the restrictions of $\mathbf{H}$ and $\mathbf{K}$ to $\langle \mathbf{v}_0 \rangle$ are zero. Let's choose some orthonormal base $\mathbf{E}_1, \ldots, \mathbf{E}_{n-1}$ in hyperplane α and let's complete it by unitary vector $\mathbf{E}_n$ directed along the vector $\mathbf{v}_0$. Operators $\mathbf{H}$ and $\mathbf{K}$ in the base $\mathbf{E}_1, \ldots, \mathbf{E}_n$ are defined by the following symmetric matrices:

$$H = \begin{Vmatrix} H^1_1 & \ldots & H^1_{n-1} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ H^{n-1}_1 & \ldots & H^{n-1}_{n-1} & 0 \\ 0 & \ldots & 0 & 0 \end{Vmatrix}, \tag{9.31}$$

$$K = \begin{Vmatrix} K^1_1 & \ldots & K^1_{n-1} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ K^{n-1}_1 & \ldots & K^{n-1}_{n-1} & 0 \\ 0 & \ldots & 0 & 0 \end{Vmatrix}. \tag{9.32}$$

Matrix of projector $\mathbf{P}$ in the base $\mathbf{E}_1, \ldots, \mathbf{E}_n$ has similar structure:

$$P = \begin{Vmatrix} 1 & \ldots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & 1 & 0 \\ 0 & \ldots & 0 & 0 \end{Vmatrix}. \tag{9.33}$$

Operator $\mathbf{K}$ is some fixed operator with matrix (9.32), but $\mathbf{H}$, as it follows from lemma 9.2, is an arbitrary operator with matrix of the form (9.31). Using this arbitrariness, let's choose the matrix of operator $\mathbf{H}$ to be the diagonal matrix with distinct elements $H^1_1, \ldots, H^{n-1}_{n-1}$, $H^n_n = 0$:

$$H = \begin{Vmatrix} H^1_1 & \ldots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & H^{n-1}_{n-1} & 0 \\ 0 & \ldots & 0 & 0 \end{Vmatrix}. \tag{9.34}$$

Commutativity of operators $\mathbf{K}$ and $\mathbf{H}$ means that their matrices (9.32) and (9.34) are also commutating. This yields $K^i_j\,H^j_j = H^i_i\,K^i_j$. Diagonal elements in matrix (9.34) are distinct: $H^i_i \neq H^j_j$ for $i \neq j$. Therefore $K^i_j = 0$ for $i \neq j$, i. e. matrix of operator $\mathbf{K}$ is diagonal in the base of vectors $\mathbf{E}_1, \ldots, \mathbf{E}_n$. This reduces matrix (9.32) to the following form:

$$K = \begin{Vmatrix} K^1_1 & \ldots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & K^{n-1}_{n-1} & 0 \\ 0 & \ldots & 0 & 0 \end{Vmatrix}. \tag{9.35}$$

Now let's use again the arbitrariness of operator $\mathbf{H}$. In this case we choose it so that its matrix (9.31) has the form:

$$H = \begin{Vmatrix} 0 & \ldots & 1 & \ldots & 0 \\ \vdots & \ddots & \vdots & & \vdots \\ 1 & \ldots & 0 & \ldots & 0 \\ \vdots & & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & \ldots & 0 \end{Vmatrix}. \tag{9.36}$$

The only nonzero elements in matrix (9.36) are $H^1_i = 1$ and $H^i_1 = 1$, where $1 < i < n$. Commutativity of $\mathbf{K}$ and $\mathbf{H}$ implies commutativity of their matrices (9.35) and (9.36). This yields $K^1_1\, H^1_i = H^1_i\, K^i_i$ and $K^i_i\, H^i_1 = H^i_1\, K^1_1$. These relationships are equivalent to $K^i_i = K^1_1$. Due to arbitrariness in i we get

$$K^1_1 = \ldots = K^{n-1}_{n-1} = \lambda. \tag{9.37}$$

The relationship (9.37) reduces matrix (9.35) to the following form:

$$K = \begin{Vmatrix} \lambda & \ldots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & \lambda & 0 \\ 0 & \ldots & 0 & 0 \end{Vmatrix}.$$

Now it's obvious that such matrix commutates with any matrix of the form (9.31). Comparing it with (9.33), we get $\mathbf{K} = \lambda \cdot \mathbf{P}$. We state this result as a third lemma.

LEMMA 9.3. *Operator $\mathbf{K}$ satisfying the relationships (9.22) commutates with arbitrary symmetric operator $\mathbf{H}$ satisfying the relationships (9.18) if and only if it differs from projector $\mathbf{P}$ only by some numeric multiple: $\mathbf{K} = \lambda \cdot \mathbf{P}$.*

Thus, by means of theorem 9.4 and by means of three lemmas 9.1, 9.2, and 9.3 we reduced (9.19) to the operator equality

$$\mathbf{K} = \lambda \cdot \mathbf{P}. \tag{9.38}$$

Let's write (9.38) in local coordinates. In order to do it we use formula (9.21) that determines components of operator **K**. As a result we get the following relationship:

$$\sum_{m=1}^{n}\sum_{i=1}^{n}\sum_{j=1}^{n} g^{\sigma m}\, P^j_m\, \tilde{\nabla}_j F_i\, P^i_\varepsilon = \lambda\, P^\sigma_\varepsilon. \tag{9.39}$$

In formula (9.39) one can lower index σ and one can raise index ε. If we take into account symmetry of operator **P**, we can write

$$\sum_{i=1}^{n}\sum_{j=1}^{n} P^j_\sigma\, \tilde{\nabla}_j F^i\, P^\varepsilon_i = \lambda\, P^\varepsilon_\sigma. \tag{9.40}$$

In the equations (9.40) we have numeric parameter λ. Let's find it by calculating traces of operators in both sides of the equality (9.38):

$$\operatorname{tr}\mathbf{K} = \lambda\cdot\operatorname{tr}\mathbf{P}. \tag{9.41}$$

It's easy to calculate trace of projector: $\operatorname{tr}\mathbf{P} = n-1$. It is nonzero, since we consider the case $n \geqslant 3$. Finding λ from (9.41), we can write (9.40) in the form that has no indefinite parameters:

$$\sum_{i=1}^{n}\sum_{j=1}^{n} P^j_\sigma\, \tilde{\nabla}_j F^i\, P^\varepsilon_i = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{m=1}^{n} \frac{P^j_m\, \tilde{\nabla}_j F^i\, P^m_i}{n-1}\, P^\varepsilon_\sigma. \tag{9.42}$$

Similar to (9.20), the equations (9.42) are the partial differential equations for the force field **F** of Newtonian dynamical system. The above calculations can be reverted. From (9.42) we could get (9.38). Then from (9.38) and (9.22) we could get the equality (9.29), which now is written as

$$\mathbf{H}\circ\mathbf{K} = \mathbf{K}\circ\mathbf{H} = \lambda\cdot\mathbf{H}.$$

Hence operator $\mathbf{M} = \mathbf{H}\circ\mathbf{K} = \lambda\cdot\mathbf{H}$ is symmetric. Due to lemma 9.1 this is equivalent to (9.19). Combining (9.19) and (9.20), we get the equality (9.12), which, in turn, is equivalent to (8.21). The equality (8.21) is the compatibility condition for the system of Pfaff equations (7.16). It provides solvability of these equations for arbitrary choice of the point $p_0 \in S$, and for arbitrary choice of numeric parameter $\nu_0 \neq 0$ in normalization condition (9.1). As a result we can formulate the main theorem of this section, which was proved by the above considerations.

Theorem 9.5. *Newtonian dynamical system on Riemannian manifold of the dimension $n \geqslant 3$ satisfies additional normality condition if and only if its force field satisfies the equations* (9.20) *and* (9.42) *at all points $q = (p, \mathbf{v})$ of tangent bundle TM, except for those, where $\mathbf{v} = 0$.*

The equations (9.20) and (9.42) are called ıadditional normality conditions. They were first derived in [38] (see also preprint [34]) for the case $M = \mathbb{R}^n$. The above derivation of the equations (9.20) and (9.42) is based on the theorem 9.4, which is not yet proved.

§ 10. Proof of the theorem on a second quadratic form.

In theorem 9.4 on the second fundamental form we consider a point p_0 on Riemannian manifold M and nonzero vector $\mathbf{v}_0$ at this point. Vector $\mathbf{v}_0$ defines a hyperplane α in $T_{p_0}(M)$ perpendicular to $\mathbf{v}_0$ and an operator $\mathbf{P}$ of orthogonal projection onto the hyperplane α. Then various hypersurfaces passing through p_0 and tangent to α at that point are considered. Basic tangent vectors $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_{n-1}$ at the point p_0 for any of such hypersurfaces are in the hyperplane α and form the base in it. Theorem 9.4 (which we need to prove) asserts that for any symmetric linear operator $\mathbf{H}$ in $T_{p_0}(M)$ satisfying the relationships (9.18) its matrix is defined by vectors $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_{n-1}$ and by second fundamental form of some hypersurface S according to the formula (9.11). Vector $\mathbf{v}_0$ and hyperplane α define the expansion of tangent space $T_{p_0}(M)$ into a direct sum of two subspaces:

$$T_{p_0}(M) = \alpha \oplus \langle \mathbf{v}_0 \rangle. \tag{10.1}$$

WE have already considered this expansion (see (9.30)). Let $\mathbf{H}$ be some arbitrary symmetric in metric $\mathbf{g}$ on M linear operator $\mathbf{H}\colon T_{p_0}(M) \to T_{p_0}(M)$ satisfying the relationships (9.18). From $\mathbf{H} = \mathbf{P} \circ \mathbf{H}$ we find that subspace α is invariant under the action of $\mathbf{H}$. From $\mathbf{H} = \mathbf{H} \circ \mathbf{P}$, in turn, we get that subspace $\langle \mathbf{v}_0 \rangle$ in (10.1) is also invariant under the action of $\mathbf{H}$, the restriction of $\mathbf{H}$ to $\langle \mathbf{v}_0 \rangle$ being zero. Let's denote by $\mathbf{B}$ the restriction of $\mathbf{H}$ to hyperplane α:

$$\mathbf{B} = \mathbf{H}\Big|_{\alpha}.$$

Such restriction is a symmetric linear operator $\mathbf{B}\colon \alpha \to \alpha$ such that $\mathbf{H} = \mathbf{B} \circ \mathbf{P}$. And conversely, for any symmetric linear operator $\mathbf{B}\colon \alpha \to \alpha$ the composition $\mathbf{H} = \mathbf{B} \circ \mathbf{P}$ is symmetric operator $\mathbf{H}\colon T_{p_0}(M) \to T_{p_0}(M)$ satisfying the relationships (9.18).

Let $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_{n-1}$ be some arbitrary base in hyperplane α. Consider a matrix with the following components

$$b_{ij} = (\boldsymbol{\tau}_i \,|\, \mathbf{B}\boldsymbol{\tau}_j). \tag{10.2}$$

Matrix (10.2) is symmetric, i. e. $b_{ij} = b_{ji}$, this is due to the symmetry of operator $\mathbf{B}$. Matrix with components (10.2) can be obtained from the matrix of operator $\mathbf{B}$ in the base $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_{n-1}$ by means of lowering index procedure:

$$b_{ij} = \sum_{m=1}^{n-1} \rho_{im}\, b^m_j. \tag{10.3}$$

Here ρ_{im} are components of metric tensor for the metric $\mathbf{g}$ restricted to the hyperplane α. Formula (10.3) for b_{ij} can be inverted as follows:

$$b^m_k = \sum_{i=1}^{n-1} \rho^{mi}\, b_{ik}. \tag{10.4}$$

Operator $\mathbf{B}$ is reconstructed by matrix (10.2) according to the formula

$$\mathbf{B}\mathbf{y} = \sum_{a=1}^{n-1}\sum_{k=1}^{n-1}\sum_{m=1}^{n-1} (\boldsymbol{\tau}_a \,|\, \mathbf{y})\, \rho^{ak}\, b^m_k\, \boldsymbol{\tau}_m. \tag{10.5}$$

Formula (10.5) and formula (10.4) inverting (10.3) mean that defining symmetric operator $\mathbf{B}$ in hyperplane α is equivalent to fixing some base $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_{n-1}$ in α and choosing some symmetric matrix b_{ij}. Substituting (10.5) into the formula $\mathbf{H} = \mathbf{B} \circ \mathbf{P}$, for components of operator $\mathbf{H}$ we get the expression, which exactly the same as in (9.11). Therefore theorem 9.4 is the consequence of the following more simple auxiliary theorem.

Theorem 10.1. *Suppose that in tangent space $T_{p_0}(M)$ at some point p_0 on Riemannian manifold of the dimension $n \geqslant 3$ we choose some nonzero vector $\mathbf{v}_0$, mark hyperplane α perpendicular to $\mathbf{v}_0$, mark some base $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_{n-1}$ in α, and choose some symmetric $(n-1)\times(n-1)$ matrix b. Then there exists some hypersurface S passing through the point p_0 tangent to α and there exist some local coordinates $u^1, \ldots, u^{n-1}$ on S such that vectors $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_{n-1}$ are coordinate tangent vectors to S at the point p_0, while b coincides with the matrix of second fundamental form for S at this point.*

Before proving the theorem let's consider the following lemma that states the fact, which is well-known in geometry.

Lemma 10.1. *For any point p_0 on Riemannian manifold M there exist local coordinates $x^1, \ldots, x^n$ such that component of metric connection Γ^k_{ij} vanishes at the point p_0 in these local coordinates.*

Proof. Suppose that $\tilde{x}^1, \ldots, \tilde{x}^n$ are some arbitrary local coordinates in some neighborhood of the point p_0. Without loss of generality we can assume that

$$\tilde{x}^1(p_0) = \ldots = \tilde{x}^n(p_0) = 0.$$

Let $\tilde{\Gamma}^k_{ij}(p_0)$ be components of metric connection at the point p_0 in these local coordinates. Let's define new local coordinates $x^1, \ldots, x^n$ by the following formula

$$x^k = \tilde{x}^k + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \tilde{\Gamma}^k_{ij}(p_0)\, \tilde{x}^i\, \tilde{x}^j. \tag{10.6}$$

For the components of transition matrix S at the point p_0 from (10.6) we get

$$S^k_i(p_0)=\frac{\partial x^k}{\partial\tilde{x}^i}\,\biggr|_{p_0}=\delta^k_i.$$

This means that S is unitary matrix at the point p_0. Matrix $T=S^{-1}$ then is also unitary matrix at the point p_0. In order to calculate components of metric connection Γ^k_{ij} in newly defined local coordinates let's use formula (4.11) from Chapter III:

$$\tilde{\Gamma}^k_{ij}=\sum^n_{m=1}\sum^n_{a=1}\sum^n_{c=1}T^k_m\,S^a_i\,S^c_j\,\Gamma^m_{ac}+\sum^n_{m=1}T^k_m\,\frac{\partial S^m_i}{\partial\tilde{x}^j}. \tag{10.7}$$

If we take into account that S and T are unitary matrices at the point p_0, then for components of metric connection $\Gamma^k_{ij}(p_0)$ and $\tilde{\Gamma}^k_{ij}(p_0)$ from (10.7) we get

$$\tilde{\Gamma}^k_{ij}(p_0)=\Gamma^k_{ij}(p_0)+\frac{\partial^2x^k}{\partial\tilde{x}^i\,\partial\tilde{x}^j}\,\biggr|_{p_0}. \tag{10.8}$$

Differentiating (10.6), we calculate second derivative in (10.8):

$$\frac{\partial^2x^k}{\partial\tilde{x}^i\,\partial\tilde{x}^j}\,\biggr|_{p_0}=\tilde{\Gamma}^k_{ij}(p_0). \tag{10.9}$$

Now from (10.8) and (10.9) we see that components of metric connection Γ^k_{ij} for newly constructed local coordinates (10.6) are zero at the point p_0, as it was stated in lemma. Lemma is proved. □

Remark. Suppose that at the point p_0 we have n vectors $\boldsymbol{\tau}_1,\,\ldots\,,\,\boldsymbol{\tau}_n$ forming the base in tangent space $T_{p_0}(M)$. Local coordinates $x^1,\,\ldots\,,\,x^n$ with $\Gamma^k_{ij}(p_0)=0$ can be chosen so that corresponding coordinate tangent vectors

$$\mathbf{E}_1=\frac{\partial}{\partial x^1},\ \ldots\ ,\ \mathbf{E}_n=\frac{\partial}{\partial x^n} \tag{10.10}$$

at the point p_0 will coincide with vectors $\boldsymbol{\tau}_1,\,\ldots\,,\,\boldsymbol{\tau}_n$. Indeed, the condition of vanishing Γ^k_{ij} at the point p_0 is invariant under the linear change of local coordinates

$$x^k=\sum^n_{i=1}S^k_i\,\tilde{x}^i,\qquad\qquad \tilde{x}^i=\sum^n_{i=1}T^i_k\,x^k,$$

where S and $T=S^{-1}$ are some constant matrices. While vectors (10.10) under such linear change of local coordinates are transformed as follows:

$$\tilde{\mathbf{E}}_i=\sum^n_{i=1}S^k_i\,\mathbf{E}_k,\qquad\qquad \mathbf{E}_k=\sum^n_{i=1}T^i_k\,\tilde{\mathbf{E}}_i.$$

Therefore we can bring vectors $\mathbf{E}_1, \ldots, \mathbf{E}_n$ to the coincidence with $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_n$, keeping condition $\Gamma^k_{ij}(p_0) = 0$ fulfilled.

Proof of theorem 10.1. In theorem 10.1 we consider the point p_0 with nonzero vector $\mathbf{v}_0$ at this point and the base $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_{n-1}$ in hyperplane $\alpha \subset T_{p_0}(M)$ perpendicular to the vector $\mathbf{v}_0$. By denoting

$$\boldsymbol{\tau}_n = \frac{\mathbf{v}_0}{|\mathbf{v}_0|}$$

we complete base $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_{n-1}$ up to the base $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_n$ in $T_{p_0}(M)$. Let's apply lemma 10.1 and choose local coordinates $x^1, \ldots, x^n$ in the neighborhood of p_0 on M such that $\Gamma^k_{ij}(p_0) = 0$ and such that

$$\mathbf{E}_1 = \boldsymbol{\tau}_1,\ \ldots,\ \mathbf{E}_{n-1} = \boldsymbol{\tau}_{n-1}, \qquad \mathbf{E}_n = \boldsymbol{\tau}_n = \frac{\mathbf{v}_0}{|\mathbf{v}_0|}.$$

Moreover, without loss of generality we can assume that $x^1(p_0) = \ldots = x^n(p_0) = 0$. Now let's define hypersurface S by the equation

$$x^n = \frac{1}{2}\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} b_{ij}\, x^i\, x^j \tag{10.11}$$

in local coordinates chosen above. Here b_{ij} are components of constant symmetric matrix b from the statement of theorem 10.1. We can write (10.11) in parametric form by introducing local coordinates $u^1, \ldots, u^{n-1}$ on S:

$$\begin{aligned} &x^k = u^k \quad \text{for} \quad k = 1, \ldots, n-1,\\ &x^n = \frac{1}{2}\sum_{i=1}^{n-1}\sum_{j=1}^{n-1} b_{ij}\, u^i\, u^j. \end{aligned} \tag{10.12}$$

Let's differentiate parametric equations (10.12) with respect to the variable u^i:

$$\frac{\partial x^k}{\partial u^i} = 1 \quad \text{for} \quad i = k, \qquad \frac{\partial x^k}{\partial u^i} = 0 \quad \text{for} \quad i \neq k < n. \tag{10.13}$$

By differentiating x^n with respect to u^i we get the formula

$$\frac{\partial x^n}{\partial u^i} = \sum_{j=1}^{n-1} b_{ij}\, u^j. \tag{10.14}$$

Hypersurface S defined by the equations (10.12) in parametric form passes through the point p_0; this point has zero local coordinates on S: $u^1(p_0) = \ldots = u^{n-1}(p_0) = 0$.

Partial derivatives (10.13) and (10.14) form components of i-th coordinate tangent vector $\boldsymbol{\tau}_i$ of S in local coordinates $x^1, \ldots, x^n$ on M:

$$\tau_i^k = \begin{cases} 1 & \text{for } \ i = k, \\ 0 & \text{for } \ i \neq k < n, \end{cases} \qquad \tau_i^n = \sum_{j=1}^{n-1} b_{ij}\, u^j. \tag{10.15}$$

Substituting coordinates $u^1(p_0) = \ldots = u^{n-1}(p_0)$ of the point p_0 into these formulas, we get $\tau_i^n = 0$. This means that S is tangent to α, while coordinate tangent vectors to S at the point p_0 coincide with the base $\mathbf{E}_1 = \boldsymbol{\tau}_1, \ldots, \mathbf{E}_{n-1} = \boldsymbol{\tau}_{n-1}$ in hyperplane α. This proves first proposal in theorem 10.1.

In order to prove second proposal in theorem 10.1 let's calculate the matrix of second fundamental form of hypersurface S at the point p_0, applying derivational formula of Weingarten (7.9) for this purpose:

$$\nabla_{u^j} \boldsymbol{\tau}_i = \sum_{m=1}^{n-1} \theta_{ij}^m\, \boldsymbol{\tau}_m + \beta_{ij}\, \mathbf{n}. \tag{10.16}$$

Let's calculate components of vector $\nabla_{u^j} \boldsymbol{\tau}_i$ in left hand side of derivational formula (10.16) according to formula (5.8) from Chapter IV:

$$\nabla_{u^j} \tau_i^k = \frac{\partial \tau_i^k}{\partial u^j} + \sum_{m=1}^{n} \sum_{r=1}^{n} \tau_j^m\, \Gamma_{mr}^k\, \tau_i^r. \tag{10.17}$$

At the point $p = p_0$ we have $\Gamma_{mr}^k(p_0) = 0$ due to special choice of local coordinates $x^1, \ldots, x^n$ on M. For τ_i^k we apply formula (10.15). Then

$$\nabla_{u^j} \tau_i^k(p_0) = 0 \ \text{ for } \ k < n, \qquad \nabla_{u^j} \tau_i^n(p_0) = b_{ij}.$$

Let's write these relationships in vectorial form, and let's calculate components of unitary normal vector $\mathbf{n}$ to S at the point p_0:

$$\nabla_{u^j} \boldsymbol{\tau}_i(p_0) = \begin{Vmatrix} 0 \\ \vdots \\ 0 \\ b_{ij} \end{Vmatrix}, \qquad \mathbf{n}(p_0) = \begin{Vmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{Vmatrix}. \tag{10.18}$$

The relationships (10.15) for the quantities τ_j^i, written in vectorial form, yield

$$\boldsymbol{\tau}_1(p_0) = \begin{Vmatrix} 1 \\ \vdots \\ 0 \\ 0 \end{Vmatrix}, \ \ldots, \ \boldsymbol{\tau}_{n-1}(p_0) = \begin{Vmatrix} 0 \\ \vdots \\ 1 \\ 0 \end{Vmatrix}. \tag{10.19}$$

Now, substituting (10.18) and (10.19) into vectorial equality (10.16), we find the values of parameters θ^m_{ij} β_{ij} at the point p_0:

$$\theta^k_{ij}(p_0) = 0, \qquad\qquad \beta_{ij} = b_{ij}.$$

So, we see that matrix of second fundamental form β for the hypersurface S defined by the equation (10.11) at the point p_0 coincides with matrix b. Theorem 10.1 is now proved. □

Along with theorem 10.1 we have proved theorem 9.4 on the second fundamental form. This follows from considerations preceding the statement of theorem 10.1.

§ 11. Newtonian dynamical systems admitting the normal shift.

Having proved the theorem on second quadratic form, let's return to the results of § 6 and § 9. Central points of these two sections are definitions 6.1 and 9.1 that introduce weak and additional normality conditions for Newtonian dynamical systems on Riemannian manifolds. We managed to write each of these two conditions in form of systems of partial differential equations for the force field $\mathbf{F}$ of the dynamical system (see equations (6.16) and (6.17) for weak normality condition, and equations (9.20) and (9.42) for additional normality condition).

Definition 11.1. Weak normality condition from definition 6.1 and additional normality condition from definition 9.1, combined together, form **complete normality** condition for Newtonian dynamical system with force field $\mathbf{F}$ on Riemannian manifold M.

The equations (6.16), (6.17), (9.20), and (9.42) form complete system of normality equations for the force field of dynamical system $\mathbf{F}$. The whole set of these equations is equivalent to complete normality condition from definition 11.1. Identically zero field $\mathbf{F} = 0$ satisfies all these equations, therefore geodesic flow is a simples example of dynamical system that satisfies complete normality condition.

Let's consider some Newtonian dynamical system with force field $\mathbf{F}$ satisfying complete normality condition. Let S be some hypersurface in the manifold M. Let's mark some point $p_0 \in S$ and choose some local coordinates $u^1, \ldots, u^{n-1}$ in the neighborhood of the point p_0 on S. Suppose S' to be some smaller piece of hypersurface S containing the point p_0. Let's draw the trajectories of dynamical system

$$\dot{x}^k = v^k, \qquad\qquad \nabla_t v^k = F^k, \tag{11.1}$$

passing through each point p of S'. We define them by the following initial data:

$$x^k(t)\Big|_{t=0} = x^k(p), \qquad\qquad v^k(t)\Big|_{t=0} = \nu(p)\cdot n^k(p). \tag{11.2}$$

This arranges the shift $f_t\colon S' \to S'_t$ of $S' \subset S$ along the trajectories of the dynamical system (11.1). Piece S' in S can always be chosen belonging to the transformation class (see definition 2.1). Due to additional normality condition being the part of complete normality condition we can choose the function $\nu(p)$ on S' such that the shift $f_t : S' \to S'_t$ approximates normal shift up to a first order derivatives. This means that all functions of deviation $\varphi_1, \ldots, \varphi_{n-1}$ satisfy initial conditions

$$\varphi_k\Big|_{t=0} = 0, \qquad \dot\varphi_k\Big|_{t=0} = 0 \tag{11.3}$$

on initial hypersurface S' (see conditions (7.5) above). Function ν thereby can be chosen belonging to transformation class (see definition 2.2) and normalized by

$$\nu(p_0) = \nu_0, \tag{11.4}$$

where ν_0 is an arbitrary nonzero real number.

Weak normality condition, which also is a part of complete normality condition, means that all functions of deviation $\varphi_1, \ldots, \varphi_{n-1}$ satisfy linear homogeneous differential equations of the second order in t on trajectories of shift:

$$\ddot\varphi_k - \mathcal{A}(t)\,\dot\varphi_k - \mathcal{B}(t)\,\varphi_k = 0. \tag{11.5}$$

Combining differential equations (11.5) and initial data (11.5), we find that functions of deviation $\varphi_1, \ldots, \varphi_{n-1}$ for the shift $f_t : S' \to S'_t$ are identically zero. This, in turn, means that the shift we are considering now is a normal shift in the sense of definition 2.3.

Definition 11.2. Newtonian dynamical system on Riemannian manifold M is called the system **admitting the normal shift**, if for any hypersurface S in M, for any point $p_0 \in S$, and for any real number $\nu_0 \neq 0$ there is a piece S' of hypersurface S belonging to transformation class and containing p_0, and there exists a function $\nu(p)$ belonging to transformation class on S' and normalized by (11.4) such that the shift $f_t\colon S' \to S'_t$ defined by this function is a normal shift along the trajectories of dynamical system.

Definition 11.2 states the central concept of this thesis, the concept of Newtonian dynamical system **admitting the normal shift**. The condition stated i this definition was first introduced in [39]. There it was called **strong normality** condition. In earlier papers [34], [35], [38], and [58] we used more simple version of this condition, which did not include normalization (11.4). This condition was called **normality condition**. It is more natural from geometric point of view, but it is not in a good agreement with the Pfaff equations that arises in our theory. Therefore presently, saying dynamical system **admitting the normal shift**, we imply the definition 11.2.

§ 12. Equivalence of strong and complete normality conditions.

Complete normality condition, which consists of weak normality condition from definition 6.1 and additional normality condition from definition 9.1, is sufficient for the strong normality condition from definition 11.2 to be fulfilled. This follows from above considerations preceding the statement of definition 11.2. Thus, Newtonian dynamical system satisfying both weak and additional normality conditions appears to be admitting the normal shift. It is applicable for arranging normal shift of any preassigned hypersurface. This result can be strengthened in form of the following theorem.

THEOREM 12.1. **Complete** *and* **strong** *normality conditions for Newtonian dynamical systems on Riemannian manifolds are* **equivalent** *to each other.*

PROOF. As we showed in § 11 above, complete normality condition is **sufficient** for the strong normality condition to be fulfilled. Now we are to show that it is **necessary** as well. Suppose that strong normality condition for Newtonian dynamical system (11.1) is fulfilled. This means that for any hypersurface S, for any point $p_0 \in S$, and for any real number $\nu_0 \neq 0$ there is a piece S' of hypersurface S and there exists a function $\nu(p)$ on S' normalized by the condition (11.4) and such that all functions of deviations on shift trajectories are identically zero:

$$\varphi_k(t) = 0 \quad \text{for all} \quad k = 1, \dots, n-1. \tag{12.1}$$

This is the very condition that provides normality of shift $f_t : S' \to S'$. From (12.2) we get that functions of deviations and their first derivatives are zero at initial instant of time, i. e. the initial conditions (11.3) are fulfilled. Hence strong normality condition implies additional normality condition to be fulfilled.

Now we shall show that strong normality condition implies weak normality condition to be fulfilled as well. Let's fix some trajectory $p(t)$ of Newtonian dynamical system (11.1) and let's mark some point $p_0 = p(0)$ on it. Suppose that velocity vector $\mathbf{v}_0 = \mathbf{v}(0)$ at this point is nonzero: $\mathbf{v}_0 \neq 0$. Then in tangent space $T_{p_0}(M)$ we can take hyperplane α perpendicular to the vector v_0. Let's consider various hyperplanes S in M passing through the point p_0. For our preassigned trajectory $p(t)$ to be one of the trajectories of normal shift for S it should be perpendicular to S at the point p_0, i. e. S should be tangent to hyperplane α at this point. Let's define unitary normal vector at p_0 and a real number $\nu_0 \neq 0$:

$$\mathbf{n} = \frac{\mathbf{v}_0}{|\mathbf{v}_0|}, \qquad\qquad \nu_0 = |\mathbf{v}_0|.$$

This defines smooth field of unitary normal vectors on S in some neighborhood of p_0. Relying upon strong normality condition of dynamical system, let's choose some piece S' of hypersurface S containing p_0, and choose some function $\nu(p)$ on S' that satisfies the condition (11.4) and initiates normal shift $f_t : S' \to S'_t$. Let $u^1, \dots, u^{n-1}$ be local coordinates on S'. These local coordinates define variation vectors $\boldsymbol{\tau}_1, \dots, \boldsymbol{\tau}_{n-1}$ and deviation functions

$$\varphi_k = (\boldsymbol{\tau}_i \,|\, \mathbf{v}), \quad \text{where} \quad k = 1, \dots, n-1. \tag{12.2}$$

Time derivatives of variation functions (12.2) are calculated by formula (4.7). Here this formula is written in the following form:

$$\dot{\varphi}_k = (\mathbf{F}\,|\,\boldsymbol{\tau}_k) + (\mathbf{v}\,|\,\nabla_t \boldsymbol{\tau}_k). \tag{12.3}$$

Covariant derivatives $\nabla_t \boldsymbol{\tau}_k$ in (12.3) at the initial instant of time $t = 0$ are calculated by formula (7.14). Using this formula, we take into account that function $\nu(p)$ on initial hypersurface S' satisfies the equation (7.16). Hence for covariant derivatives $\nabla_t \boldsymbol{\tau}_k$ in formula (12.3) we obtain

$$\nabla_t \boldsymbol{\tau}_k \Big|_{t=0} = -\frac{(\mathbf{F}\,|\,\boldsymbol{\tau}_k)}{\nu} \cdot \mathbf{n} - \nu \cdot \sum_{m=1}^{n-1} b_k^m\, \boldsymbol{\tau}_m. \tag{12.4}$$

Normal vector $\mathbf{n}$ of hypersurface S' is directed along the vector of velocity; the function $\nu(p)$ determines the modulus of initial velocity on S'. Therefore

$$\nabla_t \boldsymbol{\tau}_k \Big|_{t=0} = -\frac{(\mathbf{F}\,|\,\boldsymbol{\tau}_k)}{|\mathbf{v}|^2} \cdot \mathbf{v} - |\mathbf{v}| \cdot \sum_{m=1}^{n-1} b_k^m\, \boldsymbol{\tau}_m. \tag{12.5}$$

Strong normality condition, that provides normality of shift $f_t\colon S' \to S'_t$, implies identical in t vanishing of all functions of deviations. In particular this means

$$\varphi_k \Big|_{t=0} = 0, \qquad\qquad \dot{\varphi}_k \Big|_{t=0} = 0. \tag{12.6}$$

Vanishing of φ_k and $\dot{\varphi}_k$ in (12.6) is granted by orthogonality $\boldsymbol{\tau}_k \perp \mathbf{v}$ upon substituting (12.5) into (12.3). Let's consider second derivatives

$$\ddot{\varphi}_k \Big|_{t=0} = 0. \tag{12.7}$$

Their vanishing is also the consequence of $\varphi_k(t) = 0$. We calculate second order derivatives in the left hand side of (12.7) by formula (5.14). As a result we get

$$\boldsymbol{\beta}(\boldsymbol{\tau}_k) - \frac{\boldsymbol{\alpha}(\mathbf{N})}{|\mathbf{v}|}\,(\mathbf{F}\,|\,\boldsymbol{\tau}_k) - |\mathbf{v}| \sum_{m=1}^{n-1} b_k^m\, \boldsymbol{\alpha}(\boldsymbol{\tau}_m) = 0. \tag{12.8}$$

Here covectorial fields $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are defined by their components (5.13). When referred to the marked point p_0, the quantities b_k^m can be interpreted as components of symmetric operator $\mathbf{B}$ in hyperplane α orthogonal to the vector of velocity $\mathbf{v} = \mathbf{v}_0$ (see formula (9.14)). Therefore equations (12.8) can be written as

$$\boldsymbol{\beta}(\boldsymbol{\tau}_k) - \frac{\boldsymbol{\alpha}(\mathbf{N})}{|\mathbf{v}|}\,(\mathbf{F}\,|\,\boldsymbol{\tau}_k) - |\mathbf{v}|\, \boldsymbol{\alpha}(\mathbf{B}\,\boldsymbol{\tau}_k) = 0. \tag{12.9}$$

For further analysis of the equations (12.9) we use the arbitrariness in the choice of hypersurface S passing through the point p_0 and perpendicular to vector $\mathbf{v}_0$ at this point. Let's apply theorem 10.1. According to this theorem vector $\boldsymbol{\tau}_k$ can be replaced by arbitrary vector $\boldsymbol{\tau}$ from hyperplane α, while operator $\mathbf{B}$ in (12.9) can be understood as arbitrary symmetric operator $\mathbf{B}\colon \alpha \to \alpha$. Hence vector $\mathbf{B}\,\boldsymbol{\tau}_k$ in (12.9) can be replaced by some other arbitrary vector from hyperplane α, which do not depend on $\boldsymbol{\tau}$. Due to these reasons equation (12.9) breaks into two parts

$$\boldsymbol{\beta}(\boldsymbol{\tau}) - \frac{\boldsymbol{\alpha}(\mathbf{N})}{|\mathbf{v}|}\,(\mathbf{F}\,|\,\boldsymbol{\tau}) = 0, \qquad\qquad \boldsymbol{\alpha}(\boldsymbol{\tau}) = 0,$$

where $\boldsymbol{\tau} \in \alpha$ is arbitrary vector perpendicular to $\mathbf{v}_0$ at the point p_0. We can avoid restriction $\boldsymbol{\tau} \perp \mathbf{v}_0$, if we substitute $\boldsymbol{\tau}$ by $\mathbf{P}\boldsymbol{\tau}$:

$$\boldsymbol{\beta}(\mathbf{P}\boldsymbol{\tau}) = \frac{\boldsymbol{\alpha}(\mathbf{N})}{|\mathbf{v}|}\,(\mathbf{F}\,|\,\mathbf{P}\boldsymbol{\tau}), \qquad\qquad \boldsymbol{\alpha}(\mathbf{P}\boldsymbol{\tau}) = 0. \tag{12.10}$$

Here $\boldsymbol{\tau}$ is arbitrary vector at the point p_0 on M. Second equation (12.10) coincides with first equation (6.11), since $\boldsymbol{\tau}$ in (12.10) and $\nabla_t\boldsymbol{\tau}$ in (6.11) stand for arbitrary vectors at marked point p_0 on the trajectory of shift. First equation (12.10) coincides with second equation (6.11). In order to see it one should only denote

$$\mathcal{A} = \frac{\boldsymbol{\alpha}(\mathbf{N})}{|\mathbf{v}|}, \qquad\qquad \mathcal{B} = \frac{\boldsymbol{\beta}(\mathbf{N})}{|\mathbf{v}|} - \mathcal{A}\,\frac{(\mathbf{F}\,|\,\mathbf{N})}{|\mathbf{v}|} \tag{12.11}$$

(compare with (6.10) in § 6). Now let's use arbitrariness in the choice of point p_0 and in the choice of vector $\mathbf{v}_0$ at this point. This means that the equations (12.10) coinciding with weak normality equations (6.17) and (6.17) are fulfilled at all points $q = (p, \mathbf{v})$ of tangent bundle TM, where $\mathbf{v} \neq 0$.

Now let's take some arbitrary function of deviation φ on the trajectory $p(t)$ and form linear combination of its time derivatives

$$\ddot{\varphi} - \mathcal{A}\,\dot{\varphi} - \mathcal{B}\,\varphi, \tag{12.12}$$

using parameters $\mathcal{A}$ and $\mathcal{B}$ from (12.11) as coefficients in it. Here we do not associate vector of variation $\boldsymbol{\tau}$ in $\varphi = (\mathbf{v}\,|\,\boldsymbol{\tau})$ with any hypersurface, and therefore we do not restrict it by the condition $\mathbf{v} \perp \boldsymbol{\tau}$, as it was in the case of normal shift. Here $\boldsymbol{\tau} = \boldsymbol{\tau}(t)$ is arbitrary vector function that varies in t according to the differential equation (3.10). From (3.10) we derive the equalities (5.12) and (5.14) for $\dot{\varphi}$ and $\ddot{\varphi}$. Substituting them into (12.12) and taking into account (12.10) and (12.11), we find that linear combination is identically zero: $\ddot{\varphi} - \mathcal{A}\,\dot{\varphi} - \mathcal{B}\,\varphi = 0$. This means that arbitrary function of deviation satisfies homogeneous linear ordinary differential equation of the second order with respect to parameter t on trajectories of dynamical system. According to definition 6.1 this is exactly the weak normality condition for that dynamical system. Theorem 12.1 is proved.

CHAPTER VI

PROBLEM OF METRIZABILITY.

§ 1. Geodesic flows of conformally equivalent metrics.

Concept of dynamical system **admitting the normal shift** is a central concept of this thesis. For Newtonian dynamical system on Riemannian manifolds of the dimension $n \geqslant 2$ it is expressed by **strong normality** condition (see definition 11.2 in Chapter V). This is purely geometric condition. Theorem 12.1 in Chapter V reduces **strong normality** condition to the couple of other geometric conditions: **weak normality** condition and **additional normality** condition. These two conditions, joint together, form **complete normality** condition. Theorem 6.2 in Chapter V transforms **weak normality** condition to the system of partial differential equations for the force field $\mathbf{F}$ of Newtonian dynamical system:

$$\sum_{i=1}^{n}\left(v^{-1}\,F_i+\sum_{j=1}^{n}\tilde\nabla_i\left(N^j\,F_j\right)\right)P^i_k=0. \tag{1.1}$$

$$\begin{gathered}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\nabla_iF_j+\nabla_jF_i-2\,v^{-2}\,F_i\,F_j\right)N^j\,P^i_k\,+\\ +\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{F^j\,\tilde\nabla_jF_i}{v}-\sum_{r=1}^{n}\frac{N^r\,N^j\,\tilde\nabla_jF_r}{v}\,F_i\right)P^i_k=0.\end{gathered} \tag{1.2}$$

Additional normality condition separates two quite different subcases: **two-dimensional case** $\dim M=n=2$ and **multidimensional case** $\dim M=n\geqslant 3$. In two dimensional case additional normality condition is fulfilled unconditionally for any Newtonian dynamical system (see theorem 9.1 in Chapter V). We shall not consider two dimensional case at all, it is considered in **other thesis** [36]. In multidimensional case theorem 9.5 in Chapter V transforms **additional normality** condition to the system of partial differential equations for the force field $\mathbf{F}$:

$$\begin{gathered}\sum_{i=1}^{n}\sum_{j=1}^{n}P^i_\varepsilon\,P^j_\sigma\left(\sum_{m=1}^{n}N^m\,\frac{F_i\,\tilde\nabla_mF_j}{v}-\nabla_iF_j\right)=\\ =\sum_{i=1}^{n}\sum_{j=1}^{n}P^i_\varepsilon\,P^j_\sigma\left(\sum_{m=1}^{n}N^m\,\frac{F_j\,\tilde\nabla_mF_i}{v}-\nabla_jF_i\right).\end{gathered} \tag{1.3}$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} P^{j}_{\sigma}\,\tilde{\nabla}_{j}F^{i}\,P^{\varepsilon}_{i}=\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{m=1}^{n}\frac{P^{j}_{m}\,\tilde{\nabla}_{j}F^{i}\,P^{m}_{i}}{n-1}\,P^{\varepsilon}_{\sigma}. \tag{1.4}$$

Once normality equations (1.1), (1.2), (1.3), and (1.4) are written, the problem of their compatibility arises. We begin to study this problem by constructing some simplest solutions for these equations. Existence of such solutions follows from geometric nature of the equations (1.1), (1.2), (1.3), and (1.4). Let M be Riemannian manifold of the dimension $n \geqslant 3$ and let $\mathbf{g}$ be metric tensor of M. Geodesic flow of metric $\mathbf{g}$ is a Newtonian dynamical system admitting the normal shift in metric $\mathbf{g}$ (see § 1 in Chapter V). Let's consider the metric conformally equivalent to metric $\mathbf{g}$:

$$\tilde{\mathbf{g}} = e^{-2f}\,\mathbf{g}. \tag{1.5}$$

Here f is some smooth function on M (non-extended scalar field). Geodesic flow of metric (1.5) is obviously a dynamical system admitting the normal shift in metric (1.5) by itself. However, angles, measured in metric $\mathbf{g}$, have the same measure as in metric $\tilde{\mathbf{g}}$, i. e. normal shift in metric $\tilde{\mathbf{g}}$ remains to be normal shift with respect to basic metric $\mathbf{g}$ of manifold M. Hence geodesic flow of metric (1.5) is a dynamical system admitting the normal shift in metric $\mathbf{g}$.

Components of metric connection $\tilde{\Gamma}^{k}_{ij}$ for conformally equivalent metric (1.5) are calculated by standard formula

$$\tilde{\Gamma}^{k}_{ij}=\frac{1}{2}\sum_{s=1}^{n}\tilde{g}^{ks}\left(\frac{\partial\tilde{g}_{sj}}{\partial x^{i}}+\frac{\partial\tilde{g}_{is}}{\partial x^{j}}-\frac{\partial\tilde{g}_{ij}}{\partial x^{s}}\right). \tag{1.6}$$

Substituting $\tilde{g}_{ij} = e^{-2f}\,g_{ij}$ and $\tilde{g}^{ij} = e^{2f}\,g^{ij}$ into the formula (1.6), we get

$$\tilde{\Gamma}^{k}_{ij}=\Gamma^{k}_{ij}-\nabla_{i}f\,\delta^{k}_{j}-\nabla_{j}f\,\delta^{k}_{i}+g_{ij}\sum_{s=1}^{n}g^{ks}\,\nabla_{s}f. \tag{1.7}$$

Let's substitute (1.7) into the equations of geodesic lines for metric $\tilde{\mathbf{g}}$:

$$\ddot{x}^{k}+\sum_{i=1}^{n}\sum_{j=1}^{n}\tilde{\Gamma}^{k}_{ij}\,\dot{x}^{i}\,\dot{x}^{j}=0. \tag{1.8}$$

In basic metric $\mathbf{g}$ of manifold M the equations (1.8) are written as the equations of Newtonian dynamical system

$$\dot{x}^{k}=v^{k}, \qquad\qquad \nabla_{t}v^{k}=F^{k} \tag{1.9}$$

with the force field $\mathbf{F}$ given by the following components:

$$F^{k}=-|\mathbf{v}|^{2}\sum_{s=1}^{n}g^{ks}\,\nabla_{s}f+2\sum_{s=1}^{n}\nabla_{s}f\,v^{s}\,v^{k}. \tag{1.10}$$

Force field (1.10) satisfies all normality equations (1.1), (1.2), (1.3), and (1.4). This fact follows from geometric nature of the field (1.10) as described above. But one can check it by direct substitution of (1.10) into the equations (1.1), (1.2), (1.3), and (1.4). Functions (1.10) form parametric family of solutions of these normality equations. This family is parameterized by one scalar function on the manifold M.

§ 2. Inheriting the trajectories and trajectory equivalence.

Let $f_t : S \to S_t$ be the shift of hypersurface S along trajectories of Newtonian dynamical system (1.9) with force field $\mathbf{F}$. The same shift can be implemented by Newtonian dynamical system with some other force field $\tilde{\mathbf{F}}$, provided the trajectories of first system (those which are used in shift procedure) are among trajectories of the second one. Let's generalize this situation by the following definition.

Definition 2.1. Suppose that we have two Newtonian dynamical systems with force fields $\mathbf{F}$ and $\tilde{\mathbf{F}}$ on Riemannian manifold M. We say that second system **inherits** trajectories of first system, if each trajectory of second system as line (up to a regular reparametrization) coincides with some trajectory of first dynamical system.

Definition 2.2. Two Newtonian dynamical systems on Riemannian manifold M inheriting trajectories of each other are called **trajectory equivalent**

The concepts of **trajectory inheriting** and **trajectory equivalence** can be traced back to the classical paper [50]. The concept of **geodesic equivalence** for Riemannian and affine manifolds is nearly related concept (see [55–57]). Fore general definition of trajectory equivalence were stated in [43–48], where it was used for topological classification of dynamical systems.

Definition 2.3. Newtonian dynamical system on Riemannian manifold M is called **metrizable**, if it inherits trajectories of dynamical system with force field (1.10) defined by conformally equivalent metric $\tilde{\mathbf{g}} = e^{-2f}\,\mathbf{g}$.

Suppose that we have chosen some point p_0 of Riemannian manifold M and some vector $\tilde{\mathbf{v}}_0 \neq 0$ at this point. Then we have a trajectory $\tilde{p}(t)$ of second dynamical system such that $\tilde{p}(0) = p_0$ and $\tilde{\mathbf{v}}(0) = \mathbf{v}_0$, where $\tilde{\mathbf{v}}(t)$ is a velocity vector on this trajectory. If second dynamical system inherits trajectories of first system, then there exists the trajectory $p(\tau)$ of the first system coinciding with $\tilde{p}(t)$. This means that there exists the regular reparametrization $\tau = T(t)$ such that $p(T(t) = \tilde{p}(t)$. Without loss of generality we can assume that $T(0) = 0$. Velocity vectors on common trajectory of these two dynamical systems are bound by relationship

$$\tilde{\mathbf{v}}(t) = \frac{dT}{dt}\,\mathbf{v}(T(t)). \tag{2.1}$$

Let's apply covariant differentiation ∇_t to both sides of the equality (2.1) considered as vector fields on the curve $p(t)$. In order to apply such covariant differentiation we should choose local coordinates $x^1, \ldots, x^n$ in the neighborhood of marked point p_0 and should use formula (4.1) from Chapter IV. We should also take into account that $\nabla_\tau \mathbf{v} = \mathbf{F}$ and $\nabla_t \tilde{\mathbf{v}} = \tilde{\mathbf{F}}$. Upon completing these calculations we get

$$\tilde{\mathbf{F}}(p, \tilde{\mathbf{v}}) = \left(\frac{dT}{dt}\right)^2 \mathbf{F}(p, \mathbf{v}) + \frac{d^2T}{dt^2}\,\mathbf{v}. \tag{2.2}$$

The relationship (2.2) binds force fields of two dynamical systems. It is non-local, since $\mathbf{F}$ and $\tilde{\mathbf{F}}$ are taken in two different points $q = (p, \mathbf{v})$ and $\tilde{q} = (p, \tilde{\mathbf{v}})$ on tangent bundle TM. But there are some cases when it reduces to the local relationship. Let $\mathbf{F}$ be homogeneous function in its vectorial argument $\mathbf{v}$, and let γ be degree of homogeneity. Then from (2.1) we derive

$$\mathbf{F}(p, \mathbf{v}) = \left(\frac{dT}{dt}\right)^{-\gamma} \mathbf{F}(p, \tilde{\mathbf{v}}). \tag{2.3}$$

Since $\tau = T(t)$ is regular reparametrization, we have

$$\frac{dT}{dt} \neq 0. \tag{2.4}$$

Inequality (2.4) provides correctness of (2.3). Suppose that $\gamma = 2$. This is the very case applicable to the force field (1.10) that corresponds to geodesic flow of conformally equivalent metric $\tilde{\mathbf{g}} = e^{-2f}\,\mathbf{g}$. Other cases $\gamma \neq 2$ were considered in [41], they are not interesting for us now.

Let's substitute (2.3) with $\gamma = 2$ into the formula (2.2) and let's express vector $\mathbf{v}$ through the vector $\tilde{\mathbf{v}}$ in it. Then (2.2) is written as

$$\tilde{\mathbf{F}}(p, \tilde{\mathbf{v}}) = \mathbf{F}(p, \tilde{\mathbf{v}}) + \frac{d^2T}{dt^2} \left(\frac{dT}{dt}\right)^{-1} \tilde{\mathbf{v}}. \tag{2.5}$$

Due to the arbitrariness in choice of point p and in choice of vector $\tilde{\mathbf{v}} \neq 0$ at this point the relationship (2.5) mean that the field $\tilde{\mathbf{F}}$ differs from $\mathbf{F}$ by a vector directed along the vector of velocity. Let's express this fact as follows:

$$\tilde{\mathbf{F}}(p, \mathbf{v}) = \mathbf{F}(p, \mathbf{v}) + \frac{H(p, \mathbf{v})}{|\mathbf{v}|}\,\mathbf{v}. \tag{2.6}$$

Here $H = H(p, \mathbf{v})$ is some arbitrary scalar field from extended algebra of tensor fields on M. We replaced $\tilde{\mathbf{v}}$ by $\mathbf{v}$ for to simplify the appearance of formula.

Theorem 2.1. *Consider some two Newtonian dynamical systems on Riemannian manifold M. Suppose that the force field of first system is homogeneous function of degree 2 with respect to velocity vector $\mathbf{v}$ in fibers of tangent bundle TM. Second dynamical system inherits trajectories of the first system if and only if its force field $\tilde{\mathbf{F}}$ is given by formula* (2.6).

Proof. If second dynamical system inherits trajectories of the first one, then its force field is given by (2.6). This was shown above. Conversely, suppose that force field of the second dynamical system is defined by formula (2.6). Let's take the trajectory $\tilde{p}(t)$ of the second system, and ret's arrange reparametrization $\tau = T(t)$ of this curve defined by the solution of the following ordinary differential equation:

$$\ddot{T} = \frac{H(\tilde{p}(t), \tilde{\mathbf{v}}(t))}{|\tilde{\mathbf{v}}(t)|}\,\dot{T}. \tag{2.7}$$

Solution of the equation (2.7) can be normalized by the condition

$$\dot{T}\Big|_{t=0} = a \neq 0$$

that provides regularity of reparametrization $T\colon t \to \tau$. Denote by $\theta\colon \tau \to t$ the inverse reparametrization and take $p(\tau) = \tilde{p}(\theta(\tau))$. Then $\tilde{p}(t) = p(T(t))$, and the relationship (2.1) is fulfilled. Differentiating (2.1), we get

$$\tilde{\mathbf{F}}(\tilde{p}, \tilde{\mathbf{v}}) = \dot{T}^2\, \nabla_\tau \mathbf{v} + \ddot{T}\, \mathbf{v}. \tag{2.8}$$

Now substitute $\ddot{T}$, obtained from (2.7), into the formula (2.8). Then take into account the relationship (2.1) by itself and formula (2.6) for $\tilde{\mathbf{F}}$. As a result we get

$$\mathbf{F}(\tilde{p}, \tilde{\mathbf{v}}) = \dot{T}^2\, \nabla_\tau \mathbf{v}. \tag{2.9}$$

Let's apply formula (2.1) written as $\tilde{\mathbf{v}} = \dot{T}\,\mathbf{v}$ and take into account homogeneity of force field $\mathbf{F}(\tilde{p}, \tilde{\mathbf{v}})$ in its vectorial argument. The degree of homogeneity $\gamma = 2$. Then the equality (2.9) is brought to

$$\nabla_\tau \mathbf{v} = \mathbf{F}(\tilde{p}, \mathbf{v}). \tag{2.10}$$

This relationship, which we have just derived, means that curve $\tilde{p}(t)$ after reparametrization $p(\tau) = \tilde{p}(\theta(\tau))$ becomes the trajectory of Newtonian dynamical system with force field $\mathbf{F}$. □

THEOREM 2.2. *Newtonian dynamical system on Riemannian manifold* $\mathbf{M}$ *is metrizable if and only if its force field has the components given by formula*

$$F^k = -|\mathbf{v}|^2 \sum_{s=1}^{n} g^{ks}\, \nabla_s f + 2 \sum_{s=1}^{n} \nabla_s f\, v^s\, v^k + \frac{H}{|\mathbf{v}|}\, v^k, \tag{2.11}$$

where f is an ordinary (not extended) scalar field, while H is a scalar field from extended algebra of tensor fields on M.

Theorem 2.2 is a simple consequence of previous theorem 2.1. It doesn't require the separate proof.

§ 3. Problem of metrizability.

Problem of metrizability considered in [41] and [59] is stated as follows: **Which** of Newtonian dynamical systems admitting the normal shift **are metrizable**, i. e. which of them have trajectories coinciding with trajectories of geodesic flow for conformally equivalent metric $\tilde{\mathbf{g}} = e^{-2f}\,\mathbf{g}$. Theorem 2.2 gives easy way for solving this problem. Instead of looking for metrizable dynamical systems among those admitting the normal shift, we can look for systems admitting the normal shift in the set of metrizable systems. This is easier, since according to the theorem 2.2 force field of metrizable dynamical systems has quite definite form (2.11).

Let's substitute (2.11) into the normality equations (1.1), (1.2), (1.3), and (1.4). This will require substantial amount of calculations. Let's begin with (1.1). Here

$$\sum_{j=1}^{n} N^j\,F_j = H + |\mathbf{v}|\sum_{s=1}^{n} v^s\,\nabla_s f.$$

Applying covariant derivative $\tilde{\nabla}_i$ to this equality, we use formulas (5.5) and (5.6) from Chapter V. As a result we obtain

$$\sum_{j=1}^{n}\tilde{\nabla}_i(N^j\,F_j) = \tilde{\nabla}_i H + N_i\sum_{s=1}^{n} v^s\,\nabla_s f + |\mathbf{v}|\,\nabla_i f.$$

Upon contracting with the operator of projection $\mathbf{P}$ we get

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\tilde{\nabla}_i(N^j\,F_j)\,P^i_k = \sum_{i=1}^{n}\tilde{\nabla}_i H\,P^i_k + |\mathbf{v}|\sum_{i=1}^{n}\nabla_i f\,P^i_k. \tag{3.1}$$

Moreover, we have the following relationship being immediate consequence of (2.11):

$$\sum_{i=1}^{n}|\mathbf{v}|^{-1}\,F_i\,P^i_k = -|\mathbf{v}|\sum_{i=1}^{n}\nabla_i f\,P^i_k. \tag{3.2}$$

If we add left hand sides of (3.2) and (3.1), we obtain the expression coinciding with left hand side of normality equation (1.1). Therefore, when applied to the force field (2.11), the normality equation (1.1) gives

$$\sum_{i=1}^{n}\tilde{\nabla}_i H\,P^i_k = 0. \tag{3.3}$$

As a second step, let's substitute (2.11) into the equation (1.2). Upon opening brackets in left hand side of (1.2) we get five terms. Let's calculate third term:

$$\begin{aligned}-\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{2\,F_i\,F_j\,N^j\,P^i_k}{|\mathbf{v}|^2} &= 2\,H\sum_{i=1}^{n}\nabla_i f\;\,P^i_k\,+\\ &+2\,|\mathbf{v}|\sum_{i=1}^{n}\sum_{s=1}^{n} v^s\,\nabla_s f\,\nabla_i f\,P^i_k.\end{aligned} \tag{3.4}$$

Then find derivatives $\nabla_i F_j$ for the force field (2.11):

$$\nabla_i F_j = -|\mathbf{v}|^2\,\nabla_i\nabla_j f + 2\sum_{s=1}^{n}\nabla_i\nabla_s f\,v^s\,v_j + \nabla_i H\,N_j. \tag{3.5}$$

For first and second term in left hand side of the equation (1.2) from (3.5) we get

$$\begin{aligned}&\sum_{i=1}^{n}\sum_{j=1}^{n}\nabla_iF_j\,N^j\,P^i_k=|\mathbf{v}|^2\sum_{i=1}^{n}\sum_{j=1}^{n}\nabla_i\nabla_jf\,N^j\,P^i_k+\sum_{i=1}^{n}\nabla_iH\,P^i_k,\\&\sum_{i=1}^{n}\sum_{j=1}^{n}\nabla_jF_i\,N^j\,P^i_k=-|\mathbf{v}|^2\sum_{i=1}^{n}\sum_{j=1}^{n}\nabla_j\nabla_if\,N^j\,P^i_k.\end{aligned}$$

Let's add (3.4) and two above equalities. The result is as follows:

$$\begin{gathered}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\nabla_iF_j+\nabla_jF_i-\frac{2\,F_i\,F_j}{|\mathbf{v}|}\right)N^j\,P^i_k=\sum_{i=1}^{n}\nabla_iH\,P^i_k+\\+2\,H\sum_{i=1}^{n}\nabla_if\;P^i_k+2\,|\mathbf{v}|\sum_{i=1}^{n}\sum_{s=1}^{n}v^s\,\nabla_sf\,\nabla_if\,P^i_k.\end{gathered}\tag{3.6}$$

In deriving (3.6) we took into account that $\nabla_i\nabla_jf=\nabla_j\nabla_if$. This follows from formula (7.10) in Chapter III because tensor of torsion here is equal to zero and because f is ordinary (not extended) scalar field on the manifold M.

Further let's calculate covariant derivative $\tilde\nabla_jF_r$ for the force field of metrizable dynamical system given by formula (2.11):

$$\begin{gathered}\tilde\nabla_jF_r=2\,\nabla_jf\,v_r-2\,\nabla_rf\,v_j+\tilde\nabla_jH\,N_r+\\+2\,g_{rj}\sum_{s=1}^{n}\nabla_sf\,v^s+\frac{H}{|\mathbf{v}|}\sum_{s=1}^{n}g_{sj}\,P^s_r.\end{gathered}\tag{3.7}$$

Contracting this expression with $N^r\,N^j$ and with P^r_k, we derive

$$\begin{gathered}\sum_{j=1}^{n}\sum_{r=1}^{n}N^r\,N^j\,\tilde\nabla_jF_r=\sum_{j=1}^{n}\tilde\nabla_jH\,N^j+2\sum_{s=1}^{n}\nabla_sf\,v^s,\\\sum_{i=1}^{n}\tilde\nabla_jF_i\,P^i_k=\sum_{i=1}^{n}\left(\left(2\sum_{s=1}^{n}\nabla_sf\,v^s+\frac{H}{|\mathbf{v}|}\right)g_{ij}-2\,v_j\,\nabla_if\right)P^i_k.\end{gathered}$$

Let's combine these expressions with (3.2) and (2.11). As a result we get

$$\begin{gathered}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{F^j\,\tilde\nabla_jF_i}{|\mathbf{v}|}-\sum_{r=1}^{n}\frac{N^r\,N^j\,\tilde\nabla_jF_r}{|\mathbf{v}|}\,F_i\right)P^i_k=\\=\sum_{i=1}^{n}\nabla_if\,P^i_k\left(\sum_{j=1}^{n}\tilde\nabla_jH\,v^j-3\,H-2\,|\mathbf{v}|\sum_{s=1}^{n}\nabla_sf\,v^s\right).\end{gathered}\tag{3.8}$$

And finally, let's add the equalities (3.5) and (3.8). This yields the equality

$$\sum_{i=1}^{n}\left(\nabla_i H+\left(\sum_{j=1}^{n} v^j\,\tilde\nabla_j H-H\right)\nabla_i f\right)P^i_k=0. \tag{3.9}$$

The equality (3.9) is an ultimate result of substituting (2.11) into (1.2). It should be understood as a system of differential equations for H.

Let's study the equations (3.3) and (3.9) for extended scalar field H. In order to do it we choose some local map $(U,x^1,\ldots,x^n)$ on M and associated map $(\pi^{-1}(U),x^1,\ldots,x^n,v^1,\ldots,v^n)$ on tangent bundle. Then scalar field H is expressed by function $H(x^1,\ldots,x^n,v^1,\ldots,v^n)$, while equation (3.3) takes the form

$$\sum_{i=1}^{n}\frac{\partial H}{\partial v^i}\,P^i_k=0. \tag{3.10}$$

Recall that P^i_k are components of projector onto the hyperplane perpendicular to velocity vector. Therefore the equation (3.10) means that velocity gradient of H is projected to zero. This mean that $\tilde\nabla H$ is directed along the vector $\mathbf{N}$, which, in turn, is unit vector directed along $\mathbf{v}$. Covariant components of the vector $\mathbf{N}$ can be represented by velocity gradient of scalar field $v=|\mathbf{v}|$:

$$N_i=\tilde\nabla_i v=\frac{\partial v}{\partial v^i} \tag{3.11}$$

(see formulas (5.6) in Chapter V). Now we use the following well-known fact.

Lemma 3.1. *If velocity gradients of two scalar fields H and $v=|\mathbf{v}|$ are proportional at all point of tangent bundle TM, except for those where $\mathbf{v}=0$, then H depends only on modulus of velocity vector, i. e. $H=H(x^1,\ldots,x^n,v)$.*

Proof. Let's fix some point $p\in M$ and consider the fiber of tangent bundle over this point. In such fiber lets consider the point where velocity gradient of $v=|\mathbf{v}|$ iz nonzero. According to (3.11) this is equivalent to $\mathbf{v}\neq 0$. In the neighborhood of such point $q=(p,\mathbf{v})$ cartesian coordinates $v^1,\ldots,v^n$ in this fiber can be replaced by local coordinates $u^1,\ldots,u^n$ such that

$$\begin{cases} u^1=v(v^1,\ldots,v^n),\\ u^2=u^2(v^1,\ldots,v^n),\\ \ldots\ldots\ldots\ldots\ldots\\ u^n=u^n(v^1,\ldots,v^n).\end{cases}$$

In these coordinates $v=v(u^1,\ldots,u^n)=u^1$, therefore velocity gradient of v is represented by unitary vector: $\tilde\nabla v=(1,0,\ldots,0)$. Condition of proportionality for velocity gradients of H and v, which we have in the statement of lemma, is written as a condition of collinearity for two vector-columns, one being unitary vector-column:

$$\left\|\begin{matrix} \partial H/\partial u^1 \\ \partial H/\partial u^2 \\ \vdots \\ \partial H/\partial u^n \end{matrix}\right\|, \qquad \left\|\begin{matrix} 1 \\ 0 \\ \vdots \\ 0 \end{matrix}\right\|. \tag{3.12}$$

From collinearity of vector-columns (3.12) we derive

$$\frac{\partial H}{\partial u^2} = 0, \ \ldots, \ \frac{\partial H}{\partial u^n} = 0.$$

This means that H depend only on the variable $u^1 = v$ within the fiber over fixed point $p \in M$. If we recall that H can depend on p, then we get $H = H(x^1, \ldots, x^n, v)$. Lemma is proved. □

Let's proceed with studying the equations (3.3) and (3.9) for H in local map $(\pi^{-1}(U), x^1, \ldots, x^n, v^1, \ldots, v^n)$ on tangent bundle TM. According to lemma 3.1 the equation (3.3) written in form of (3.10) determines the dependence of H on vector $\mathbf{v}$ within fibers of tangent bundle: H depends on $\mathbf{v}$ only through its dependence on $v = |\mathbf{v}|$. Taking into account this circumstance, we reduce the equation (3.9) for $H = H(x^1, \ldots, x^n, v)$ to the following form:

$$\sum_{i=1}^{n} \left(\nabla_i H + \left(v\,\frac{\partial H}{\partial v} - H\right)\nabla_i f\right) P^i_k = 0. \tag{3.13}$$

Covariant derivatives $\nabla_i f$ are easily calculated, they coincide with partial derivatives since f doesn't depend on vector of velocity $\mathbf{v}$:

$$\nabla_i f = \frac{\partial f}{\partial x^i}. \tag{3.14}$$

In order to calculate $\nabla_i H$ we apply formula (7.3) from Chapter III. When applied to scalar field H this formula is written as follows:

$$\nabla_i H = \frac{\partial H}{\partial x^i} - \sum_{j=1}^{n}\sum_{k=1}^{n} \Gamma^k_{ij}\, v^j\, \frac{\partial H}{\partial v^k}. \tag{3.15}$$

Note that, calculating partial derivatives in (3.15), we should treat $H(x^1, \ldots, x^n, v)$ as the function of $2n$ variables $x^1, \ldots, x^n, v^1, \ldots, v^n$, where dependence of v upon these variables is given by the following expression:

$$v = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} g_{ij}(x^1, \ldots, x^n)\, v^i\, v^j}\,.$$

Therefore formula (3.15) for covariant derivatives $\nabla_i H$ is brought to the form:

$$\nabla_i H=\frac{\partial H}{\partial x^i}+\frac{\partial H}{\partial v}\sum_{j=1}^{n}\sum_{k=1}^{n}\left(\frac{1}{2v}\,\frac{\partial g_{jk}}{\partial x^i}\,v^j\,v^k-\Gamma^k_{ij}\,v^j\,N_k\right). \tag{3.16}$$

Here partial derivatives of the function $H=H(x^1,\ldots,x^n,v)$ are calculated respective to its natural set of arguments $x^1,\,\ldots\,,\,x^n$ and v. In order to simplify (3.16) we use formula (1.2) from Chapter V, that expresses Γ^k_{ij}. Then we get the following result, which could seem striking at first sight:

$$\nabla_i H=\frac{\partial H}{\partial x^i}. \tag{3.17}$$

Components of spatial gradient for extended scalar field H depending **only** on **modulus** of velocity vector are calculated as **partial derivatives** of its representation in local map $H=H(x^1,\ldots,x^n,v)$ with respect to spatial variables $x^1,\,\ldots\,,\,x^n$.

Now let's substitute (3.14) and (3.17) into the equation (3.13). As a result of this substitution we bring this equation to the form

$$\sum_{i=1}^{n}\left(\frac{\partial H}{\partial x^i}+\left(v\,\frac{\partial H}{\partial v}-H\right)\frac{\partial f}{\partial x^i}\right)P^i_k=0. \tag{3.18}$$

Note that in equation (3.18) components of projection operator $\mathbf{P}$ depend on the direction of velocity vector. Other terms

$$L_i=\frac{\partial H}{\partial x^i}+\left(v\,\frac{\partial H}{\partial v}-H\right)\frac{\partial f}{\partial x^i} \tag{3.19}$$

depend only on modulus of velocity vector $v=|\mathbf{v}|$. Keeping the value of $|\mathbf{v}|$ unchanged, we can rotate vector $\mathbf{v}$ so that it will become perpendicular to the vector $\mathbf{L}$, whose covariant components are given by formula (3.19). Then we shall get the equality $\mathbf{P}(\mathbf{L})=\mathbf{L}$ that reduces (3.18) to the following form

$$\frac{\partial H}{\partial x^i}+\left(v\,\frac{\partial H}{\partial v}-H\right)\frac{\partial f}{\partial x^i}=0. \tag{3.20}$$

Now let's change variables in (3.20) replacing $x^1,\,\ldots\,,\,x^n,v$ by $\tilde x^1,\,\ldots\,,\,\tilde x^n,\tilde v$:

$$\begin{cases}\tilde x^1=x^1,\\ \ldots\ldots\\ \tilde x^n=x^n,\\ \tilde v=v\,\exp(-f(x^1,\ldots,x^n)).\end{cases} \tag{3.21}$$

Upon making change of variables (3.21) the equations (3.18) look like

$$\frac{\partial H}{\partial\tilde x^i}-H\,\frac{\partial f}{\partial\tilde x^i}=0. \tag{3.22}$$

We can make these equations more simple by writing them as follows:

$$\frac{\partial}{\partial\tilde{x}^i}\left(H\,e^{-f}\right)=0. \tag{3.23}$$

From (3.23) now it's clear that these equations are compatible and general form of their solution is determined by arbitrary function of one variable $H(\tilde{v})$:

$$H(\tilde{x}^1,\ldots,\tilde{x}^n,\tilde{v})=H(\tilde{v})\,e^f. \tag{3.24}$$

By inverting (3.21) we can return to initial variables $x^1,\ \ldots\ ,\ x^2$:

$$H(x^1,\ldots,x^n,v)=H(v\,e^{-f})\,e^f. \tag{3.25}$$

Formula (3.25) determines general solution for the system of differential equations (3.20). We shall formulate this result in form of the following theorem.

THEOREM 3.1. *Newtonian dynamical system on Riemannian manifold M with metric $\mathbf{g}$ satisfying weak normality condition is metrizable by conformally equivalent metric $\tilde{\mathbf{g}}=e^{-2f}\,\mathbf{g}$ if and only if its force field $\mathbf{F}$ is given by formula*

$$F_k=-|\mathbf{v}|^2\,\nabla_k f+2\sum_{s=1}^{n}\nabla_s f\,v^s\,v_k+\frac{H(v\,e^{-f})\,e^f}{|\mathbf{v}|}\,v_k. \tag{3.26}$$

Theorem 3.1 is not complete solution for problem of metrizability formulated in the beginning of this section. In order to have complete solution we should substitute (3.26) into additional normality equations (1.3) and (1.4). From (3.5) we derive

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\nabla_iF_j\,P^i_\varepsilon\,P^j_\sigma=-|\mathbf{v}|^2\sum_{i=1}^{n}\sum_{j=1}^{n}\nabla_i\nabla_jf\,P^i_\varepsilon\,P^j_\sigma. \tag{3.27}$$

In a similar way from formula (3.7) we derive

$$\sum_{m=1}^{n}\sum_{j=1}^{n}N^m\,\tilde{\nabla}_mF_j\,P^j_\sigma=-2\,|\mathbf{v}|\sum_{j=1}^{n}\nabla_jf\,P^j_\sigma.$$

Let's combine this relationship with (3.2). As a result we obtain

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{m=1}^{n}P^i_\varepsilon\,P^j_\sigma\,N^m\frac{F_i\,\tilde{\nabla}_mF_j}{|\mathbf{v}|}=2\,|\mathbf{v}|^2\sum_{i=1}^{n}\sum_{j=1}^{n}\nabla_if\,\nabla_jf\,P^i_\varepsilon\,P^j_\sigma.$$

Right hand size of this equality do not change if we transpose indices ε and σ. Similar property we observe in right hand size of (3.27), it's the consequence of $\nabla_i\nabla_jf=\nabla_j\nabla_if$ (see comment to formula (3.6) above). From these two observations we easily derive that the equations (1.3) are fulfilled by force field (2.11) for arbitrary choice of function H in it.

Now let's consider the equation (1.4). Substituting (2.11) into the equation (1.4), due to (3.7) we derive the following relationship:

$$\sum_{i=1}^{n}\sum_{j=1}^{n} P^j_\sigma\,\tilde\nabla_j F_i\,P^i_\varepsilon=\left(2\sum_{i=1}^{n}\nabla_s f\,v^s+\frac{H}{|\mathbf{v}|}\right)\sum_{j=1}^{n}P^j_\sigma\,g_{j\varepsilon}.$$

Upon raising index ε in this relationship we obtain

$$\sum_{i=1}^{n}\sum_{j=1}^{n} P^j_\sigma\,\tilde\nabla_j F^i\,P^\varepsilon_i=\left(2\sum_{i=1}^{n}\nabla_s f\,v^s+\frac{H}{|\mathbf{v}|}\right)P^\varepsilon_\sigma. \tag{3.28}$$

Right hand size of the equality (3.28) differs from components of the projection operator $\mathbf{P}$ only by scalar multiple

$$\lambda=2\sum_{i=1}^{n}\nabla_s f\,v^s+\frac{H}{|\mathbf{v}|}.$$

This circumstance mean that the equations (1.4) are fulfilled by force field (2.11) unconditionally for arbitrary function H in it. Summarizing this result with previous one we formulate the following theorem.

Theorem 3.2. *Arbitrary Newtonian dynamical system with force field* (2.11) *satisfies additional normality equations* (1.3) *and* (1.4).

Combining theorems 3.1 and 3.2 we obtain complete solution for the problem of metrizability formulated in the beginning of this section.

Theorem 3.3. *Newtonian dynamical system admitting the normal shift on Riemannian manifold M with metric $\mathbf{g}$ is metrizable by conformally equivalent metric $\tilde{\mathbf{g}}=e^{-2f}\,\mathbf{g}$ if and only if its force field $\mathbf{F}$ is given by formula* (3.26)

§ 4. Wider treatment of the problem of metrizability.

Theorem 3.3 proved above solves the problem of metrizability of dynamical systems by conformally equivalent metric. In wider treatment of this problem we should give up this preassigned requirement of conformal equivalence of metrics $\mathbf{g}$ and $\tilde{\mathbf{g}}$. Problem of metrizability should sound as follows: under which condition Newtonian dynamical system admitting normal shift in metric $\mathbf{g}$ inherits trajectories of geodesic flow of another metric $\tilde{\mathbf{g}}$? This statement is natural generalization of the problem of metrizability as it was stated in § 3. However, in the sense of normal shift in metric $\mathbf{g}$ it is less motivated. The matter is that normal shift of hypersurface in metric $\mathbf{g}$ shouldn't be the normal shift in metric $\tilde{\mathbf{g}}$ (unless these metrics are conformally equivalent). This explains our previous choice of conformally equivalent metrics.

Let Γ^k_{ij} and $\tilde\Gamma^k_{ij}$ be components of metric connections corresponding to metrics $\mathbf{g}$ and $\tilde{\mathbf{g}}$ respectively. Denote by M^k_{ij} their difference:

$$M^k_{ij}=\tilde\Gamma^k_{ij}-\Gamma^k_{ij}. \tag{4.1}$$

Quantities (4.1) form the set of components for some tensor field $\mathbf{M}$ of type $(1,2)$ on the manifold M. Metric $\mathbf{g}$ will be understood as basic metric on M. Therefore geodesic flow of second metric $\tilde{\mathbf{g}}$ can be written as Newtonian dynamical system on Riemannian manifold M with metric $\mathbf{g}$. Force field of such dynamical system is defined by the following formula for its components:

$$F^k = -\sum_{i=1}^{n}\sum_{j=1}^{n} M^k_{ij}\, v^i\, v^j. \tag{4.2}$$

Definition 4.1. We say that Newtonian dynamical system on Riemannian manifold M is **metrizable** by geodesic flow of some affine connection on M, if it inherits trajectories of this geodesic flow.

The origin of affine connection defining geodesic flow in the definition 4.1 is absolutely unessential. In case when this connection is metrical, i. e. when its components $\tilde{\Gamma}^k_{ij}$ are given by formula (1.6), we say that dynamical system is metrizable by metric $\tilde{\mathbf{g}}$. If metric $\tilde{\mathbf{g}}$ is conformally equivalent to basic metric of Riemannian manifold M, i. e. if $\tilde{\mathbf{g}} = e^{-2f}\,\mathbf{g}$, then we return to the situation described by definition 2.3.

In general case described by definition 4.1 tensor field M with components M^k_{ij} in (4.2) is an arbitrary symmetric (non-extended) tensor field of type $(1,2)$ on the manifold M. Components of force (4.2) are homogeneous functions of degree $\gamma = 2$ respective to velocity vector $\mathbf{v}$. Therefore we can apply results of § 2. For the force field of metrizable dynamical system this gives

$$F^k = -\sum_{i=1}^{n}\sum_{j=1}^{n} M^k_{ij}\, v^i\, v^j + \frac{H}{|\mathbf{v}|}\, v^k. \tag{4.3}$$

Now we are to study, which of Newtonian dynamical systems, metrizable in the sense of definition 4.1, admit the normal shift? In order to explore this question we substitute (4.3) into the equations of normality (1.1), (1.2), (1.3), (1.4). Let's begin with the last equation. Therefore let's calculate $\tilde{\nabla}_j F^i$:

$$\tilde{\nabla}_j F^i = -2\sum_{s=1}^{n} M^i_{js}\, v^s + \tilde{\nabla}_j H\, N^i + \frac{H}{|\mathbf{v}|}\, P^i_j. \tag{4.4}$$

Contracting this expression with P^j_σ and P^ε_i we get

$$\sum_{i=1}^{n}\sum_{j=1}^{n} P^j_\sigma\, \tilde{\nabla}_j F^i\, P^\varepsilon_i = -2\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{s=1}^{n} P^j_\sigma\, M^i_{js}\, v^s\, P^\varepsilon_i + \frac{H}{|\mathbf{v}|}\, P^\varepsilon_\sigma.$$

Last term in right hand side of this equality differs from P^ε_σ by scalar multiple. Hence for the force field (4.2) to satisfy the equation 1.4 we should provide the equality

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{s=1}^{n} P^j_\sigma\, M^i_{js}\, v^s\, P^\varepsilon_i = \lambda\, P^\varepsilon_\sigma \tag{4.5}$$

to be fulfilled. Here λ is a scalar parameter. The value of this parameter is easily calculated by contracting (4.5) with respect to indices ε and σ:

$$\lambda=\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{s=1}^{n}\frac{P^j_i\,M^i_{js}\,v^s}{n-1}. \tag{4.6}$$

The relationships (4.5) with parameter given by (4.6) form the set of equations for the components of tensor $\mathbf{M}$ at each p on the manifold M. They are purely algebraic equations. These equations contain components of velocity vector $\mathbf{v}$, though quantities M^i_{jk} do not depend on $\mathbf{v}$. In order to solve the equations (4.5) let's introduce two operators $\mathbf{X}$ and $\mathbf{Y}$ with the following components:

$$X^i_j=\sum_{s=1}^{n}M^i_{js}\,v^s, \qquad\qquad Y^\varepsilon_j=\sum_{i=1}^{n}P^\varepsilon_i\,X^i_j. \tag{4.7}$$

Then the equation (4.5) and second relationship in (4.7) are equivalent to operator equalities $\mathbf{Y}\,\mathbf{P}=\lambda\,\mathbf{P}$ and $\mathbf{P}\,\mathbf{X}=\mathbf{Y}$ respectively.

Lemma 4.1. *The equations* (4.5) *are reduced to the relationships*

$$\sum_{s=1}^{n}M^i_{js}\,v^s=\lambda\,P^i_j+u^i\,N_j+N^i\,w_j, \tag{4.8}$$

where λ is a scalar parameter defined by (4.6)*, w_j are components of some covector, and u^i are components of some vector perpendicular to* $\mathbf{v}$.

Proof. Let's write operator equality $\mathbf{Y}\,\mathbf{P}=\lambda\,\mathbf{P}$, equivalent to (4.5), in local coordinates, and then use formulas (5.4) from Chapter V:

$$\sum_{j=1}^{n}Y^\varepsilon_j\,P^j_\sigma=\sum_{j=1}^{n}Y^\varepsilon_j\left(\delta^j_\sigma-N^j\,N_\sigma\right)=Y^\varepsilon_\sigma-\left(\sum_{j=1}^{n}Y^\varepsilon_j\,N^j\right)N_\sigma=\lambda\,P^\varepsilon_\sigma.$$

This relationship can be rewritten as follows:

$$Y^\varepsilon_\sigma=\lambda\,P^\varepsilon_j+\left(\sum_{j=1}^{n}Y^\varepsilon_j\,N^j\right)N_\sigma=\lambda\,P^\varepsilon_\sigma+u^\varepsilon\,N_\sigma. \tag{4.9}$$

By u^ε we denote components of vector $\mathbf{u}=\mathbf{Y}(\mathbf{N})=\mathbf{P}(\mathbf{X}(\mathbf{N}))$. Operator $\mathbf{P}$ is a projector to hyperplane perpendicular to the vector of velocity. Therefore $\mathbf{u}\perp\mathbf{v}$.

Further let's write in coordinates operator equality $\mathbf{P}\,\mathbf{X}=\mathbf{Y}$ and let's apply formulas (5.4) from Chapter V again:

$$\sum_{i=1}^{n}P^\varepsilon_i\,X^i_\sigma=\sum_{i=1}^{n}(\delta^\varepsilon_i-N^\varepsilon\,N_i)\,X^i_\sigma=X^\varepsilon_\sigma-\left(\sum_{j=1}^{n}N_i\,X^i_\sigma\right)N^\varepsilon=Y^\varepsilon_\sigma.$$

Denote by w_σ the coefficients of N^ε in the above formula binding $\mathbf{X}$ and $\mathbf{Y}$. They are components of some covector $\mathbf{w}$:

$$w_\sigma = \sum_{j=1}^{n} N_i\, X^i_\sigma.$$

Now the relationship $\mathbf{P}\,\mathbf{X} = \mathbf{Y}$ for $\mathbf{X}$ and $\mathbf{Y}$ can be written as

$$X^\varepsilon_\sigma = Y^\varepsilon_\sigma + w_\sigma\, N^\varepsilon. \tag{4.10}$$

Combining (4.9), (4.10), and first formula in (4.7), we get the relationship coinciding with (4.8). Lemma is proved □

Components of vector $\mathbf{u}$ and components of covector $\mathbf{w}$ in (4.8) can be calculated in explicit form. In order to find w_j we multiply (4.8) by N_i and contract with respect to index i. Thereby we take into account the condition $\mathbf{u} \perp \mathbf{N}$, which follows from orthogonality of vectors $\mathbf{u}$ and $\mathbf{v}$:

$$w_j = \sum_{i=1}^{n}\sum_{s=1}^{n} M^i_{js}\, v^s\, N_i. \tag{4.11}$$

In order to find components of vector $\mathbf{u}$ we multiply (4.8) by N^j and contract with respect to index j. This yields

$$u^i = \sum_{j=1}^{n}\sum_{s=1}^{n} N^j\, M^i_{js}\, v^s - \sum_{j=1}^{n} w_j\, N^j\, N^i. \tag{4.12}$$

Parameter λ is calculated by formula (4.6), as it was above. However, this formula now can be written as follows:

$$\lambda = \sum_{j=1}^{n}\sum_{s=1}^{n} \frac{M^j_{js}\, v^s}{n-1} - \sum_{j=1}^{n} \frac{w_j\, N^j}{n-1}. \tag{4.13}$$

The relationships (4.8), (4.11), (4.12), and (4.13), as well as the initial relationships (4.5), are the equations for the components of tensor $\mathbf{M}$ that should be fulfilled identically for all $\mathbf{v} \neq 0$. Components of tensor $\mathbf{M}$by themselves do not depend on $\mathbf{v}$. Now we shall use the arbitrariness in $\mathbf{v}$. Let's choose vector $\mathbf{v}$ such that all its components are zero, except for one of them:

$$v^s = \delta^s_r = \begin{cases} 0 & \text{for}\quad s \neq r,\\ 1 & \text{for}\quad s = r. \end{cases} \tag{4.14}$$

From (4.14) one easily calculate $|\mathbf{v}|$ and components of the vector $\mathbf{N}$ as well:

$$|\mathbf{v}|^2 = g_{rr}, \qquad N^s = \frac{\delta^s_r}{\sqrt{g_{rr}}}, \qquad N_s = \frac{g_{sr}}{\sqrt{g_{rr}}}. \tag{4.15}$$

Further let's first calculate parameters w_j and u^i by formulas (4.11) and (4.12):

$$\begin{aligned} w_j &= \frac{M_{rjr}}{\sqrt{g_{rr}}} = \sum_{k=1}^{n} \frac{M^k_{jr}\,g_{kr}}{\sqrt{g_{rr}}},\\ u^i &= \frac{M^i_{rr}}{\sqrt{g_{rr}}} - \frac{M_{rrr}}{g_{rr}\,\sqrt{g_{rr}}}\,\delta^i_r. \end{aligned} \tag{4.16}$$

In a similar way, let's calculate parameter λ by formula (4.13):

$$\lambda = \frac{1}{n-1}\left(\sum_{k=1}^{n} M^k_{kr} - \frac{M_{rrr}}{g_{rr}}\right). \tag{4.17}$$

Now let's substitute (4.15), (4.16), and (4.17) into the formulas (4.8). This yields

$$\begin{aligned} M^i_{jr} = \frac{1}{n-1}\left(\sum_{k=1}^{n} M^k_{kr} - \frac{M_{rrr}}{g_{rr}}\right)\left(\delta^i_j - \frac{\delta^i_r\,g_{jr}}{g_{rr}}\right) +\\ + \left(M^i_{rr} - \frac{M_{rrr}}{g_{rr}}\,\delta^i_r\right)\frac{g_{jr}}{g_{rr}} + M_{rjr}\,\frac{\delta^i_r}{g_{rr}}. \end{aligned} \tag{4.18}$$

Formula (4.18) gives the expressions for components of tensor **M**. Let's study the dependence of M^i_{jr} upon indices i, j, and r. For this purpose, first of all, let's introduce the following notations:

$$\mu_r = \frac{1}{n-1}\left(\sum_{k=1}^{n} M^k_{kr} - \frac{M_{rrr}}{g_{rr}}\right), \tag{4.19}$$

$$\theta^i_r = \frac{M^i_{rr}}{g_{rr}}, \tag{4.20}$$

$$\varkappa_{jr} = \frac{M_{rjr}}{g_{rr}} - \frac{M_{rrr}\,g_{jr}}{(g_{rr})^2} - \frac{g_{jr}\,\mu_r}{g_{rr}}. \tag{4.21}$$

Taking into account notations (4.19), (4.20), and (4.21), we can rewrite formula (4.18), bringing it to the following form:

$$M^i_{jr} = \mu_r\,\delta^i_j + \varkappa_{jr}\,\delta^i_r + \theta^i_r\,g_{jr}. \tag{4.22}$$

Apart from formula (4.22) we take into account symmetry of tensor **M**. Let $j \neq r$. Let's calculate the difference $M^i_{jr} - M^i_{rj}$. Equating it to zero, we get the equation

$$(\mu_r - \varkappa_{rj})\,\delta^i_j + (\varkappa_{jr} - \mu_j)\,\delta^i_r + (\theta^i_r - \theta^i_j)\,g_{jr} = 0. \tag{4.23}$$

The equation (4.23) substantially simplifies, if we assume that $i \neq j$ and $i \neq r$. In this case we have $(\theta^i_r - \theta^i_j)\,g_{jr} = 0$. Without loss of generality we can choose local coordinates so that all components of metric tensor are nonzero: $g_{jr} \neq 0$. Then

$$\theta^i_r = \theta^i_j \quad \text{for} \quad i \neq j \quad \text{and} \quad i \neq r. \tag{4.24}$$

Remember that we consider multidimensional case $n \geqslant 3$, two-dimensional case is considered in **other thesis** (see [36]). In multidimensional case $n \geqslant 3$ the range of indices is large enough for to satisfy inequalities $i \neq r$, $i \neq j$, and $j \neq r$ simultaneously. Therefore the relationships (4.24) are not trivial. They mean that the quantities θ^i_r in (4.22) do not depend on r for all $r \neq i$. Let's write this fact as

$$\theta^i_r = \begin{cases} z^i & \text{for} \quad r \neq i, \\ \theta^r_r & \text{for} \quad r = i. \end{cases} \tag{4.25}$$

It is convenient to bring (4.25) to the following form

$$\theta^i_r = z^i + (\theta^r_r - z^r)\,\delta^i_r. \tag{4.26}$$

Apart from the case, when $i \neq j$ and $i \neq r$, in (4.23) have two other cases, when $i = j$ or $i = r$. In both cases $j \neq r$, since for $j = r$ the equation (4.23) turns to identity $0 = 0$. Considering the equation (4.23) in both these cases, we get the same result, which is written as follows:

$$\varkappa_{jr} = \mu_j + (z^r - \theta^r_r)\,g_{jr} \quad \text{for} \quad j \neq r. \tag{4.27}$$

Let's substitute (4.26) and (4.27) into (4.22). As a result we obtain

$$M^i_{jr} = \mu_r\,\delta^i_j + \mu_j\,\delta^i_r + z^i\,g_{jr} \quad \text{for} \quad j \neq r. \tag{4.28}$$

Components M^i_{rr} corresponding to the case, when $j = r$, can be determined from formula (4.20). This yields $M^i_{rr} = \theta^i_r\,g_{rr}$. If we take into account (4.26), then we get

$$M^i_{rr} = (\theta^r_r - z^r)\,g_{rr}\,\delta^i_r + z^i\,g_{rr}. \tag{4.29}$$

For the sake of uniformity in the above formulas (4.28) and (4.29) it is convenient to introduce the following notations:

$$\tilde{\mu}_r = \frac{(\theta^r_r - z^r)\,g_{rr}}{2}.$$

Then (4.28) and (4.29) can be united into one formula

$$M^i_{jr} = \begin{cases} \mu_r\,\delta^i_j + \mu_j\,\delta^i_r + z^i\,g_{jr} & \text{for} \quad j \neq r, \\ \tilde{\mu}_r\,\delta^i_j + \tilde{\mu}_j\,\delta^i_r + z^i\,g_{jr} & \text{for} \quad j = r. \end{cases} \tag{4.30}$$

LEMMA 4.2. *Tensor* **M** *satisfies the equations* (4.5) *if and only if its components are defined by formula*

$$M^i_{jr} = \mu_r\,\delta^i_j + \mu_j\,\delta^i_r + z^i\,g_{jr}. \tag{4.31}$$

PROOF. By deriving formula (4.30) we have already done the part of proof for lemma 4.2. Now we are to prove that $\mu_i = \tilde{\mu}_i$ in this formula. Let's substitute the expression (4.30) into (4.5) and into the formula (4.6) for λ. This yields

$$\sum_{s=1}^{n}\mu_s\,v^s\,P^\varepsilon_\sigma + \sum_{j=1}^{n} 2\,(\tilde{\mu}_j - \mu_j)\,v^j\,P^\varepsilon_j\,P^j_\sigma = \lambda\,P^\varepsilon_\sigma, \tag{4.32}$$

$$\lambda = \sum_{s=1}^{n}\mu_s\,v^s + \sum_{j=1}^{n}\frac{2\,(\tilde{\mu}_j - \mu_j)\,v^j\,P^j_j}{n-1}. \tag{4.33}$$

By substituting (4.33) into (4.32) we bring the equation (4.5) to the form

$$\sum_{j=1}^{n}(\tilde{\mu}_j - \mu_j)\,v^j\,P^\varepsilon_j\,P^j_\sigma = \sum_{j=1}^{n}\frac{(\tilde{\mu}_j - \mu_j)\,v^j\,P^j_j\,P^\varepsilon_\sigma}{n-1}. \tag{4.34}$$

Similar to (4.5), the equations (4.34) derived from (4.5) should be fulfilled identically for all **v**. Therefore we can use again the trick with special choice of **v** in them. However, the choice of (4.14) yields nothing new, bringing (4.34) to the identities. We are to modify this trick by considering vector **v** that slightly differs from (4.14):

$$v^s = \delta^s_r + \xi^s. \tag{4.35}$$

Assuming ξ^s in (4.35) to be small parameters, let's consider Taylor expansions for all terms in (4.34). Thereby we shall hold only terms linear in ξ^s. For $|\mathbf{v}|$ we have

$$|\mathbf{v}|^2 = g_{rr} + 2\sum_{k=1}^{n} g_{rk}\,\xi^k + \ldots = g_{rr} + 2\,\xi_r + \ldots\,. \tag{4.36}$$

Further let's calculate covariant and contravariant components of unitary vector **N** directed along the velocity vector **v**:

$$\begin{aligned} N^s &= \frac{\delta^s_r}{\sqrt{g_{rr}}}\left(1 - \frac{\xi_r}{g_{rr}}\right) + \frac{\xi^s}{\sqrt{g_{rr}}} + \ldots\,,\\ N_s &= \frac{g_{sr}}{\sqrt{g_{rr}}}\left(1 - \frac{\xi_r}{g_{rr}}\right) + \frac{\xi_s}{\sqrt{g_{rr}}} + \ldots\,. \end{aligned} \tag{4.37}$$

Using (4.37) and using formulas (5.4) from Chapter V, we calculate components of orthogonal projection operator **P**:

$$P^i_j = \delta^i_j - \frac{\delta^i_r\,g_{jr}}{g_{rr}}\left(1 - 2\,\frac{\xi_r}{g_{rr}}\right) - \frac{\xi^i\,g_{jr} + \xi_j\,\delta^i_r}{g_{rr}} + \ldots\,. \tag{4.38}$$

For the sake of brevity in formula (4.34) we introduce the following notations:

$$m_j = \tilde{\mu}_j - \mu_j. \tag{4.39}$$

From (4.35) we get $v^j = \delta^j_r + \xi^j$. Let's substitute this expression into the equations (4.34) and take into account above notations:

$$P^\varepsilon_r\, m_r\, P^r_\sigma - \frac{P^r_r\, m_r\, P^\varepsilon_\sigma}{n-1} = \sum_{j=1}^{n} m_j\, \xi^j \left(\frac{P^j_j\, P^\varepsilon_\sigma}{n-1} - P^\varepsilon_j\, P^j_\sigma\right). \tag{4.40}$$

Right hand side of (4.40) contains small parameters ξ^j as multiplicands. Hence for components of projector $\mathbf{P}$ in right hand side of (4.40) we can hold only zero order terms from their Taylor expansions (4.38):

$$\begin{gathered}\sum_{j=1}^{n} m_j\, \xi^j\, \frac{P^j_j\, P^\varepsilon_\sigma}{n-1} = \sum_{j=1}^{n} m_j\, \xi^j \left(1 - \frac{\delta^j_r\, g_{jr}}{g_{rr}}\right) \frac{P^\varepsilon_\sigma}{n-1} + \ldots\,,\\ \sum_{j=1}^{n} m_j\, \xi^j\, P^\varepsilon_j\, P^j_\sigma = \sum_{j=1}^{n} m_j\, \xi^j\, P^\varepsilon_j \left(\delta^j_\sigma - \frac{\delta^j_r\, g_{r\sigma}}{g_{rr}}\right) + \ldots\,.\end{gathered}$$

Let's accomplish summation in right hand sides of the obtained expansions. Then

$$\begin{gathered}\sum_{j=1}^{n} m_j\, \xi^j\, \frac{P^j_j\, P^\varepsilon_\sigma}{n-1} = \sum_{j\neq r} m_j\, \xi^j\, \frac{P^\varepsilon_\sigma}{n-1} + \ldots\,,\\ \sum_{j=1}^{n} m_j\, \xi^j\, P^\varepsilon_j\, P^j_\sigma = m_\sigma\, \xi^\sigma\, P^\varepsilon_\sigma - m_r\, \xi^r\, P^\varepsilon_r\, \frac{g_{r\sigma}}{g_{rr}} + \ldots\,.\end{gathered} \tag{4.41}$$

Now we take into account that zero order terms in the expansions of P^ε_r are zero. Indeed, from formula (4.38) we find that

$$P^\varepsilon_r = \frac{\xi_r}{g_{rr}}\, \delta^\varepsilon_r - \xi^\varepsilon + \ldots\,. \tag{4.42}$$

Let's substitute the expansions (4.41) and (4.42) into the equations (4.40). Collecting first order terms in resulting equality, we obtain

$$\begin{gathered}m_r \left(\frac{\delta^\varepsilon_r}{g_{rr}}\, \xi_r - \xi^\varepsilon\right) \left(\delta^r_\sigma - \frac{g_{\sigma r}}{g_{rr}}\right) + m_\sigma\, \xi^\sigma \left(\delta^\varepsilon_\sigma - \frac{\delta^\varepsilon_r\, g_{r\sigma}}{g_{rr}}\right) =\\ = \left(\sum_{j\neq r} \frac{m_j\, g_{rr} + m_r\, g_{rj}}{g_{rr}\,(n-1)}\, \xi^j\right) \left(\delta^\varepsilon_\sigma - \frac{\delta^\varepsilon_r\, g_{r\sigma}}{g_{rr}}\right).\end{gathered}$$

Remember that we consider case $n \geqslant 3$. Two-dimensional case is considered in **other thesis** (see [36]). In multidimensional case $n \geqslant 3$ we can choose three distinct values for ε, r, and σ, i. e. $\varepsilon \neq r$, $r \neq \sigma$, and $\sigma \neq \varepsilon$. This choice substantially simplifies the above equalities. Now they look like

$$\frac{m_r\,g_{\sigma r}}{g_{rr}}\,\xi^{\varepsilon} = 0. \tag{4.43}$$

As we already mentioned above, without loss of generality we can choose local coordinates so that all components of metric tensor $g_{\sigma r}$ are nonzero. Parameters $\xi^1, \ldots, \xi^n$ are arbitrary real numbers, they can be chosen nonzero as well. Hence from (4.43) and (4.39) we get $m_r = 0$ and $\tilde{\mu}_r = \mu_r$. Thus, we proved that formula (4.35) follows from the equations (4.5). And conversely, (4.5) follows from (4.31). This is proved immediately by substituting (4.31) into the equations (4.5). □

Combining the results of lemma 4.1 and lemma 4.2, we get the theorem.

Theorem 4.1. *On Riemannian manifold M of the dimension $n \geqslant 3$ the force field* (4.5) *satisfies the equations of normality* (1.4) *if and only if components of tensor $\mathbf{M}$ in* (4.5) *are defined by formula* (4.31).

Parameters z^i and μ_j in (4.31) are the components of some vector field $\mathbf{z}$ and some covector field $\boldsymbol{\mu}$ respectively. In order to prove it let's consider the following two expressions in formula (4.31):

$$\begin{gathered}\sum_{r=1}^{n}\sum_{s=1}^{n} g^{ir}\,M^s_{sr} = (n+1)\,\mu^i + z^i,\\ \sum_{r=1}^{n}\sum_{s=1}^{n} g^{sr}\,M^i_{sr} = 2\mu^i + n\,z^i.\end{gathered} \tag{4.44}$$

By means of (4.44) we can express parameters z^i and μ_j explicitly through components of tensor field $\mathbf{M}$:

$$\begin{gathered}z^i = \sum_{r=1}^{n}\sum_{s=1}^{n}\frac{(n+1)\,g^{sr}\,M^i_{sr} - 2\,g^{ir}\,M^s_{sr}}{n^2+n-2},\\ \mu_j = \sum_{i=1}^{n}\sum_{r=1}^{n}\sum_{s=1}^{n}\frac{n\,g^{ir}\,M^s_{sr} - g^{sr}\,M^i_{sr}}{n^2+n-2}\,g_{ij}.\end{gathered} \tag{4.45}$$

Formulas (4.45) makes transparent vectorial and covectorial nature of the quantities z^i and μ_j in (4.31).

Let's substitute (4.31) into the formula (4.3) for the force field $\mathbf{F}$. As a result we get the following expression for covariant components of $\mathbf{F}$:

$$F_i = -2\sum_{j=1}^{n}\mu_j\,v^j\,v_i - z_i\,|\mathbf{v}|^2 + H\,N_i. \tag{4.46}$$

Next step consist in substituting (4.46) into the normality equations (1.3). Let's first calculate $\tilde{\nabla}_m F_j$ and $\nabla_i F_j$:

$$\begin{aligned}\tilde{\nabla}_m F_j = &- 2\,\mu_m\,v_j - \sum_{s=1}^{n} 2\,\mu_s\,v^s\,g_{mj} -\\ &- 2\,z_j\,v_m + \tilde{\nabla}_m H\,N_j + \sum_{s=1}^{n}\frac{H}{|\mathbf{v}|}\,P^s_m\,g_{sj},\\ \nabla_i F_j = &-\sum_{s=1}^{n} 2\,\nabla_i\mu_s\,v^s\,v_j - \nabla_i z_j\,|\mathbf{v}|^2 + \nabla_i H\,N_j.\end{aligned}$$

Then accomplish the following calculations:

$$\begin{aligned}&\sum_{i=1}^{n}\sum_{j=1}^{n}\nabla_i F_j\,P^i_\varepsilon\,P^j_\sigma = -|\mathbf{v}|^2\sum_{i=1}^{n}\sum_{j=1}^{n}\nabla_i z_j\,P^i_\varepsilon\,P^j_\sigma,\\ &\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{m=1}^{n} N^m\,\frac{F_i\,\tilde{\nabla}_m F_j}{|\mathbf{v}|}\,P^i_\varepsilon\,P^j_\sigma = -|\mathbf{v}|^2\sum_{i=1}^{n}\sum_{j=1}^{n} z_i\,z_j\,P^i_\varepsilon\,P^j_\sigma.\end{aligned}$$

If we take into account the above calculations, then, substituting components of force field (4.46) into (1.3), we get

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(\nabla_i z_j - \nabla_j z_i)\,P^i_\varepsilon\,P^j_\sigma = 0. \tag{4.47}$$

Components of projector field in (4.47) depend on $\mathbf{v}$, while difference $\nabla_i z_j - \nabla_j z_i$ doesn't depend. Let's recall again that here we consider multidimensional case $n \geqslant 3$. Two-dimensional case is considered in **other thesis** [36]. Suppose that $\mathbf{a}$ and $\mathbf{b}$ are two arbitrary vectors from tangent space to M at some point p. For $n \geqslant 3$ we can direct $\mathbf{v}$ so that it will be perpendicular to $\mathbf{a}$ and $\mathbf{b}$. Then

$$\sum_{\varepsilon=1}^{n} P^i_\varepsilon\,a^\varepsilon = a^i, \qquad\qquad \sum_{\sigma=1}^{n} P^j_\sigma\,b^\sigma = b^j.$$

Contracting (4.47) with the product $a^\varepsilon\,b^\sigma$, now we get

$$\varphi(\mathbf{a},\mathbf{b}) = \sum_{i=1}^{n}\sum_{j=1}^{n}(\nabla_i z_j - \nabla_j z_i)\,a^i\,b^j = 0.$$

Since $\mathbf{a}$ and $\mathbf{b}$ are arbitrary vectors, the above equation means that skew symmetric bilinear form φ with components $\varphi_{ij} = \nabla_i z_j - \nabla_j z_i$ is identically zero. Hence

$$\nabla_i z_j - \nabla_j z_i = 0. \tag{4.48}$$

Note that quantities $z_1, \ldots, z_n$ in (4.48), which are covariant components of vector field $\mathbf{z}$, do not depend on $\mathbf{v}$. Let's calculate $\nabla_i z_j$ and $\nabla_j z_i$ according top the formula (7.3) from Chapter III, and let's take into account the symmetry of connection components Γ^k_{ij} for metric connection. As a result from (4.48) we derive

$$\frac{\partial z_i}{\partial x^j} = \frac{\partial z_j}{\partial x^i}. \tag{4.49}$$

The relationship (4.49) means that differential 1-form $\mathbf{z}$ with components $z_1, \ldots, z_n$ is closed. Its known that any closed form is locally exact (see proof in [79]). When applied to differential 1-form $\mathbf{z}$, this fact yields

$$z_i = \frac{\partial f}{\partial x^i} = \nabla_i f, \tag{4.50}$$

where f is a scalar field, which is defined locally in the neighborhood of any point.

Let's substitute the expression (4.50) for z_i into the formula (4.46). The result of this substitution is written as follows:

$$F_i = -\nabla_i f\,|\mathbf{v}|^2 + 2\sum_{s=1}^{n} \nabla_s f\, v^s\, v_i + \left(H - \sum_{s=1}^{n} 2\,|\mathbf{v}|\,(\mu_s + \nabla_s f)\, v^s\right) N_i.$$

Now it's convenient to make the following redesignation, since the equations (1.3) and (1.4) do not specify the form of scalar field $\mathbf{H}$:

$$H - \sum_{s=1}^{n} 2|\mathbf{v}|\,(\mu_s + \nabla_s)\, v^s \longrightarrow H.$$

As a result of this redesignation we get simpler formula for the force field $\mathbf{F}$:

$$F_i = -\nabla_i f\,|\mathbf{v}|^2 + 2\sum_{s=1}^{n} \nabla_s f\, v^s\, v_i + H\, N_i. \tag{4.51}$$

In essential, formula (4.51) coincides with formula (2.11), which we studied earlier. The only difference is that field F here is defined locally. Substituting formula (4.51) into weak normality equations (1.1) and (1.2), we can specify H in it:

$$F_i = -|\mathbf{v}|^2\,\nabla_i f + 2\sum_{s=1}^{n} \nabla_s f\, v^s\, v_i + H(v\, e^{-f})\, e^f\, N_i. \tag{4.52}$$

This is the result of § 3 (see formula (3.26) and theorem 3.1). Main result of this section is formulated as follows.

Theorem 4.1. *Newtonian dynamical system on Riemannian manifold M of the dimension $n \geqslant 3$ admitting the normal shift is metrized by geodesic flow of affine connection Γ if and only if locally this connection is a metric connection for metric $\tilde{\mathbf{g}} = e^{-2f}\,\mathbf{g}$ conformally equivalent to basic metric of manifold M.*

This result justifies our first approach, when we restricted ourselves to considering only conformally equivalent metrics. It was obtained in paper [85], which is not yet published.

§ 5. Examples of non-metrizable dynamical systems.

Some examples of dynamical systems admitting the normal shift in Euclidean space $\mathbb{R}^2$ were constructed in paper [35] (see also preprint [34]), where the theory of such systems originates. Later it was found that some of them are non-metrizable. This indicates non-triviality of theory in the dimension $n = 2$.

Problem of finding multidimensional non-metrizable systems was open for a while. The importance of solving this problem was pointed out by academician A. T. Fomenko, when I reported at the seminar in Moscow State University in the beginning of 1994. Some example of multidimensional non-metrizable dynamical systems were found in [42]. Here we shall describe these examples.

Suppose that $A = A(v)$ is some smooth scalar function of one variable, and suppose that $\boldsymbol{\mu}$ is some vector field on Riemannian manifold M of the dimension $n \geqslant 3$. Let's consider the force field $\mathbf{F}$ on M with the following components:

$$F^k = A(|\mathbf{v}|)\left(\sum_{s=1}^{n} 2\,\mu^s\,N_s\,N^k - \mu^k\right). \tag{5.1}$$

Theorem 5.1. *Newtonian dynamical system with force field* (5.1) *on Riemannian manifold M of the dimension $n \geqslant 3$ is a system admitting the normal shift if and only if $\boldsymbol{\mu}$ is (at least locally) a gradient of some scalar field, i. e. $\mu_i = \nabla_i f$.*

Proof. Theorem 12.1 from Chapter V reduces proof to substituting (5.1) into normality equations (1.1), (1.2), (1.3), and (1.4). Let's begin with the first of these equations. In (1.1) we need to evaluate the following expressions:

$$\begin{aligned}&\sum_{j=1}^{n} N^j\,F_j = A\sum_{s=1}^{n}\mu_s\,N^s,\\ &\sum_{i=1}^{n}\frac{F_i\,P^i_q}{|\mathbf{v}|} = -\sum_{i=1}^{n}\frac{A\,\mu_i\,P^i_q}{|\mathbf{v}|}.\end{aligned} \tag{5.2}$$

Let's differentiate first of these two equalities (5.2). Applying $\tilde\nabla_i$, we get

$$\sum_{j=1}^{n}\tilde\nabla_i(N^j\,F_j) = A'\sum_{s=1}^{n}\mu_s\,N^s\,N_i + A\sum_{s=1}^{n}\frac{\mu_s\,P^s_i}{|\mathbf{v}|}. \tag{5.3}$$

Here A' is a derivative of the function $A = A(v)$ with respect to its unique argument v. By contracting (5.3) with P^i_q we obtain

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\tilde\nabla_i(N^j\,F_j)\,P^i_q = \sum_{i=1}^{n}\frac{A\,\mu_i\,P^i_q}{|\mathbf{v}|}. \tag{5.4}$$

Comparing (5.4) with second equality (5.2), we find that (5.1) satisfies the equation (1.1) identically.

Now let's substitute (5.1) into (1.2). This require larger amount of calculations. For the first, let's evaluate covariant derivatives $\nabla_i F_j$ and $\tilde{\nabla}_r F_j$:

$$\nabla_i F_j = A\left(\sum_{s=1}^{n} 2\,\nabla_i \mu_s\, N^s\, N_j - \nabla_i \mu_j\right), \tag{5.5}$$

$$\begin{aligned}\tilde{\nabla}_j F_i &= A'\left(\sum_{s=1}^{n} 2\,\mu_s\, N^s\, N_i - \mu_i\right) N_j\, + \\ &+ \sum_{s=1}^{n} A\left(\frac{2\,\mu_s\, P^s_j\, N_i}{|\mathbf{v}|} + \sum_{k=1}^{n} \frac{2\,\mu_s\, N^s\, P^k_i\, g_{kj}}{|\mathbf{v}|}\right).\end{aligned} \tag{5.6}$$

Then, using formulas (5.5) and (5.6), we evaluate the following expressions:

$$\begin{aligned}&\sum_{i=1}^{n}\sum_{j=1}^{n} \nabla_i F_j\, N^j\, P^i_k = A\sum_{i=1}^{n}\sum_{j=1}^{n} \nabla_i \mu_j\, N^j\, P^i_k,\\ &\sum_{i=1}^{n}\sum_{j=1}^{n} \nabla_j F_i\, N^j\, P^i_k = -A\sum_{i=1}^{n}\sum_{j=1}^{n} \nabla_j \mu_i\, N^j\, P^i_k,\end{aligned} \tag{5.7}$$

$$\sum_{j=1}^{n}\sum_{r=1}^{n} \tilde{\nabla}_j F_r\, N^j\, N^r = A'\sum_{s=1}^{n} \mu_s\, N^s. \tag{5.8}$$

Apart from (5.7) and (5.8), we need to evaluate few more expressions:

$$\sum_{i=1}^{n} \tilde{\nabla}_j F_i\, P^i_k = -A'\sum_{s=1}^{n} \mu_s\, P^s_k\, N_j + \sum_{s=1}^{n}\sum_{r=1}^{n} \frac{2\,A\,\mu_s\, N^s\, P^r_j\, g_{rk}}{|\mathbf{v}|},$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \tilde{\nabla}_j F_i\, P^i_k = -\left(A\,A' + \frac{2\,A^2}{|\mathbf{v}|}\right)\sum_{r=1}^{n}\sum_{s=1}^{n} \mu_s\, N^s\, \mu_r\, P^r_k,$$

$$-\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{2\,F_i\, F_j\, N^j\, P^i_k}{|\mathbf{v}|^2} = \frac{2\,A^2}{|\mathbf{v}|}\sum_{r=1}^{n}\sum_{s=1}^{n} \mu_s\, N^s\, \mu_r\, P^r_k,$$

$$-\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{r=1}^{n} \frac{N^r\, N^j\, \tilde{\nabla}_j F_r}{|\mathbf{v}|}\, F_i\, P^i_k = A\,A'\sum_{r=1}^{n}\sum_{s=1}^{n} \mu_s\, N^s\, \mu_r\, P^r_k.$$

Taking into account all above calculations, we reduce (1.2) to the following equation

$$\sum_{i=1}^{n}\sum_{j=1}^{n} (\nabla_i \mu_j - \nabla_j \mu_i)\, N^j\, P^i_k = 0. \tag{5.9}$$

Note that (5.9) is the equation with respect to μ_i. It doesn't contain A. The quantities $\nabla_i\mu_j$ and $\nabla_j\mu_i$ in (5.9) do not depend on $\mathbf{v}$. All dependence on velocity vector is concentrated in N^j and P^i_k. Suppose that $\mathbf{a}$ and $\mathbf{b}$ are two arbitrary vectors from tangent space $T_p(M)$ at some point $p\in M$. We can direct $\mathbf{v}$ along vector $\mathbf{a}$ in (5.9). Then we have the following relationships:

$$a^j=\alpha\,N^j,\qquad\qquad b^i=\beta\,N^i+\sum_{k=1}^{n}P^i_k\,b^k. \tag{5.10}$$

First relationship (5.10) simply means that $\mathbf{a}$ and $\mathbf{N}$ are collinear vectors. Second relationship means that $\mathbf{b}$ is represented as a sum of two components, one directed along $\mathbf{N}$, and other being perpendicular to $\mathbf{N}$, i. e. $\mathbf{b}=\beta\,\mathbf{N}+\mathbf{P}(\mathbf{b})$. Now denote $\varphi_{ij}=\nabla_i\mu_j-\nabla_j\mu_i$. Then let's do the following calculations:

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\varphi_{ij}\,b^i\,a^j=\alpha\,\beta\sum_{i=1}^{n}\sum_{j=1}^{n}\varphi_{ij}\,N^i\,N^j+$$
$$+\,\alpha\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}(\nabla_i\mu_j-\nabla_j\mu_i)\,N^j\,P^i_k\,b^k=0.$$

Here took into account (5.9) and used skew symmetry: $\varphi_{ij}=-\varphi_{ji}$. As a result of above calculations we obtained the relationship

$$\varphi(\mathbf{b},\mathbf{a})=\sum_{i=1}^{n}\sum_{j=1}^{n}\varphi_{ij}\,b^i\,a^j=0,$$

which means that bilinear form φ with components φ_{ij} is identically zero. Therefore

$$\nabla_i\mu_j=\nabla_j\mu_i. \tag{5.11}$$

The equation (5.11) is exact analog of the equation (4.48). By the same arguments as in § 4 from (5.11) we derive the following formula for the components of $\boldsymbol{\mu}$:

$$\mu_i=\frac{\partial f}{\partial x^i}=\nabla_i f. \tag{5.12}$$

Here f is a scalar field, which is defined locally in the neighborhood of each point p on the manifold M.

Next step consists in substituting (5.1) into the equation (1.3). With this purpose, let's do appropriate calculations:

$$\sum_{j=1}^{n}\sum_{m=1}^{n}P^j_\sigma\,N^m\tilde{\nabla}_m F_j=-A'\sum_{j=1}^{n}\mu_j\,P^j_\sigma. \tag{5.13}$$

Let's combine the equality (5.13) and second relationship in (5.2). This yields

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{m=1}^{n} P^i_\varepsilon\,P^j_\sigma\,N^m\,\frac{F_i\,\tilde\nabla_m F_j}{|\mathbf{v}|} = A\,A'\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\mu_i\,P^i_\varepsilon\,\mu_j\,P^j_\sigma}{|\mathbf{v}|}.$$

Now we evaluate the expression obtained by contracting $\nabla_i F_j$ with P^i_ε and P^j_σ:

$$-\sum_{i=1}^{n}\sum_{j=1}^{n} P^i_\varepsilon\,P^j_\sigma\,\nabla_i F_j = \sum_{i=1}^{n}\sum_{j=1}^{n} P^i_\varepsilon\,P^j_\sigma\,\nabla_i\mu_j.$$

Taking into account above calculations, we can write (1.3) in form of the following equations for the components of $\boldsymbol{\mu}$:

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(\nabla_i\mu_j-\nabla_j\mu_i)\,P^i_\varepsilon\,P^j_\sigma = 0. \tag{5.14}$$

These equations are fulfilled unconditionally due to the relationships (5.11) and (5.12). Thus, the equations (1.3) do not impose additional limitations upon the choice of parameters A and $\boldsymbol{\mu}$ in (5.1).

Last step in proving theorem 5.1 consists in substituting (5.1) into the equations (1.4). For to do it let's evaluate the expression from left hand side of this equation:

$$\sum_{i=1}^{n}\sum_{j=1}^{n} P^j_\sigma\,\tilde\nabla_j F^i\,P^\varepsilon_i = \sum_{s=1}^{n}\frac{2\,A\,\mu_s\,N^s}{|\mathbf{v}|}\,P^\varepsilon_\sigma. \tag{5.15}$$

Right hand side of (5.15) differs from P^ε_σ by numeric multiple

$$\lambda = \sum_{s=1}^{n}\frac{2\,A\,\mu_s\,N^s}{|\mathbf{v}|}.$$

Right hand side of (1.4) is equal to $\lambda\,P^\varepsilon_\sigma$ as well. Now one can see that the equations (1.4) are fulfilled by (5.1) unconditionally. They do not impose additional limitations on the choice of A and $\boldsymbol{\mu}$. Thus, the only restriction due to the normality equations consists in (5.11). This restriction is resolved explicitly in form of (5.12). Theorem 5.1 is proved. □

In paper [42] we chose flat space $M=\mathbb{R}^n$ with standard Euclidean metric. Therefore, choosing constant field $\boldsymbol{\mu}$ with $\mu_i = \text{const}$, in [42] we had the situation, when relationships (5.11) are unconditionally fulfilled.

Now suppose that vector field $\boldsymbol{\mu}$ is chosen according to the theorem 5.1 just proved. Then dynamical system with force field (5.1) is a system admitting the normal shift. Is it metrizable or not? Force field of metrizable dynamical system that admits normal shift is given by formula (3.26). If formulas (5.1) and (3.26) define the same force field, then we should have the following equality:

$$\begin{aligned}|\mathbf{v}|^2\left(\sum_{s=1}^{n}2\,\nabla_s f\,N^s\,N_k-\nabla_k f\right)+H(|\mathbf{v}|\,e^{-f})\,e^f\,N_k=\\=A(|\mathbf{v}|)\left(\sum_{s=1}^{n}2\,\mu_s\,N^s\,N_k-\mu_k\right).\end{aligned}\tag{5.16}$$

Let's multiply (5.16) by N^k, and then contract it with respect to index k. As a result we get the equality binding two special scalar fields from extended algebra:

$$H(|\mathbf{v}|\,e^{-f})\,e^f=\sum_{s=1}^{n}(\mu_s\,A(|\mathbf{v}|)-|\mathbf{v}|^2\,\nabla_s f)\,N^s.\tag{5.17}$$

Left hand side of (5.17) doesn't depend on $\mathbf{v}$. Difference $\mu_s\,A(|\mathbf{v}|)-|\mathbf{v}|^2\,\nabla_s f$ in right hand side of (5.17) doesn't depend on $\mathbf{v}$ as well. Therefore we can keep $|\mathbf{v}|$ unchanged and rotate vector $\mathbf{v}$ so that right hand side of (5.17) turns to be zero. Hence H is identically zero function. This reduces (5.17) to the form

$$\sum_{s=1}^{n}(\mu_s\,A(|\mathbf{v}|)-|\mathbf{v}|^2\,\nabla_s f)\,N^s=0.\tag{5.18}$$

Due to the arbitrariness in the choice of $\mathbf{N}$ from (5.18) we derive

$$\mu_s\,A(|\mathbf{v}|)-|\mathbf{v}|^2\,\nabla_s f=0.$$

It's obvious that this covectorial equality breaks into two parts:

$$\mu_s=\mathrm{const}\cdot\nabla_s f,\qquad\qquad A(v)=\frac{v^2}{\mathrm{const}}.\tag{5.19}$$

As a result of (5.19) we conclude: *Newtonian dynamical system with force field* (5.1) *is metrizable if and only if parameter A in* (5.1) *coincides with $|\mathbf{v}|^2$ up to a constant multiple.* Any other choice of this parameter, say $A=|\mathbf{v}|^3$, and the choice $\mu_s=\nabla_s f$ give an example of non-metrizable dynamical system.

CHAPTER VII

REDUCTION OF NORMALITY EQUATIONS FOR THE DIMENSION $n \geqslant 3$.

§ 1. Scalar ansatz[1].

Further analysis of Newtonian dynamical systems admitting the normal shift on Riemannian manifold M requires the analysis of corresponding differential equations for the force field $\mathbf{F}$, which are called **normality equations**. They are subdivided into two groups of equations. First group is formed by **weak normality** equations (see (6.16) and (6.17) in Chapter V). They are written as

$$\sum_{i=1}^{n}\left(v^{-1}\,F_i+\sum_{j=1}^{n}\tilde{\nabla}_i\left(N^j\,F_j\right)\right)P^i_k=0, \tag{1.1}$$

$$\begin{gathered}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\nabla_iF_j+\nabla_jF_i-2\,v^{-2}\,F_i\,F_j\right)N^j\,P^i_k+\\ +\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{F^j\,\tilde{\nabla}_jF_i}{v}-\sum_{r=1}^{n}\frac{N^r\,N^j\,\tilde{\nabla}_jF_r}{v}\,F_i\right)P^i_k=0.\end{gathered} \tag{1.2}$$

Second group is formed by **additional normality conditions** (see (6.16) and (6.17) in Chapter V). They are written only in multidimensional case $n \geqslant 3$:

$$\begin{gathered}\sum_{i=1}^{n}\sum_{j=1}^{n}P^i_\varepsilon\,P^j_\sigma\left(\sum_{m=1}^{n}N^m\,\frac{F_i\,\tilde{\nabla}_mF_j}{v}-\nabla_iF_j\right)=\\ =\sum_{i=1}^{n}\sum_{j=1}^{n}P^i_\varepsilon\,P^j_\sigma\left(\sum_{m=1}^{n}N^m\,\frac{F_j\,\tilde{\nabla}_mF_i}{v}-\nabla_jF_i\right),\end{gathered} \tag{1.3}$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n}P^j_\sigma\,\tilde{\nabla}_jF^i\,P^\varepsilon_i=\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{m=1}^{n}\frac{P^j_m\,\tilde{\nabla}_jF^i\,P^m_i}{n-1}\,P^\varepsilon_\sigma. \tag{1.4}$$

Normality equations (1.1), (1.2), (1.3), (1.4) assume that we choose some local map $(U,x^1,\dots,x^n)$ on M and associated map $(\pi^{-1}(U),x^1,\dots,x^n,v^1,\dots,v^n)$ on tangent bundle TM (see (4.2) in Chapter II). This choice determines auxiliary parameters in these equations. These are: components of unit vector $\mathbf{N}$ directed along the

[1] **Ansatz**, word of german origin, in Russian mathematical texts means **substitution**, i. e. something to be substituted into the equation with the hope that it's a solution.

vector of velocity; components of orthogonal projector **P** to the hyperplane perpendicular to the velocity vector; scalar field $v = |\mathbf{v}|$ of modulus of velocity vector.

We begin the study of normality equations (1.1), (1.2), (1.3), (1.4) with the first of them, i. e. first we consider the equations (1.1). Denote by A the following sum:

$$A = \sum_{i=1}^{n} N^i\, F_i. \tag{1.5}$$

The quantity A is an extended scalar field on M. Scalar field A is interpreted as a projection of **F** to the direction of velocity vector **v**. Taking into account formula (1.5), let's rewrite the equation (1.1) as follows:

$$\sum_{i=1}^{n}\left(v^{-1}\,F_i + \tilde{\nabla}_i A\right) P^i_k = 0. \tag{1.6}$$

Now let's open brackets in the equations (1.6) and recall explicit expression for the components of projector **P** (see formula (5.4) in Chapter V). This yields

$$\begin{aligned}\sum_{i=1}^{n}\frac{F_i\,P^i_k}{|\mathbf{v}|} + \sum_{i=1}^{n}\tilde{\nabla}_i A\,P^i_k = \sum_{i=1}^{n}\frac{F_i\,\delta^i_k}{|\mathbf{v}|} - \sum_{i=1}^{n}\frac{F_i\,N^i\,N_k}{|\mathbf{v}|} +\\ + \sum_{i=1}^{n}\tilde{\nabla}_i A\,P^i_k = \frac{F_k}{|\mathbf{v}|} - \sum_{i=1}^{n}\frac{A\,N_k}{|\mathbf{v}|} + \sum_{i=1}^{n}\tilde{\nabla}_i A\,P^i_k = 0.\end{aligned} \tag{1.7}$$

The last equality in (1.7) allows us to express components of force vector **F** through scalar field A. Indeed, we easily find that

$$F_k = A\,N_k - |\mathbf{v}|\sum_{i=1}^{n}\tilde{\nabla}_i A\,P^i_k. \tag{1.8}$$

Due to what was said above this formula is called **scalar ansatz**.

Scalar ansatz turns to identity first normality equation (1.1). Let's apply it to second equation, i. e. let's substitute (1.8) into (1.2). For $\nabla_i F_j$ we get

$$\nabla_i F_j = \nabla_i A\,N_j - |\mathbf{v}|\sum_{s=1}^{n}\nabla_i\tilde{\nabla}_s A\,P^s_j. \tag{1.9}$$

In deriving (1.9) we used formulas (5.5), (5.6), (5.7), (5.8), and (5.9) from Chapter V. The same formulas are used in evaluating $\tilde{\nabla}_i F_j$:

$$\begin{aligned}\tilde{\nabla}_j F_i = \tilde{\nabla}_j A\,N_i + \frac{A\,P_{ij}}{|\mathbf{v}|} + \sum_{s=1}^{n}\nabla_s A\,N^s\,P_{ij} +\\ + \sum_{s=1}^{n}\tilde{\nabla}_s A\,P^s_j\,N_i - \sum_{s=1}^{n}\tilde{\nabla}_s A\,P^s_i\,N_j - |\mathbf{v}|\sum_{s=1}^{n}\tilde{\nabla}_j\tilde{\nabla}_s A\,P^s_i.\end{aligned} \tag{1.10}$$

For the sake of brevity we denoted by P_{ij} covariant components of projector $\mathbf{P}$:

$$P_{ij}=\sum_{s=1}^{n}P_i^s\,g_{sj}=g_{ij}-N_i\,N_j. \tag{1.11}$$

Now let's substitute (1.8) into the following expressions that have entries of $\mathbf{F}$:

$$\sum_{i=1}^{n}F_i\,P_k^i=-|\mathbf{v}|\sum_{s=1}^{n}\tilde{\nabla}_sA\,P_k^s, \tag{1.12}$$

$$-\sum_{i=1}^{n}\sum_{i=1}^{n}\frac{2\,F_i\,F_j}{|\mathbf{v}|^2}\,N^j\,P_k^i=\sum_{s=1}^{n}\frac{2\,A\,\tilde{\nabla}_sA\,P_k^s}{|\mathbf{v}|}. \tag{1.13}$$

Further we use formula (1.9) in order to evaluate the following expressions:

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\nabla_iF_j\,N^j\,P_k^i=\sum_{s=1}^{n}\nabla_sA\,P_k^s, \tag{1.14}$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\nabla_jF_i\,N^j\,P_k^i=-|\mathbf{v}|\sum_{r=1}^{n}\sum_{s=1}^{n}\nabla_r\tilde{\nabla}_sA\,N^r\,P_k^s. \tag{1.15}$$

And finally, we use formula (1.10) in order to accomplish the following calculations:

$$\begin{aligned}\sum_{i=1}^{n}\frac{\tilde{\nabla}_jF_i}{|\mathbf{v}|}\,P_k^i=\frac{A\,P_{kj}}{|\mathbf{v}|^2}&+\sum_{s=1}^{n}\frac{\tilde{\nabla}_sA\,N^s\,P_{kj}}{|\mathbf{v}|}-\\&-\sum_{s=1}^{n}\frac{\tilde{\nabla}_sA\,P_k^s\,N_j}{|\mathbf{v}|}-\sum_{s=1}^{n}\tilde{\nabla}_j\tilde{\nabla}_sA\,P_k^s,\end{aligned} \tag{1.16}$$

$$-\sum_{j=1}^{n}\sum_{r=1}^{n}\frac{N^r\,N^j\,\tilde{\nabla}_jF_r}{|\mathbf{v}|}=-\sum_{s=1}^{n}\frac{\tilde{\nabla}_sA\,N^s}{|\mathbf{v}|}. \tag{1.17}$$

Before substituting (1.16) into the equation (1.2) we should contract it with F^j, while the equality (1.17) should be multiplied by (1.12). Upon completing these evaluations we get

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{F^j\,\tilde{\nabla}_jF_i}{|\mathbf{v}|}\,P_k^i&=|\mathbf{v}|\sum_{q=1}^{n}\sum_{s=1}^{n}\sum_{r=1}^{n}P^{qr}\,\tilde{\nabla}_qA\,\tilde{\nabla}_r\tilde{\nabla}_sA\,P_k^s-\\&-\sum_{s=1}^{n}\left(\sum_{r=1}^{n}N^r\left(\tilde{\nabla}_rA\,\tilde{\nabla}_sA+A\,\tilde{\nabla}_r\tilde{\nabla}_sA\right)+\frac{2\,A\,\tilde{\nabla}_sA}{|\mathbf{v}|}\right)P_k^s.\end{aligned} \tag{1.18}$$

$$-\sum_{j=1}^{n}\sum_{r=1}^{n}\frac{N^r\,N^j\,\tilde{\nabla}_jF_r}{|\mathbf{v}|}=\sum_{s=1}^{n}\sum_{r=1}^{n}N^r\,\nabla_rA\,\nabla_sA\,P_k^s. \tag{1.19}$$

For the sake of brevity in (1.18) by P^{qr} we denoted purely contravariant components of the field of orthogonal projectors $\mathbf{P}$:

$$P^{qr} = \sum_{s=1}^{n} g^{qs}\, P^r_s = g^{qr} - N^q\, N^r. \tag{1.20}$$

Now let's add the expressions (1.13), (1.14), (1.15), (1.18), (1.19) and let's take into account the equation (1.2). Then (1.2) appears to be written as the following equation for the scalar field A introduced by formula (1.5):

$$\begin{gathered}\sum_{s=1}^{n}\left(\nabla_s A + |\mathbf{v}|\sum_{q=1}^{n}\sum_{r=1}^{n} P^{qr}\,\tilde\nabla_q A\,\tilde\nabla_r\tilde\nabla_s A\, - \right.\\ \left.-\sum_{r=1}^{n} N^r\, A\,\tilde\nabla_r\tilde\nabla_s A - |\mathbf{v}|\sum_{r=1}^{n} N^r\,\nabla_r\tilde\nabla_s A\right) P^s_k = 0.\end{gathered}\tag{1.21}$$

This equation (1.21) is a result of applying scalar ansatz (1.8) to the second normality equation (1.2).

Next step consists in applying scalar ansatz to the third normality equation (1.3). First of all, let's evaluate the following expressions by virtue of (1.8):

$$-\sum_{i=1}^{n}\sum_{j=1}^{n} P^i_\varepsilon\, P^j_\sigma \nabla_j F_i = |\mathbf{v}|\sum_{s=1}^{n} P^r_\sigma\,\nabla_r\tilde\nabla_s A\, P^s_\varepsilon,$$

$$\sum_{i=1}^{n}\sum_{m=1}^{n} P^i_\varepsilon\, N^m\,\frac{\tilde\nabla_m F_i}{|\mathbf{v}|} = -\sum_{s=1}^{n}\frac{\tilde\nabla_s A\, P^s_\varepsilon}{|\mathbf{v}|} - \sum_{s=1}^{n}\sum_{q=1}^{n} N^q\,\tilde\nabla_q\tilde\nabla_s A\, P^s_\varepsilon.$$

Taking into account (1.12), from these two equalities we derive

$$\begin{gathered}\sum_{i=1}^{n}\sum_{j=1}^{n} P^i_\varepsilon\, P^j_\sigma\left(\sum_{m=1}^{n} N^m\,\frac{F_i\,\tilde\nabla_m F_j}{v} - \nabla_i F_j\right) = \\ = \sum_{s=1}^{n}\sum_{r=1}^{n} P^r_\sigma\, P^s_\varepsilon\left(\tilde\nabla_r A\,\tilde\nabla_s A + |\mathbf{v}|\,\nabla_r\tilde\nabla_s A + |\mathbf{v}|\sum_{q=1}^{n}\tilde\nabla_r A\, N^q\,\tilde\nabla_q\tilde\nabla_s A\right).\end{gathered}$$

Now normality equations (1.3) are written in form of the following equations for scalar field A that determines force field according to (1.8):

$$\begin{gathered}\sum_{s=1}^{n}\sum_{r=1}^{n} P^r_\sigma\, P^s_\varepsilon\left(\nabla_r\tilde\nabla_s A + \sum_{q=1}^{n}\tilde\nabla_r A\, N^q\,\tilde\nabla_q\tilde\nabla_s A\right) = \\ = \sum_{s=1}^{n}\sum_{r=1}^{n} P^r_\sigma\, P^s_\varepsilon\left(\nabla_s\tilde\nabla_r A + \sum_{q=1}^{n}\tilde\nabla_s A\, N^q\,\tilde\nabla_q\tilde\nabla_r A\right).\end{gathered}\tag{1.22}$$

The last step consists in applying scalar ansatz to the fourth normality equation (1.4). With this goal, let's evaluate the following expression by virtue of (1.8):

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{n} P^j_\sigma\,\tilde{\nabla}_j F^i\,P^\varepsilon_i &= \sum_{q=1}^{n} N^q\,\tilde{\nabla}_q A\,P^\varepsilon_\sigma\,+\\ &+\frac{A}{|\mathbf{v}|}\,P^\varepsilon_\sigma - |\mathbf{v}|\sum_{r=1}^{n}\sum_{s=1}^{n} P^r_\sigma\,\tilde{\nabla}_r\tilde{\nabla}_s A\,P^{s\varepsilon}.\end{aligned} \tag{1.23}$$

Here $P^{s\varepsilon}$ are defined by (1.20). First two terms in right hand side of (1.23) differ from P^ε_σ only by scalar multiple. Therefore they are unessential for resulting equation:

$$\sum_{r=1}^{n}\sum_{s=1}^{n} P^r_\sigma\,\tilde{\nabla}_r\tilde{\nabla}_s A\,P^{s\varepsilon} = \lambda\,P^\varepsilon_\sigma. \tag{1.24}$$

Here λ is a scalar parameter that could be defined by the following formula:

$$\lambda = \sum_{r=1}^{n}\sum_{s=1}^{n}\frac{P^{rs}\,\tilde{\nabla}_r\tilde{\nabla}_s A}{n-1}. \tag{1.25}$$

The equations (1.24) with scalar parameter λ from (1.25) are obtained from (1.4) by applying scalar ansatz (1.8). They play key role in further analysis of whole set of normality equations.

§ 2. Fiber spherical coordinates.

The equations (1.21), (1.22), (1.24) form complete set of normality equations for scalar field A. They are equivalent to (1.1), (1.2), (1.3), (1.4) due to scalar ansatz (1.8). Let's study last one of them. In order to do this we write explicit expressions for covariant derivatives $\tilde{\nabla}_r$ and $\tilde{\nabla}_s$, applying formula (7.4) from Chapter III:

$$\sum_{r=1}^{n}\sum_{s=1}^{n} P^r_\sigma\,P^s_\varepsilon\,\frac{\partial^2 A}{\partial v^r\,\partial v^s} = \lambda\,P_{\sigma\varepsilon}. \tag{2.1}$$

The equations (2.1) contain partial derivatives with respect to variables $v^1, \ldots, v^n$ in the fiber of tangent bundle over each point $p \in M$, but they do not contain partial derivatives respective to variables $x^1, \ldots, x^n$ in base manifold M. Fiber $\pi^{-1}(p)$ is n-dimensional linear space with metric $\mathbf{g}$. Components of this metric g_{ij} in coordinates $v^1, \ldots, v^n$ are constants within fixed fiber. Therefore equations (2.1) are the equations in linear space with Euclidean metric.

Let's fix some point $p \in M$. Condition $|\mathbf{v}| = \text{const}$ defines a sphere of the radius $v = |\mathbf{v}|$ in tangent space $T_p(M) = \pi^{-1}(p)$ being the fiber of tangent bundle. Tangent space $T_p(M)$ is disjoint union of spheres of various radii and the point $\mathbf{v} = 0$, which can be treated as a sphere of zero radius. Let $u^1, \ldots, u^{n-1}$ be local coordinates on the sphere of unit radius S^{n-1}, and suppose that vector-function

$\mathbf{v} = \mathbf{N}(u^1, \ldots, u^{n-1})$ realize its embedding into the tangent space $T_p(M)$. Let's add one more coordinate $u^n = v = |\mathbf{v}|$ to $u^1, \ldots, u^{n-1}$. Then the relationships

$$\begin{aligned} v^1 &= u^n\, N^1(u^1, \ldots, u^{n-1}),\\ &\ldots\ldots\ldots\ldots\ldots\ldots\\ v^n &= u^n\, N^n(u^1, \ldots, u^{n-1}). \end{aligned} \tag{2.2}$$

define transition function binding Cartesian coordinates $v^1, \ldots, v^n$ and spherical coordinates $u^1, \ldots, u^n$ in $T_p(M)$. Such spherical coordinates $u^1, \ldots, u^n$ are called **fiber spherical** coordinates, since they are defined separately in the fiber of tangent bundle over fixed point $p \in M$. In fact, components $N^1, \ldots, N^n$ of the vector $\mathbf{N}$ in (2.2) depend on coordinates $x^1, \ldots, x^n$ of the point p as well. However, in the analysis of the equations (2.1) this circumstance plays no role, since p is fixed.

Let's make the change of variables (2.2) in the equations (2.1). As it's used to do in such situation, we denote by S and T transition matrices corresponding to the change of variables (2.2). For their components we have

$$S^k_i = \frac{\partial v^k}{\partial u^i}, \qquad T^k_i = \frac{\partial u^k}{\partial v^i}. \tag{2.3}$$

Partial derivatives $\partial/\partial v^i$ in (2.1) coincide with covariant derivatives with respect to metric connection for metric $\mathbf{g}$, which is flat Euclidean metric in $T_p(M)$ with constant components g_{ij} in Cartesian coordinates $v^1, \ldots, v^n$. In curvilinear coordinates components of metric $\mathbf{g}$ are not constant, therefore corresponding connection components ϑ^k_{ij} are nonzero, and the equations (2.1) take the form

$$\sum_{r=1}^{n}\sum_{s=1}^{n} \tilde{P}^r_\sigma\, \tilde{P}^s_\varepsilon \left(\frac{\partial^2 A}{\partial u^r\, \partial u^s} - \sum_{k=1}^{n} \vartheta^k_{rs}\, \frac{\partial A}{\partial u^k}\right) = \lambda\, \tilde{P}_{\sigma\varepsilon}. \tag{2.4}$$

Components of projection operator $\mathbf{P}$ are transformed to curvilinear coordinates $u^1, \ldots, u^n$ by standard formula

$$\tilde{P}^k_q = \sum_{i=1}^{n}\sum_{j=1}^{n} T^k_i\, S^j_q\, P^i_j,$$

where transition matrix components are defined by (2.3).

Spherical coordinates (2.2) have some features. Vector $\mathbf{N} = \mathbf{N}(u^1, \ldots, u^{n-1})$, that defines transition to spherical coordinates $u^1, \ldots, u^n$ in (2.2), coincides with normal vector of spheres $u^n = \text{const}$. Due to

$$S^i_n = \frac{\partial v^i}{\partial u^n} = N^i$$

vector $\mathbf{N}$ is n-th base vector in moving frame of curvilinear coordinates $u^1, \ldots, u^n$. Other vectors of this frame are perpendicular to $\mathbf{N}$ and tangent to spheres the $u^n =$

const. This determines the structure of components of metric tensor in spherical coordinates $u^1, \ldots, u^n$ defined by (2.2):

$$\tilde{g}_{ij} = \begin{Vmatrix} \tilde{g}_{11} & \ldots & \tilde{g}_{1\,n-1} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ \tilde{g}_{n-1\,1} & \ldots & \tilde{g}_{n-1\,n-1} & 0 \\ 0 & \ldots & 0 & 1 \end{Vmatrix}. \tag{2.5}$$

Quantities $u^1, \ldots, u^{n-1}$ can be considered as local coordinates on sphere $u^n =$ const. In this approach we find that upper diagonal block in matrix (2.5) is made of components of metric tensor for induced metric on sphere $|\mathbf{v}| = u^n = \text{const}$.

Bloc structure of matrix (2.5) determines similar structure of components of metric connection ϑ^k_{rs}. For i, j, k ranging $1 \leqslant i, j, k \leqslant n-1$ the quantities ϑ^k_{rs} coincide with components of metric connection for induced metric on spheres $u^n = \text{const}$. Other components ϑ^k_{rs} are calculated explicitly. Part of them are zero:

$$\vartheta^i_{nn} = \vartheta^n_{in} = \vartheta^n_{ni} = 0 \quad \text{for all} \quad i = 1, \ldots, n. \tag{2.6}$$

Moreover, for $1 \leqslant i, j \leqslant n-1$ and for $k = n$ we have

$$\vartheta^n_{ij} = -\frac{\tilde{g}_{ij}}{|\mathbf{v}|}, \qquad\qquad \vartheta^i_{nj} = \vartheta^i_{jn} = \frac{\delta^i_j}{|\mathbf{v}|}. \tag{2.7}$$

Operator $\mathbf{P}$ is a projector to hyperplane perpendicular to the vector $\mathbf{N}$ that coincides with n-th coordinate vector in moving frame of curvilinear coordinates $u^1, \ldots, u^n$. Therefore, taking into account the structure of metric tensor (2.5), we get

$$\tilde{N}^i = \tilde{N}_i = \begin{cases} 0 & \text{for} \quad i < n, \\ 1 & \text{for} \quad i = n. \end{cases}$$

In order to evaluate components of projector field $\mathbf{P}$ we use formula (5.4) from Chapter VII: $\tilde{P}^i_j = \delta^i_j - \tilde{N}^i\,\tilde{N}_j$. This yields

$$\tilde{P}^i_j = \begin{Vmatrix} 1 & \ldots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & 1 & 0 \\ 0 & \ldots & 0 & 0 \end{Vmatrix}. \tag{2.8}$$

The relationships (2.6), (2.7), and (2.8) allow to make the equations (2.4) more specific. For $1 \leqslant \varepsilon, \sigma \leqslant n-1$ we have

$$\frac{\partial^2 A}{\partial u^\varepsilon\,\partial u^\sigma} - \sum_{k=1}^{n-1} \vartheta^k_{\varepsilon\sigma}\,\frac{\partial A}{\partial u^k} + \frac{\tilde{g}_{\varepsilon\sigma}}{|\mathbf{v}|}\,\frac{\partial A}{\partial u^n} = \lambda\,\tilde{g}_{\varepsilon\sigma}. \tag{2.9}$$

In case of $\varepsilon = n$ and in case of $\sigma = n$ the equations (2.4) turn to identity. They add nothing to (2.9). As for λ, it is an arbitrary scalar parameter. Upon substituting

$$\lambda = \mu + \frac{1}{|\mathbf{v}|}\,\frac{\partial A}{\partial u^n} = \mu + \frac{1}{|\mathbf{v}|}\sum_{i=1}^{n} N^i\,\tilde{\nabla}_i A \tag{2.10}$$

into (2.9) we replace λ by other scalar parameter μ. This simplifies our equations

$$\frac{\partial^2 A}{\partial u^\varepsilon\,\partial u^\sigma} - \sum_{k=1}^{n-1} \vartheta^k_{\varepsilon\sigma}\,\frac{\partial A}{\partial u^k} = \mu\,\tilde{g}_{\varepsilon\sigma}. \tag{2.11}$$

Due to $1 \leqslant \varepsilon, \sigma \leqslant n-1$ the equations (2.11) describe only the dependence of A upon coordinate $u^1, \ldots, u^{n-1}$ on spheres $|\mathbf{v}| = \text{const}$. The quantities $\vartheta^k_{\varepsilon\sigma}$ in (2.11) are components of metric connection on such spheres. Therefore we can write (2.11) in terms of covariant derivatives on spheres

$$\bar{\nabla}_\varepsilon\bar{\nabla}_\sigma A = \mu\,\tilde{g}_{\varepsilon\sigma}. \tag{2.12}$$

In multidimensional case $n \geqslant 3$, which we are studying now, system of equations (2.12) remains overdetermined yet. This allows us to consider it's differential consequences. To do it we use well-known identity (see [77], or identity (7.12) in Chapter III):

$$[\bar{\nabla}_i,\,\bar{\nabla}_j]\bar{\nabla}_q A = -\sum_{k=1}^{n-1} \tilde{R}^k_{qij}\,\bar{\nabla}_k A. \tag{2.13}$$

Here $\tilde{R}^k_{qij}$ are the components of curvature tensor for induced metric on sphere $|\mathbf{v}| = v = \text{const}$. This sphere is a space of constant sectional curvature (see [13]). Curvature tensor of the space of constant sectional curvature is determined by one scalar quantity. In case of sphere this quantity is its radius:

$$\tilde{R}^k_{qij} = \frac{\delta^k_i\,\tilde{g}_{qj} - \delta^k_j\,\tilde{g}_{qi}}{|\mathbf{v}|^2}.$$

This specifies right hand side in the identity (2.13). Now this identity is written as

$$[\bar{\nabla}_i,\,\bar{\nabla}_j]\bar{\nabla}_q A = \frac{\tilde{g}_{qi}\,\bar{\nabla}_j A - \tilde{g}_{qj}\,\bar{\nabla}_i A}{|\mathbf{v}|^2}. \tag{2.14}$$

Left hand side of (2.14) is calculated by virtue of (2.12):

$$[\bar{\nabla}_i,\,\bar{\nabla}_j]\bar{\nabla}_q A = \bar{\nabla}_i\bar{\nabla}_j\bar{\nabla}_q A - \bar{\nabla}_j\bar{\nabla}_i\bar{\nabla}_q A = \tilde{g}_{jq}\,\bar{\nabla}_i\mu - \tilde{g}_{iq}\,\bar{\nabla}_j\mu.$$

Upon substituting this expression into (2.14) we obtain the required differential consequence of the equations (2.12):

$$\tilde{g}_{qj}\,\bar{\nabla}_i\mu - \tilde{g}_{qi}\,\bar{\nabla}_j\mu = -\frac{\tilde{g}_{qj}\,\bar{\nabla}_i A - \tilde{g}_{qi}\,\bar{\nabla}_j A}{|\mathbf{v}|^2}. \tag{2.15}$$

These equations (2.15) can be made more simple, if we multiply them by g^{qj} and contract with respect to indices q and j. This yields

$$(n-2)\,\bar{\nabla}_i\mu = -\frac{(n-2)}{|\mathbf{v}|^2}\bar{\nabla}_i A.$$

In multidimensional case $n \geqslant 3$, which we are studying now, the multiple $(n-2)$ in the above equation is nonzero. Hence

$$\bar{\nabla}_i A = -|\mathbf{v}|^2\,\bar{\nabla}_i\mu. \tag{2.16}$$

We are to note that differential consequences (2.16), obtained just above, are of lower order than initial equations (2.12) by themselves.

Next step consists in substituting (2.16) back to the equations (2.12). As a result of this substitution we obtain the following equations:

$$\bar{\nabla}_i\bar{\nabla}_j\mu = -\frac{\tilde{g}_{ij}}{|\mathbf{v}|^2}\,\mu. \tag{2.17}$$

Similar to (2.12), the equations (2.17) can be treated as the equations on spheres $|\mathbf{v}| = \text{const}$ in tangents space $T_p(M)$ at the fixed point p. They describe the dependence ob μ upon spherical coordinates $u^1, \ldots, u^{n-1}$, but do not determine the dependence of μ on radial variable $u^n = |\mathbf{v}|$.

Theorem 2.1. *Each smooth function μ on sphere of the radius $u^n = |\mathbf{v}| = \text{const}$ satisfying differential equations* (2.17) *has the form*

$$\mu = \sum_{i=1}^{n} m_i\,N^i, \tag{2.18}$$

where $m_1, \ldots, m_n$ are some constants, and $N^1, \ldots, N^n$ are components of vector **N** *that determines embedding of sphere into Euclidean space according to* (2.2).

Proof. The equations (2.17) have tensorially covariant form, i. e. they are written in covariant derivatives. They do not change their appearance under the changes of local coordinates on spheres $|\mathbf{v}| = \text{const}$. Right hand side of (2.18) also is not sensible to the changes of local coordinates on spheres. Therefore we can choose some special (most convenient) coordinates, assuming that metric $\mathbf{g}$ in the fiber $T_p(M)$ over fixed point p in Cartesian coordinates $v^1, \ldots, v^n$ is brought to canonical form, so that its matrix is determined by Kronecker delta symbol:

$$g_{ij} = \delta_{ij} = \begin{cases} 1 & \text{for } \ i = j, \\ 0 & \text{for } \ i \neq j. \end{cases} \tag{2.19}$$

This can be done at the expense of proper choice of local coordinates $x^1, \ldots, x^n$ on the base manifold M. Now spherical coordinates $u^1, \ldots, u^{n-1}$ can be chosen so that in metric (2.19) change of variables (2.2) specifies by the following relationships:

$$\begin{aligned}
v^1 &= v\,\sin(u_{n-1})\,\sin(u_{n-2})\cdot\ldots\cdot\sin(u_3)\sin(u_2)\,\sin(u_1),\\
v^2 &= v\,\sin(u_{n-1})\,\sin(u_{n-2})\cdot\ldots\cdot\sin(u_3)\sin(u_2)\,\cos(u_1),\\
v^3 &= v\,\sin(u_{n-1})\,\sin(u_{n-2})\cdot\ldots\cdot\sin(u_3)\cos(u_2),\\
&\ldots\ldots\ldots\ldots\ldots\ldots\\
v^{n-1} &= v\,\sin(u_{n-1})\,\cos(u_{n-2}),\\
v^n &= v\,\cos(u_{n-1}).
\end{aligned} \tag{2.20}$$

Here $v = |\mathbf{v}| = u^n$ is the radius of sphere. The following inequalities determine ranges for spherical coordinates $u^1, \ldots, u^n$:

$$\begin{aligned}
&0 \leqslant u^1 < 2\pi,\\
&0 \leqslant u^2 < \pi,\\
&\ldots\ldots\ldots\\
&0 \leqslant u^{n-1} < \pi,\\
&0 \leqslant u^n < 2 < +\infty.
\end{aligned}$$

In spherical coordinates $u^1, \ldots, u^n$ given by (2.20) the matrix of metric tensor is diagonal. Its diagonal elements are positive: $g_{ii} > 0$. They define the quantities $H_i = \sqrt{g_{ii}}$, which are called **scaling factors** or **Lame coefficients**:

$$\begin{aligned}
H_1 &= v\,\sin(u_{n-1})\,\sin(u_{n-2})\cdot\ldots\cdot\sin(u_3)\sin(u_2),\\
H_2 &= v\,\sin(u_{n-1})\,\sin(u_{n-2})\cdot\ldots\cdot\sin(u_3),\\
&\ldots\ldots\ldots\ldots\ldots\ldots\\
H_{n-2} &= v\,\sin(u_{n-1}),\\
H_{n-1} &= v,\\
H_n &= 1.
\end{aligned} \tag{2.21}$$

Components of metric connection ϑ^k_{ij} are defined according to well-known formula

$$\vartheta^k_{ij} = \frac{1}{H_k}\,\frac{\partial H_k}{\partial u^j}\,\delta_{ik} + \frac{1}{H_k}\,\frac{\partial H_k}{\partial u^i}\,\delta_{jk} - \frac{H_i}{(H_k)^2}\,\frac{\partial H_i}{\partial u^k}\,\delta_{ij},$$

which is derived from formula (1.6) in Chapter VI. Most connection components ϑ^k_{ij} appear to be zero. This follows from explicit formula for them:

$$\vartheta^k_{ij} = \begin{cases} \cot(u_j)\,\delta^k_i & \text{for } \ i < j \leqslant n-1,\\ \cot(u_i)\,\delta^k_j & \text{for } \ j < i \leqslant n-1.\end{cases}$$

The case of coinciding indices $i = j$ subdivides into two cases. For the first of them, when $k \leqslant i$, we have $\vartheta^k_{ii} = 0$. Tn second subcase, when $i < k \leqslant n-1$, we get

$$\vartheta^k_{ii} = -\frac{\cos(u^k)}{\sin(u^k)}\prod_{s=i+1}^{k}\sin^2(u^s).$$

Using these formulas, we can write the equations (2.17) explicitly. We obtain the equations arranged in three groups. First group corresponds to the case $i < j \leqslant n-1$:

$$\mu_{ij} = \frac{\cos(u^j)}{\sin(u^j)}\,\mu_i. \tag{2.22}$$

The equations of second group correspond to the case $i = j \leqslant n-2$:

$$\mu_{ii} + \sum_{k=i+1}^{n-1}\frac{\cos(u^k)}{\sin(u^k)}\left(\prod_{s=i+1}^{k}\sin^2(u^s)\right)\mu_k = -\left(\prod_{s=i+1}^{n-1}\sin^2(u^s)\right)\mu. \tag{2.23}$$

Here by μ_i and μ_{ij} de denoted partial derivatives of μ of the first and second order with respect to variables u^i and u^j:

$$\mu_i = \frac{\partial\mu}{\partial u^i}, \qquad\qquad \mu_{ij} = \frac{\partial\mu}{\partial u^i\,\partial u^j}.$$

And finally, third group of equations corresponds to $i = j = n-1$. It consists of exactly one equation,which is very simple:

$$\mu_{ii} = -\mu, \ \text{ where } \ i = n-1. \tag{2.24}$$

In further proof of theorem we use the method of reducing dimension $n \to n-1$. Let's examine the equation (2.24). It describes the dependence of μ on the last coordinate on sphere u^{n-1}. One can easily write the general solution of this equation:

$$\mu = \tilde{\mu}\,\sin(u^{n-1}) + m_n\,\cos(u^{n-1}). \tag{2.25}$$

Parameters $\tilde{\mu}$ and m_n can depend on $u^1, \ldots, u^{n-2}$, but they do not depend on u^{n-1}. Let's substitute (2.25) into the equations (2.22), taking $j = n-1$ and $i < n-1$:

$$\begin{aligned}\frac{\partial\tilde{\mu}}{\partial u^i}\,\cos(u^{n-1}) - \frac{\partial\tilde{m}_n}{\partial u^i}\,\sin(u^{n-1}) = \\ = \frac{\cos(u^{n-1})}{\sin(u^{n-1})}\left(\frac{\partial\tilde{\mu}}{\partial u^i}\,\sin(u^{n-1}) + \frac{\partial\tilde{m}_n}{\partial u^i}\,\cos(u^{n-1})\right).\end{aligned}$$

Opening brackets and collecting similar terms in both sides of this equality, we get

$$\frac{\partial\tilde{m}_n}{\partial u^i} = 0 \ \text{ for all } \ i < n-1.$$

This means that parameter m_n in (2.25) does not depend on $u^1, \ldots, u^{n-1}$. It is a constant on sphere $|\mathbf{v}| = v = \text{const}$.

Keeping in mind that m_n is constant, let's substitute the expression (2.25) for μ into the equations (2.23). For $i = n-2$ we obtain

$$\begin{gathered}\tilde{\mu}_{ii}\,\sin(u^{n-1}) + \frac{\cos(u^{n-1})}{\sin(u^{n-1})}\,\sin^2(u^{n-1})\cdot\bigl(\tilde{\mu}\,\cos(u^{n-1})\ -\\ -\ m_n\,\sin(u^{n-1})\bigr) = -\sin^2(u^{n-1})\cdot\bigl(\tilde{\mu}\,\sin(u^{n-1}) + m_n\,\cos(u^{n-1})\bigr).\end{gathered}$$

Opening brackets and collecting similar terms, we get the equation

$$\tilde{\mu}_{ii} = -\tilde{\mu}, \ \ \text{where} \ \ i = n-2. \tag{2.26}$$

If the dimension n is high enough ($n > 3$), we have more equations, these are the equations (2.22) with $i < j \leqslant n-2$ and the equations (2.23) with $i \leqslant n-3$. Substituting (2.25) into the equations (2.22) for $i < j \leqslant n-2$, we get

$$\tilde{\mu}_{ij} = \frac{\cos(u^j)}{\sin(u^j)}\,\tilde{\mu}_i. \tag{2.27}$$

Substituting (2.25) into (2.23), we get more bulk expressions:

$$\begin{gathered}\left(\tilde{\mu}_{ii} + \sum_{k=i+1}^{n-2}\frac{\cos(u^k)}{\sin(u^k)}\left(\prod_{s=i+1}^{k}\sin^2(u^s)\right)\tilde{\mu}_k\right)\cdot\sin(u^{n-1}) +\\ +\frac{\cos(u^{n-1})}{\sin(u^{n-1})}\left(\prod_{s=i+1}^{n-1}\sin^2(u^s)\right)\cdot\bigl(\tilde{\mu}\,\cos(u^{n-1}) - m_n\,\sin(u^{n-1})\bigr) =\\ = -\left(\prod_{s=i+1}^{n-1}\sin^2(u^s)\right)\cdot\bigl(\tilde{\mu}\,\sin(u^{n-1}) - m_n\,\cos(u^{n-1})\bigr).\end{gathered}$$

Let's open brackets and collect similar terms in the above expression. All entries of parameter m_n then cancel out. Further let's take into account the identity $\cos^2(u^{n-1}) + \sin^2(u^{n-1}) = 1$ and divide both sides of resulting equality by $\sin(u^{n-1}) \neq 0$. As a result we get the equation for $\tilde{\mu}$, where $i \leqslant n-3$:

$$\begin{gathered}\tilde{\mu}_{ii} + \sum_{k=i+1}^{n-2}\frac{\cos(u^k)}{\sin(u^k)}\left(\prod_{s=i+1}^{k}\sin^2(u^s)\right)\tilde{\mu}_k =\\ = -\left(\prod_{s=i+1}^{n-2}\sin^2(u^s)\right)\tilde{\mu}.\end{gathered} \tag{2.28}$$

The equations (2.26), (2.27), and (2.28) are analogous to (2.24), (2.22), and (2.23) with the only difference that n is replaced by $n-1$ and μ is replaced by $\tilde{\mu}$. This means that the equations (2.24), (2.22), and (2.23) admit the reduction $n \to n-1$ by substituting (2.25). Let's consider the series of reductions:

$$n \to n-1 \to n-2 \to \ \ldots \ \to 3 \to 2. \tag{2.29}$$

All reductions in the series (2.29) are similar to each other, except for the last one. In making last reduction we break the condition $n > 3$. Therefore the equations (2.27) and (2.28) do not arise, and we get the only equation (2.26) with $i = 1$:

$$\tilde{\mu}_{11} = -\tilde{\mu}. \tag{2.30}$$

General solution of the equation (2.30) is determined by two constants m_1 and m_2:

$$\tilde{\mu} = m_1\,\sin(u^1) + m_2\,\cos(u^1). \tag{2.31}$$

Now let's step back along the series of reductions (2.29), using (2.25). As a result we get initial function μ

$$\mu = m_1 \prod_{s=1}^{n-1} \sin(u^s) + \sum_{k=2}^{n} m_k \left(\prod_{s=k}^{n-1} \sin(u^s)\right) \cos(u^{k-1}). \tag{2.32}$$

Here $m_1, \ldots, m_n$ are constants. Comparing (2.2) and (2.20), we get

$$\begin{aligned}
&N^1 = \prod_{s=1}^{n-1} \sin(u^s),\\
&N^k = \left(\prod_{s=k}^{n-1} \sin(u^s)\right) \cos(u^{k-1}) \ \text{ for } \ 1 < k < n,\\
&N^n = \cos(u^{n-1}).
\end{aligned}$$

And finally, comparing the above expression for components of vector $\mathbf{N}$ with (2.32) we conclude, that formula (2.14) is exactly the same as formula (2.18) in the statement of theorem 2.1. Proof is over. □

Having proved the theorem 2.1 and thereby having studied the equation (2.17), let's return to the equations (2.16). Let's write them as

$$\bar{\nabla}_i(A + |\mathbf{v}|^2\,\mu) = 0. \tag{2.33}$$

Covariant derivative $\bar{\nabla}_i$, when applied to scalar field $A + |\mathbf{v}|^2\,\mu$, coincides with partial derivative $\partial/\partial u^i$ with respect to local coordinate u^i on sphere $|\mathbf{v}| = \text{const}$. Therefore from the equation (2.33) we derive

$$A + |\mathbf{v}|^2\,\mu = \text{const}\,. \tag{2.34}$$

Note that right hand side of (2.34) and parameters $m_1, \ldots, m_n$ in (2.18) are constant only within each particular sphere $|\mathbf{v}| = \text{const}$ in $T_p(M)$. Therefore, denoting $b_i = -|\mathbf{v}|\, m_i$, we can formulate the following theorem.

Theorem 2.2. *Scalar field A from extended algebra of tensor fields on Riemannian manifold $\mathbf{M}$, which is smooth everywhere on TM, except for the points of zero section $\mathbf{v} = 0$, satisfies the equations* (1.24) *if and only if it is given by formula*

$$A = a + \sum_{i=1}^{n} b_i\, v^i, \tag{2.35}$$

where extended scalar field a and the components b_i of extended covectorial field $\mathbf{b}$ depend only on modulus of velocity vector $v = |\mathbf{v}|$ in fibers of tangent bundle TM.

Necessity of the conclusion of theorem 2.2 is already proved by discussion preceding the statement of the theorem. In order to prove sufficiency one should only substitute (2.35) into the equations (1.24). Let's use that $a = a(x^1, \ldots, x^n, v)$ and $b_i = b_i(x^1, \ldots, x^n, v)$, where $v = |\mathbf{v}|$, and denote by a' and b'_i the derivatives

$$a' = \frac{\partial a}{\partial v}, \qquad\qquad b'_i = \frac{\partial b_i}{\partial v}.$$

Then for the derivatives $\tilde{\nabla}_s A$ and $\tilde{\nabla}_r \tilde{\nabla}_s A$ in the equations (1.24) we get

$$\tilde{\nabla}_s A = \left(a' + \sum_{i=1}^{n} b'_i\, v^i\right) N_s + b_s, \tag{2.36}$$

$$\begin{aligned} \tilde{\nabla}_r \tilde{\nabla}_s A = &\left(a'' + \sum_{i=1}^{n} b''_i\, v^i\right) N_r\, N_s + \\ &+ b'_s\, N_r + b'_r\, N_s + \left(\frac{a'}{v} + \sum_{i=1}^{n} b'_i\, N^i\right) P_{rs}. \end{aligned} \tag{2.37}$$

Further let's contract (2.37) with P^r_σ, and with $P^{s\varepsilon}$ respective to indices r and s:

$$\sum_{r=1}^{n} \sum_{s=1}^{n} P^r_\sigma\, \tilde{\nabla}_r \tilde{\nabla}_s A\, P^{s\varepsilon} = \left(\frac{a'}{v} + \sum_{i=1}^{n} b'_i\, N^i\right) P^\varepsilon_\sigma.$$

The relationship just obtained shows that the quantity A from (2.35) satisfies the equations (1.24), where scalar parameter λ is given by

$$\lambda = \frac{a'}{v} + \sum_{i=1}^{n} b'_i\, N^i.$$

We conclude the study of the equations (1.24) by calculating μ on a base of formula (2.10) binding λ and μ:

$$\mu = \lambda - \frac{1}{|\mathbf{v}|} \sum_{i=1}^{n} N^i \, \tilde{\nabla}_i A = -\sum_{i=1}^{n} \frac{b_i}{|\mathbf{v}|} \, N^i.$$

This expression is in agreement with formula (2.18), if we remember that $b_i = |\mathbf{v}| \, m_i$.

§ 3. Specification of scalar ansatz.

Normality equations (1.24), as it follows from theorem 2.2, determine the dependence of A upon velocity vector $\mathbf{v}$. Dependence of A upon spatial coordinates $x^1, \ldots, x^n$ should be specified by substituting (2.35) into other two normality equations (1.21) and (1.22). Let's make calculations needed for it:

$$\nabla_s A = \nabla_s a + \sum_{i=1}^{n} \nabla_s b_i \, v^i, \tag{3.1}$$

$$\nabla_r \tilde{\nabla}_s A = \left(\nabla_r a' + \sum_{i=1}^{n} \nabla_r b'_i \, v^i\right) N_s + \nabla_r b_s. \tag{3.2}$$

From (2.36) and (2.37) we derive the following relationships:

$$\sum_{r=1}^{n} P^r_\sigma \, \tilde{\nabla}_r A = \sum_{r=1}^{n} P^r_\sigma \, b_r, \tag{3.3}$$

$$\sum_{s=1}^{n} \sum_{q=1}^{n} P^s_\varepsilon \, N^q \, \tilde{\nabla}_q \, \tilde{\nabla}_s A = \sum_{s=1}^{n} P^s_\varepsilon \, b'_s. \tag{3.4}$$

Let's combine (3.3) and (3.4). As a result we get the relationship

$$\sum_{s=1}^{n} \sum_{r=1}^{n} \sum_{q=1}^{n} P^r_\sigma \, P^s_\varepsilon \, \tilde{\nabla}_r A \, N^q \, \tilde{\nabla}_q \tilde{\nabla}_s A = \sum_{s=1}^{n} \sum_{r=1}^{n} P^r_\sigma \, P^s_\varepsilon \, b_r \, b'_s.$$

Then let's multiply both sides of (3.2) by $P^r_\sigma \, P^s_\varepsilon$ and let's contract it with respect to r and s. This yields one more relationship:

$$\sum_{s=1}^{n} \sum_{r=1}^{n} P^r_\sigma \, P^s_\varepsilon \, \nabla_r \tilde{\nabla}_s A = \sum_{s=1}^{n} \sum_{r=1}^{n} P^r_\sigma \, P^s_\varepsilon \, \nabla_r b_s.$$

Now let's add two above relationships. This determines the result of substituting

(2.35) into the left hand side of the normality equation (1.22):

$$\begin{gathered}\sum_{s=1}^{n}\sum_{r=1}^{n}P^r_\sigma\,P^s_\varepsilon\left(\nabla_r\tilde\nabla_sA+\sum_{q=1}^{n}\tilde\nabla_rA\,N^q\,\tilde\nabla_q\tilde\nabla_sA\right)=\\ =\sum_{s=1}^{n}\sum_{r=1}^{n}P^r_\sigma\,P^s_\varepsilon\,\nabla_rb_s+\sum_{s=1}^{n}\sum_{r=1}^{n}P^r_\sigma\,P^s_\varepsilon\,b_r\,b'_s.\end{gathered}\tag{3.5}$$

In a similar way we can calculate right hand side of normality equation (1.22):

$$\begin{gathered}\sum_{s=1}^{n}\sum_{r=1}^{n}P^r_\sigma\,P^s_\varepsilon\left(\nabla_s\tilde\nabla_rA+\sum_{q=1}^{n}\tilde\nabla_sA\,N^q\,\tilde\nabla_q\tilde\nabla_rA\right)=\\ =\sum_{s=1}^{n}\sum_{r=1}^{n}P^r_\sigma\,P^s_\varepsilon\,\nabla_sb_r+\sum_{s=1}^{n}\sum_{r=1}^{n}P^r_\sigma\,P^s_\varepsilon\,b_s\,b'_r.\end{gathered}\tag{3.6}$$

From (3.5) and (3.6) we conclude that the equation (1.22) reduces to

$$\sum_{s=1}^{n}\sum_{r=1}^{n}P^r_\sigma\,P^s_\varepsilon\left(\nabla_rb_s+b_r\,b'_s-\nabla_sb_r-b_s\,b'_r\right)=0.\tag{3.7}$$

For the further analysis of the obtained equations (3.7) we should use the peculiarity of extended covector field $\mathbf{b}$. Components $b_1, \ldots, b_n$ of this field depend only on modulus of velocity vector, they do not depend on its direction. In calculating derivatives ∇_rb_s and ∇_sb_r in (3.7) we shall use the following theorem.

THEOREM 3.1. *Let $(U, x^1, \ldots, x^n)$ be some local map on Riemannian manifold M, and let $(\pi^{-1}(U), x^1, \ldots, x^n, v^1, \ldots, v^n)$ be associated local map on tangent bundle TM. If components of extended tensor field $\mathbf{X}$ in U depend only on modulus of velocity vector $\mathbf{v}$, then components of spatial gradient in U are calculated by formula*

$$\begin{gathered}\nabla_mX^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}=\frac{\partial X^{i_1\ldots\,i_r}_{j_1\ldots\,j_s}}{\partial x^m}+\sum_{k=1}^{r}\sum_{a_k=1}^{n}\Gamma^{i_k}_{m\,a_k}\,X^{i_1\ldots\,a_k\ldots\,i_r}_{j_1\ldots\ldots\ldots\,j_s}-\\ -\sum_{k=1}^{s}\sum_{b_k=1}^{n}\Gamma^{b_k}_{m\,j_k}X^{i_1\ldots\ldots\ldots\,i_r}_{j_1\ldots\,b_k\ldots\,j_s}.\end{gathered}\tag{3.8}$$

PROOF. Formula (3.8) is obtained as a reduction of the formula (7.3) from Chapter III. For the components of tensor field $\mathbf{X}$, which is mentioned in the theorem, their natural arguments are $x^1, \ldots, x^n$ and v, where

$$v=|\mathbf{v}|=\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}g_{ij}(x^1,\ldots,x^n)\,v^i\,v^j}\,.\tag{3.9}$$

However, in formula (7.3) from Chapter III all derivatives are assumed to be calculated with respect to $x^1, \ldots, x^n, v^1, \ldots, v^n$. Therefore we should substitute

$$\frac{\partial X^{i_1\ldots\, i_r}_{j_1\ldots\, j_s}}{\partial v^b} \quad\text{for}\quad \frac{\partial X^{i_1\ldots\, i_r}_{j_1\ldots\, j_s}}{\partial v}\cdot\frac{\partial v}{\partial v^b}, \tag{3.10}$$

$$\frac{\partial X^{i_1\ldots\, i_r}_{j_1\ldots\, j_s}}{\partial x^m} \quad\text{for}\quad \frac{\partial X^{i_1\ldots\, i_r}_{j_1\ldots\, j_s}}{\partial x^m}+\frac{\partial X^{i_1\ldots\, i_r}_{j_1\ldots\, j_s}}{\partial v}\cdot\frac{\partial\, v}{\partial x^m}. \tag{3.11}$$

Partial derivatives $\partial\, v/\partial v^b$ and $\partial\, v/\partial x^m$ are calculated by virtue of (3.9). Upon calculating these derivatives and upon making substitutions (3.10) and (3.11) into the formula (7.3) from Chapter III we find two terms there:

$$\sum_{a=1}^{n}\sum_{b=1}^{n}\frac{X^{i_1\ldots\, i_r}_{j_1\ldots\, j_s}}{\partial v}\,\frac{1}{2}\,\frac{\partial g_{ab}}{\partial x^m}\,\frac{v^a\,v^b}{v} \quad\text{and}\quad -\sum_{a=1}^{n}\sum_{b=1}^{n} v^a\,\Gamma^b_{ma}\,\frac{\partial X^{i_1\ldots\, i_r}_{j_1\ldots\, j_s}}{\partial v}\,N_b.$$

If we take into account explicit formula for components of metric connection Γ^b_{ma} (see (1.2) in Chapter V), then we easily see that these two terms cancel out each other. As a result formula (7.3) from Chapter III reduces to (3.8). □

Remark. Calculations similar to that, which we used in proving theorem 3.1, were applied to scalar field H (see formula (3.16) in Chapter VI).

Corollary. *Components $\omega_{rs} = \nabla_r b_s + b_r\, b'_s - \nabla_s b_r - b_s\, b'_r$ of skew-symmetric extended tensor field $\boldsymbol{\omega}$ in* (3.7) *depend only on modulus of velocity vector* $\mathbf{v}$ *within fibers of tangent bundle TM.*

This fact is an immediate consequence of formula (3.8). It allows us to make further simplifications in the equations (3.7). Let $\mathbf{c}$ and $\mathbf{d}$ be two arbitrary vectors from tangent space $T_p(M)$. In multidimensional case $n \geqslant 3$ we can keep $|\mathbf{v}|$ unchanged and rotate velocity vector $\mathbf{v}$ so that will be perpendicular to $\mathbf{c}$ and $\mathbf{d}$ simultaneously. Then the following equalities will be fulfilled

$$\sum_{\sigma=1}^{n} P^r_\sigma\, c^\sigma = c^r, \qquad\qquad \sum_{\varepsilon=1}^{n} P^s_\varepsilon\, d^\varepsilon = d^s.$$

Therefore we can bring the equations (3.7) to the form

$$\sum_{s=1}^{n}\sum_{r=1}^{n} c^r\, d^s\left(\nabla_r b_s + b_r\, b'_s - \nabla_s b_r - b_s\, b'_r\right) = 0. \tag{3.12}$$

Since $\mathbf{c}$ and $\mathbf{d}$ are two arbitrary vectors, the equations (3.12) are equivalent to

$$\nabla_r b_s + b_r\, b'_s = \nabla_s b_r + b_s\, b'_r. \tag{3.13}$$

Next step consists in reducing the equations (1.21). For this purpose let's substitute (2.35) into (1.21). Due to (3.1) then we have

$$\sum_{s=1}^{n}\nabla_s A\,P^s_k=\sum_{s=1}^{n}\nabla_s a\,P^s_k+\sum_{s=1}^{n}\sum_{r=1}^{n}\nabla_s b_r\,v^r\,P^s_k. \tag{3.14}$$

Further from (2.36) and (2.37) we derive the following two relationships:

$$\sum_{r=1}^{n}P^{qr}\,\tilde{\nabla}_q A=\sum_{s=1}^{n}b_s\,P^{sr}, \tag{3.15}$$

$$|\mathbf{v}|\sum_{r=1}^{n}\sum_{s=1}^{n}P^{qr}\,\tilde{\nabla}_r\tilde{\nabla}_s A\,P^s_k=\left(a'+\sum_{r=1}^{n}b'_r\,v^r\right)P^q_k. \tag{3.16}$$

In (3.15) we have free index r, and in (3.16) we have free index is q. Let's multiply the equalities (3.15) and (3.16), then contract resulting product with respect to indices r and q upon multiplying it by g_{rq}. This yields

$$|\mathbf{v}|\sum_{q=1}^{n}\sum_{r=1}^{n}\sum_{s=1}^{n}P^{qr}\,\tilde{\nabla}_q A\,\tilde{\nabla}_r\tilde{\nabla}_s A\,P^s_k=\left(a'+\sum_{r=1}^{n}b'_r\,v^r\right)\sum_{s=1}^{n}b_s\,P^s_k.$$

Let's multiply (2.37) by $N^r\,A\,P^s_q$ and contract it with respect to indices r and s:

$$-\sum_{r=1}^{n}\sum_{s=1}^{n}N^r\,A\,\tilde{\nabla}_r\tilde{\nabla}_s A\,P^s_k=-\left(a+\sum_{r=1}^{n}b_r\,v^r\right)\sum_{s=1}^{n}b'_s\,P^s_k.$$

And finally, let's contract (3.2) with N^r and with P^s_k by indices r and s. This yields

$$-|\mathbf{v}|\sum_{r=1}^{n}\sum_{s=1}^{n}N^r\,\nabla_r\tilde{\nabla}_s A\,P^s_k=-\sum_{r=1}^{n}\sum_{s=1}^{n}v^r\,\nabla_r b_s\,P^s_k.$$

Now, in order to get the result of substituting (2.35) into the equations (1.21) we should add (3.14) and three above equalities:

$$\sum_{s=1}^{n}\left(\nabla_s a+b_s\,a'-a\,b'_s+\sum_{r=1}^{n}v^r\left(\nabla_s b_r+b_s\,b'_r-\nabla_r b_s-b_r\,b'_s\right)\right)P^s_k=0.$$

Taking into account (3.13) we find that the expression enclosed into smaller round brackets is zero. Therefore (1.21) is written as

$$\sum_{s=1}^{n}\left(\nabla_s a+b_s\,a'-a\,b'_s\right)P^s_k=0. \tag{3.17}$$

The equations (3.17) are similar to (3.7), contraction with projector components P^s_k in them can be omitted. Then the equations (3.17) are written as

$$\nabla_s a + b_s\, a' = a\, b'_s. \tag{3.18}$$

Reasons used to get (3.18) are similar to those used to bring (3.13) to the form (3.7).

Theorem 3.2. *Force field* $\mathbf{F}$ *defined by scalar ansatz* (1.8) *corresponds to Newtonian dynamical system admitting the normal shift on Riemannian manifold* M *if and only if scalar field* A *in* (1.8) *is defined by formula* (2.35), *where extended fields* a *and* $\mathbf{b}$ *depending only on modulus of velocity vector within fibers of tangent bundle* TM *are such that they satisfy the equation* (3.13) *and* (3.18).

Thus, the scalar ansatz (1.8) and its specification (2.35) reduce the system of normality equations (1.1), (1.2), (1.3), and (1.4) to the equations (3.13) and (3.18). These two equations are called **reduced normality equations**.

§ 4. Analysis of the reduced equations.

For the purpose of further study of the equations (3.13) and (3.18) let's express covariant derivatives in them through corresponding partial derivatives. In order to do it we use formula (3.8) and take into account symmetry of connection components:

$$\left(\frac{\partial}{\partial x^s} + b_s\,\frac{\partial}{\partial v}\right) a = \left(a\,\frac{\partial}{\partial v}\right) b_s, \tag{4.1}$$

$$\left(\frac{\partial}{\partial x^s} + b_s\,\frac{\partial}{\partial v}\right) b_r = \left(\frac{\partial}{\partial x^r} + b_r\,\frac{\partial}{\partial v}\right) b_s. \tag{4.2}$$

The equations (4.2) form closed system of equations with respect to $b_1, \ldots, b_n$. Let's study them separately. Consider the differential operators used in them:

$$\mathbf{L}_i = \frac{\partial}{\partial x^i} + b_i\,\frac{\partial}{\partial v}, \ \text{ where } \ i = 1, \ldots, n. \tag{4.3}$$

Now by means of direct calculations we verify that the system of equations (4.2) is exactly the commutativity condition for the operators (4.3):

$$[\mathbf{L}_s, \mathbf{L}_r] = 0 \ \text{ for } \ r, s = 1, \ldots, n. \tag{4.4}$$

Let $\mathbb{R}^+ = (0, +\infty)$ be positive half-axis on the axis of real numbers $\mathbb{R}$. Linear operators (4.3) have natural interpretation as vector field on Cartesian product $M \times \mathbb{R}^+$. Let's complete vector fields $\mathbf{L}_1, \ldots, \mathbf{L}_n$ with one more vector field $\mathbf{L}_{n+1}$, which possibly doesn't commutate with $\mathbf{L}_1, \ldots, \mathbf{L}_n$, but complete them up to a moving frame on the manifold $M \times \mathbb{R}^+$. Each field $\mathbf{L}_i$ generates its own one-parametric local group of local diffeomorphisms (see [12]). Denote by y^i parameter of such group:

$$\varphi_i(y^i) :\, M \times \mathbb{R}^+ \to M \times \mathbb{R}^+. \tag{4.5}$$

Let's fix some point $p_0 \in M \times \mathbb{R}^+$ and consider the composition of such diffeomorphisms applied to the fixed point p_0:

$$p(y^1, \ldots, y^n, w) = \varphi_1(y^1) \circ \ \ldots \ \circ \varphi_n(y^n) \circ \varphi_{n+1}(w)(p_0). \tag{4.6}$$

In left hand side of (4.6) we have the point p parameterized by the set of real numbers $y^1, \ldots, y^n$, $w = y^{n+1}$. This is equivalent to defining local coordinates on the manifold $M \times \mathbb{R}^+$ in some neighborhood of the point p_0. Commutativity condition (4.4) implies commutativity of first n maps (4.5) in composition (4.6). When applied to vector fields $\mathbf{L}_1, \ldots, \mathbf{L}_n$, this yields

$$\begin{aligned} &\mathbf{L}_1 = \frac{\partial}{\partial y^1} = \sum_{i=1}^{n} \frac{\partial x^i}{\partial y^1} \frac{\partial}{\partial x^i} + \frac{\partial v}{\partial y^1} \frac{\partial}{\partial v}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\mathbf{L}_n = \frac{\partial}{\partial y^n} = \sum_{i=1}^{n} \frac{\partial x^i}{\partial y^n} \frac{\partial}{\partial x^i} + \frac{\partial v}{\partial y^n} \frac{\partial}{\partial v}. \end{aligned} \tag{4.7}$$

Let's compare formulas (4.7) and (4.3) for the fields $\mathbf{L}_1, \ldots, \mathbf{L}_n$. Then

$$\frac{\partial x^i}{\partial y^k} = \delta^i_k = \begin{cases} 1 & \text{for} \ \ i = k, \\ 0 & \text{for} \ \ i \neq k. \end{cases} \tag{4.8}$$

From the same comparison of (4.7) and (4.3) for the functions $b_k = b_k(x^1, \ldots, x^n, v)$ expressed in variables $y^1, \ldots, y^n$, w we get

$$b_k = \frac{\partial v}{\partial y^k}. \tag{4.9}$$

The relationships (4.8) shows that transition from new coordinates $y^1, \ldots, y^n$, w back to coordinates $x^1, \ldots, x^n$, v is defied by the only one function $V(y^1, \ldots, y^n, w)$:

$$\begin{cases} x^1 = y^1, \ \ldots, \ x^n = y^n, \\ v = V(y^1, \ldots, y^n, w). \end{cases} \tag{4.10}$$

Direct transition from $x^1, \ldots, x^n$, v to new variables $y^1, \ldots, y^n$, w is also defined by the only one function $W(x^1, \ldots, x^n, v)$:

$$\begin{cases} y^1 = x^1, \ \ldots, \ y^n = x^n, \\ w = W(x^1, \ldots, x^n, v). \end{cases} \tag{4.11}$$

Functions $V(y^1, \ldots, y^n, w)$ and $W(x^1, \ldots, x^n, v)$ are bound by obvious relationships expressing the fact that transitions (4.10) and (4.11) are inverse to each other:

$$\begin{aligned} &V(x^1, \ldots, x^n, W(x^1, \ldots, x^n, v)) = v, \\ &W(x^1, \ldots, x^n, V(x^1, \ldots, x^n, w)) = w. \end{aligned} \tag{4.12}$$

For the sake of brevity let's introduce the following quite natural notations for partial derivatives of the function $V(x^1, \dots, x^n, w)$:

$$\begin{aligned} V_i(x^1, \dots, x^n, w) &= \frac{\partial V(x^1, \dots, x^n, w)}{\partial x^i}, \\ V_w(x^1, \dots, x^n, w) &= \frac{\partial V(x^1, \dots, x^n, w)}{\partial w}. \end{aligned} \tag{4.13}$$

Similar notations are used for second order partial derivatives of $V(x^1, \dots, x^n, w)$:

$$\begin{aligned} V_{ij}(x^1, \dots, x^n, w) &= \frac{\partial^2 V(x^1, \dots, x^n, w)}{\partial x^i\, \partial x^j}, \\ V_{iw}(x^1, \dots, x^n, w) &= \frac{\partial^2 V(x^1, \dots, x^n, w)}{\partial x^i\, \partial w}. \end{aligned} \tag{4.14}$$

Now we can rewrite formula (4.9) for b_k in initial variables $x^1, \dots, x^n, v$:

$$b_k = V_k(x^1, \dots, x^n, W(x^1, \dots, x^n, v)). \tag{4.15}$$

Theorem 4.1. *Functions $b_1, \dots, b_n$ satisfy nonlinear system of differential equations* (4.2) *if and only if they are defined according to the formula* (4.15) *by some function $V(x^1, \dots, x^n, w)$ such that $\partial V/\partial w \neq 0$.*

Proof. Each solution of the system of equations (4.2) is given by (4.15). This was proved above. Conversely, suppose that we have a function $V(x^1, \dots, x^n, w)$ with nonzero derivative $\partial V/\partial w$. From $\partial V/\partial w \neq 0$ we conclude that there exists (at least locally) some function $W(x^1, \dots, x^n, v)$ related to $V(x^1, \dots, x^n, w)$ according to the formulas (4.12) (see theorems on implicit functions in [86] or in [87]). From formulas (4.12) we derive the relationships

$$\frac{\partial W}{\partial v} = \frac{1}{V_w(x^1, \dots, x^n, W(x^1, \dots, x^n, v))}, \tag{4.16}$$

$$\frac{\partial W}{\partial x^s} = -\frac{V_s(x^1, \dots, x^n, W(x^1, \dots, x^n, v))}{V_w(x^1, \dots, x^n, W(x^1, \dots, x^n, v))}, \tag{4.17}$$

written in terms of above notations (4.13). Let's substitute functions V and W into (4.15) and get functions $b_1, \dots, b_n$. Then by direct substitution we verify that functions (4.15) satisfy the equations (4.2). Indeed, we have

$$\begin{aligned} \frac{\partial b_r}{\partial x^s} = V_{rs}(x^1, \dots, x^n, W(x^1, \dots, x^n, v)) + \\ + V_{rw}(x^1, \dots, x^n, W(x^1, \dots, x^n, v))\, \frac{\partial W}{\partial x^s}, \end{aligned} \tag{4.18}$$

$$\begin{aligned}(4.19)\qquad b_s\,\frac{\partial b_r}{\partial v}&=V_s(x^1,\ldots,x^n,W(x^1,\ldots,x^n,v))\times\\&\times\,V_{rw}(x^1,\ldots,x^n,W(x^1,\ldots,x^n,v))\,\frac{\partial W}{\partial v}.\end{aligned}$$

Let's add the equalities (4.18) and (4.19). Then take into account formulas (4.16) and (4.17) for partial derivatives of W. As a result we get

$$\left(\frac{\partial}{\partial x^s}+b_s\,\frac{\partial}{\partial v}\right)b_r=V_{rs}(x^1,\ldots,x^n,W(x^1,\ldots,x^n,v)).\tag{4.20}$$

Due to (4.14) indices r and s in right hand side of (4.20) can be transposed, i. e. $V_{rs}=V_{sr}$. Hence the equations (4.2) for the functions (4.15) are fulfilled. □

Having constructed general solution for the system of equations (4.2), we now turn to the equations (4.1). Let's make change of variables (4.10) in them. In left hand side of these equations we have the same differential operator $\mathbf{L}_s$ as in (4.2). In variables $y^1, \ldots, y^n, w$ it has the form $\mathbf{L}_s=\partial/\partial y^s$ (see relationships (4.7)). Then let's transform to variables $y^1, \ldots, y^n, w$ the operator from right hand side of (4.1):

$$\frac{\partial}{\partial v}=\sum_{i=1}^{n}\frac{\partial y^i}{\partial v}\,\frac{\partial}{\partial y^i}+\frac{\partial w}{\partial v}\,\frac{\partial}{\partial w}=\frac{\partial W}{\partial v}\,\frac{\partial}{\partial w}.$$

For further transformation of this expression we apply formula (4.16):

$$\frac{\partial}{\partial v}=\frac{1}{V_w(y^1,\ldots,y^n,w)}\,\frac{\partial}{\partial w}.$$

Now, in terms of notations introduced in (4.13) and (4.14), the equations (4.1) transformed to new variables $y^1, \ldots, y^n, w$ are written as follows:

$$\frac{\partial a}{\partial y^s}=\frac{V_{sw}}{V_w}\,a.\tag{4.21}$$

These equations (4.21) are easily solved, if we write them in the form

$$\frac{\partial}{\partial y^s}\left(\frac{a}{V_w}\right)=0.\tag{4.22}$$

General solution of (4.22) contain an arbitrary function of one variable $h(w)$:

$$a=h(w)\;V_w(y^1,\ldots,y^n,w).\tag{4.23}$$

Upon returning to the variables $x^1, \ldots, x^n, v$ in formula (4.23) we have

$$\begin{aligned}(4.24)\qquad a&=h(W(x^1,\ldots,x^n,v))\times\\&\times\;V_w(x^1,\ldots,x^n,W(x^1,\ldots,x^n,v)).\end{aligned}$$

For the purpose of additional verification let's substitute the expression (4.24) for a and the expressions (4.15) for b_s into the equation (4.1):

$$\frac{\partial a}{\partial x^s} = \bigl(h'(W)\,V_w + h(W)\,V_{ww}\bigr)\,\frac{\partial W}{\partial x^s} + h(W)\,V_{sw},$$

$$b_s\,\frac{\partial a}{\partial v} = V_s\,\bigl(h'(W)\,V_w + h(W)\,V_{ww}\bigr)\,\frac{\partial W}{\partial v}.$$

Then we add these equalities and take into account formulas (4.16) and (4.17) for the derivatives $\partial W/\partial x^s$ and $\partial W/\partial v$:

$$\left(\frac{\partial}{\partial x^s} + b_s\,\frac{\partial}{\partial v}\right) a = h(W)\,V_{sw}. \tag{4.25}$$

Similar operations in right hand side of the equations (4.1) yield

$$\left(a\,\frac{\partial}{\partial v}\right) b_s = h(W)\,V_w\,V_{sw}\,\frac{\partial W}{\partial v}. \tag{4.26}$$

Comparing (4.25) with (4.26) and taking into account formula (4.16) for $\partial W/\partial v$, we complete the proof of the following theorem.

Theorem 4.2. *Functions $b_1, \ldots, b_n$ and a satisfy the system of differential equations* (4.1) *and* (4.2) *if and only if they are given by formulas* (4.15) *and* (4.24).

§ 5. The general formula for the force field.

Analysis of reduced normality equations, accomplished in § 4, allows us to construct general solution for these equations. Now we are able to write **formula** defining force field for **arbitrary** Newtonian dynamical system **admitting the normal shift** on Riemannian manifold of the dimension $n \geqslant 3$. Let's substitute (2.35) into the formula (1.8) for components of force field $\mathbf{F}$. Thereby let's take into account that fields a and $\mathbf{b}$ determining parameter A in scalar ansatz depend only on modulus of velocity vector $\mathbf{v}$ within fibers of tangent bundle TM:

$$F_k = a\,N_k + |\mathbf{v}|\sum_{i=1}^{n} b_i\,\bigl(2\,N^i\,N_k - \delta^i_k\bigr). \tag{5.1}$$

Formula (5.1) completely determines the dependence of $\mathbf{F}$ on velocity vector $\mathbf{v} = v\,\mathbf{N}$. It is similar to formula (4.51) from Chapter VI. However the dependence on modulus of velocity vector in (5.1) is much more complicated. It is described by formulas (4.15) and (4.24). Finally, for $\mathbf{F}$ we have

$$\begin{aligned} F_k &= h(W)\;V_w(x^1,\ldots,x^n,W)\;N_k\,+\\ &\quad+\;|\mathbf{v}|\sum_{i=1}^{n} V_i(x^1,\ldots,x^n,W)\;\bigl(2\,N^i\,N_k - \delta^i_k\bigr). \end{aligned} \tag{5.2}$$

Here $h(W)$ is an arbitrary function of one variable, through V_i and V_w we denoted partial derivatives (4.13), while functions $V(x^1,\ldots,x^n,W)$ and $W(x^1,\ldots,x^n,v)$ are related by (4.12).

Formula (5.2) for the force field of Newtonian dynamical system admitting the normal shift contains arbitrariness given by one function of $(n+1)$ variables, namely by function $V(x^1,\ldots,x^n,w)$. Arbitrariness given by $h(w)$ can be excluded due to gauge transformation that preserves a and $\mathbf{b}$:

$$\begin{aligned}&V(x^1,\ldots,x^n,w)\to V(x^1,\ldots,x^n,\rho^{-1}(w)),\\&W(x^1,\ldots,x^n,v)\to \rho(W(x^1,\ldots,x^n,v)),\\&h(w)\to h(\rho^{-1}(w))\;\rho'(\rho^{-1}(w)).\end{aligned}\tag{5.3}$$

Hence this gauge transformation (5.3) changes V, W, and h in (5.2), but preserves force field $\mathbf{F}$. For $h(w)\neq 0$ we can choose function $\rho(w)$ such that $h(w)\,\rho'(w)=1$. Then, upon making gauge transformation (5.3), in this case we get formula

$$\begin{aligned}F_k&=V_w(x^1,\ldots,x^n,W)\;N_k\,+\\&+\,|\mathbf{v}|\sum_{i=1}^{n}V_i(x^1,\ldots,x^n,W)\;\bigl(2\,N^i\,N_k-\delta^i_k\bigr).\end{aligned}\tag{5.4}$$

Formula (5.4) is almost as general as formula (5.2). The only exception is the case, when $h=0$. This case cannot be described by (5.4).

§ 6. Effectivization of the general formula.

Formulas (5.2) and (5.4) yield explicit expressions for the force field ob Newtonian dynamical system admitting the normal shift on Riemannian manifold M. However, they have one common fault: non-efficiency due to the fact that one should calculate implicit function $W(x^1,\ldots,x^n,v)$ from the equations (4.12). In order to get more effective formula let's use relationships (4.16) and (4.17). We rewrite them as follows:

$$V_w(x^1,\ldots,x^n,W(x^1,\ldots,x^n,v))=\frac{1}{\partial W/\partial v},\tag{6.1}$$

$$V_k(x^1,\ldots,x^n,W(x^1,\ldots,x^n,v))=-\frac{\partial W/\partial x^k}{\partial W/\partial v}.\tag{6.2}$$

Substituting (6.2) into formula (4.15) for components of the field $\mathbf{b}$, we get

$$b_k=-\frac{\partial W(x^1,\ldots,x^n,v)/\partial x^k}{\partial W(x^1,\ldots,x^n,v)/\partial v}.\tag{6.3}$$

Here $W(x^1,\ldots,x^n,v)$ is an arbitrary function satisfying the only condition that the derivative in denominator of fraction (6.3) is nonzero. This function, upon substituting (6.1) into the formula (4.24), determines scalar field a as well:

$$a=\frac{h(W(x^1,\ldots,x^n,v))}{\partial W(x^1,\ldots,x^n,v)/\partial v}. \tag{6.4}$$

Now let's substitute (6.3) and (6.4) into the formula (5.1) for the field $\mathbf{F}$:

$$\begin{aligned}F_k=&\frac{h(W(x^1,\ldots,x^n,v))}{\partial W(x^1,\ldots,x^n,v)/\partial v}\ N_k+\\&-\ |\mathbf{v}|\sum_{i=1}^{n}\frac{\partial W(x^1,\ldots,x^n,v)/\partial x^i}{\partial W(x^1,\ldots,x^n,v)/\partial v}\ \bigl(2\,N^i\,N_k-\delta^i_k\bigr).\end{aligned} \tag{6.5}$$

Formula (6.5) can be rewritten in terms of covariant derivatives. We formulate this result as a theorem.

THEOREM 6.1. *Newtonian dynamical system on Riemannian manifold M of the dimension $n\geqslant 3$ admits the normal shift if and only if its force field $\mathbf{F}$ is given by*

$$F_k=\frac{h(W)\,N_k}{W_v}-|\mathbf{v}|\sum_{i=1}^{n}\frac{\nabla_i W}{W_v}\,\bigl(2\,N^i\,N_k-\delta^i_k\bigr), \tag{6.6}$$

where W is an arbitrary extended scalar field on M with nonzero derivative

$$W_v=\frac{\partial W}{\partial v}=\sum_{i=1}^{n}N^i\ \tilde\nabla_i W\neq 0$$

depending only on modulus of velocity vector $v=|\mathbf{v}|$ within fibers of tangent bundle TM, while $h=h(W)$ is an arbitrary scalar function of one variable.

In various particular cases formula (6.6) yields various special examples of dynamical systems admitting the normal shift, which were considered above. For $W=v$ and $h=0$ we get $\mathbf{F}=0$. This case corresponds to **geodesic flow** on M.

Let $f=f(x^1,\ldots,x^n)$ be some scalar field on M. Let's take $W=v\,e^{-f}$. The condition $W_v=\partial W/\partial v\neq 0$ then is fulfilled, since $W_v=e^{-f}$. Now assume that $h=H(w)$, and substitute h and W into (6.6). As a result we obtain force field

$$F_k=H(v\,e^{-f})\,e^f\,N_k-|\mathbf{v}|^2\,\nabla_k f+2\sum_{s=1}^{n}\nabla_s f\,v^s\,v_k,$$

corresponding to **metrizable dynamical system** admitting the normal shift (see formula (3.26) in Chapter VI).

Let $A=A(v)$ be some nonzero scalar function of one variable, and suppose that $f=f(x^1,\ldots,x^n)$ is some scalar field on M. We take

$$W=\exp\left(\int_0^v\frac{v\,dv}{A(v)}-f\right)$$

and choose $h(W) = 0$. Then formula (6.6) yields the field $\mathbf{F}$ of the form

$$F_k = A(|\mathbf{v}|) \sum_{i=1}^{n} \nabla_i f \left(2\, N^i\, N_k - \delta^i_k\right).$$

This field correspond s to **non-metrizable dynamical system** considered in § 5 of Chapter VI. Note that dynamical systems with force field (6.6) in generic case are also **non-metrizable**.

§ 7. Kinematics of normal shift.

Having explicit formula (6.6) for the force field of dynamical system admitting the normal shift, we can describe in more details the process of normal shift for given hypersurface S. According to the definition 11.2 from Chapter V one choose some point p_0, then on some smaller part of S' of S one should find the function ν that determines the modulus of initial velocity on trajectories of shift. This function is normalized at the point p_0 by the condition

$$\nu(p_0) = \nu_0, \tag{7.1}$$

where ν_0 is nonzero real number (see condition (11.4) in Chapter V). Let's choose local coordinates $x^1, \ldots, x^n$ on M in some neighborhood of p_0 and local coordinates $u^1, \ldots, u^{n-1}$ on S in the neighborhood of the same point. This choice determines coordinate tangent vectors $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_{n-1}$ to S. Denote by τ^i_k components of k-th tangent vector $\boldsymbol{\tau}_k$. If S is defined by parametric equations

$$\begin{aligned} &x^1 = x^1(u^1, \ldots, u^{n-1}),\\ &\ldots\ldots\ldots\ldots\ldots\ldots\\ &x^n = x^n(u^1, \ldots, u^{n-1}), \end{aligned} \tag{7.2}$$

then components of tangent vectors $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_{n-1}$ are defined by derivatives

$$\tau^i_k = \frac{\partial x^i}{\partial u^k}. \tag{7.3}$$

Function $\nu = \nu(p) = \nu(u^1, \ldots, u^{n-1})$ determines initial velocity on trajectories of shift (see (2.2) in Chapter V):

$$\mathbf{v}(t)\Big|_{t=0} = \nu(p) \cdot \mathbf{n}(p). \tag{7.4}$$

Here $\mathbf{n}(p)$ is unitary normal vector to S at the point p. Condition (7.4) provides normality of shift at the initial instant of time $t = 0$. For to keep normality of shift at all other instants of time we should choose function $\nu(u^1, \ldots, u^{n-1})$ in some special way. It should satisfy the system of differential equations (7.16) that was derived in Chapter V. These equations have the following form:

$$\frac{\partial \nu}{\partial u^k} = -\nu^{-1}\,(\mathbf{F}\,|\,\boldsymbol{\tau}_k). \tag{7.5}$$

Let's substitute force field (6.6) into the equations (7.5) and let's take into account that due to (7.4), for $t = 0$, vector $\mathbf{N}$ coincides with unitary normal vector $\mathbf{n}(p)$ on S, and $|\mathbf{v}|$ coincides with ν. As a result we get:

$$\frac{\partial \nu}{\partial u^k} = -\sum_{i=1}^{n} \frac{\nabla_i W}{W_v}\,\tau^i_k. \tag{7.6}$$

Let's multiply (7.6) by W_v and let's collect all terms in left hand side of this equation. Moreover, let's write all derivatives in explicit form. Then we obtain

$$\frac{\partial W}{\partial v}\,\frac{\partial \nu}{\partial u^k} + \sum_{i=1}^{n} \frac{\partial W}{\partial x^i}\,\frac{\partial x^i}{\partial u^k} = 0. \tag{7.7}$$

Now it is easy to detect that left hand side of (7.7) is a derivative of function $W(x^1, \ldots, x^n, v)$ with respect to the variable u^k, when functions (7.2) are substituted for arguments $x^1, \ldots, x^n$ and function $\nu(u^1, \ldots, u^{n-1})$ is substituted for argument v in this function. Therefore the equations (7.7) are easily integrated. They yield the following implicit equation for the function $\nu = \nu(p)$ on S:

$$W(x^1(p), \ldots, x^n(p), \nu(p)) = W_0 = \text{const}\,. \tag{7.8}$$

The value of constant W_0 in (7.8) is determined by normalization condition (7.1):

$$W_0 = W(x^1(p_0), \ldots, x^n(p_0), \nu_0).$$

Theorem 7.1. *Let S be arbitrary hypersurface S in Riemannian manifold M given by functions $x^1(p), \ldots, x^n(p)$ in parametric form* (7.2)*. In order to construct the normal shift of S along trajectories of Newtonian dynamical system with force field* (6.6) *we should determine the function $\nu = \nu(p)$ in* (7.4) *by functional equation* (7.8) *in implicit form.*

Having constructed the normal shift $f_t\colon S \to S_t$ of S along trajectories of dynamical system with force field (6.6), we get the one parametric family of hypersurfaces S_t. By changing reference instant of time $t \to t + t_0$ we can consider each of them as initial hypersurface. Therefore on each hypersurface S_t we have the equality

$$W(p, |\mathbf{v}|) = W(t) = \text{const} \tag{7.9}$$

similar to (7.8). However, the values of constants $W(t)$ in (7.9) can differ for different hypersurfaces S_t. Let's find the dynamics of $W(t)$ in time. In order to do this we use the formula, which is the consequence of formula (4.2) from Chapter IV applied to extended scalar field $v = |\mathbf{v}|$:

$$\frac{d|\mathbf{v}|}{dt} = \nabla_t v = \sum_{i=1}^{n} \tilde{\nabla}_i v\, \nabla_t v^i = \sum_{i=1}^{n} N_i\, F^i, \tag{7.10}$$

Let's substitute (6.6) into (7.10). As a result of this substitution we get

$$\frac{dv}{dt}=\frac{h(W)}{W_v}-\sum_{i=1}^{n}\frac{\nabla_i W}{W_v}\,\frac{dx^i}{dt}.$$

Let's multiply (7.6) by W_v and collect all terms in left hand side of this equation. Moreover, let's write all derivatives in explicit form. Then we obtain

$$\sum_{i=1}^{n}\frac{\partial W}{\partial x^i}\,\frac{dx^i}{dt}+\frac{\partial W}{\partial v}\,\frac{dv}{dt}=h(W). \tag{7.11}$$

It is easy to see that left hand side of (7.11) is a time derivative of scalar field W along the trajectories of normal shift. Thus the equality (7.11) is the very equality determining the dynamics of parameters $W(t)$ in (7.9). Let's write it as

$$\frac{dW}{dt}=h(W). \tag{7.12}$$

If function $h(W)$ in (6.6) is zero, then due to (7.12) scalar function W not only is constant on each separate hypersurface S_t, but have the same values on all such hypersurfaces. For $h(W)\neq 0$ differential equation (7.12) is easily integrated. Therefore, if we know the value of W on initial hypersurface S, then we can calculate its value $W(t)$ on any shifted hypersurface S_t.

CHAPTER VIII

NORMAL SHIFT IN FINSLERIAN GEOMETRY.

§ 1. Basic concepts of Finslerian geometry.

Finslerian geometry is more complicated geometric media, where we can develop the theory of Newtonian dynamical systems admitting the normal shift. Finslerian geometry stems from [88]. It arises in considering extremum problem for functional

$$I = \int_{t_1}^{t_2} \mathcal{F}(x^1, \ldots, x^n, v^1, \ldots, v^n)\, dt, \tag{1.1}$$

where $x^i = x^i(t)$ and $v^i = \dot{x}^i = dx^i/dt$. In geometric situation variables $x^1, \ldots, x^n$ are interpreted as local coordinates on some manifold M, while integral (1.1) is a path integral along some curve connecting two points of this manifold. In order to keep previous terminology we interpret parameter t as time variable. Time derivatives $v^i = \dot{x}^i$ for $x^1, \ldots, x^n$ are components of vector $\mathbf{v}$ tangent to the path of integration. It is called velocity vector. Function $\mathcal{F}$ in (1.1) is assumed to be satisfying the following three conditions:

Φ1). function $\mathcal{F}$ is positively homogeneous with the degree of homogeneity 1 respective to components of velocity vector, i. e.

$$\mathcal{F}(x^1, \ldots, x^n, k \cdot \mathbf{v}) = |k| \cdot \mathcal{F}(x^1, \ldots, x^n, \mathbf{v});$$

Φ2). function $\mathcal{F}$ is non-negative, i. e. $\mathcal{F}(x^1, \ldots, x^n, \mathbf{v}) \geqslant 0$, and such that

$$\mathcal{F}(x^1, \ldots, x^n, \mathbf{v}) = 0 \ \text{ implies } \ \mathbf{v} = 0;$$

Φ3). for the function $\mathcal{H}(x^1, \ldots, x^n, \mathbf{v}) = \mathcal{F}(x^1, \ldots, x^n, \mathbf{v})^2$ corresponding matrix

$$g_{ij}(x^1, \ldots, x^n, \mathbf{v}) = \frac{1}{2}\,\frac{\partial^2 \mathcal{H}}{\partial v^i\, \partial v^j}, \tag{1.2}$$

given by partial derivatives of the function $\mathcal{H}$, is positively definite.

Definition 1.1. Smooth manifold M equipped with smooth[1] extended scalar field $\mathcal{F}$ satisfying three above conditions Φ1, Φ2, Φ3 is called **Finslerian manifold**.

[1]Scalar field $\mathcal{F}$ in definition 1.1 is assumed smooth on tangent bundle TM everywhere, except for the points of zero section $\mathbf{v} = 0$.

Definition 1.1 shows that concept of extended tensor fields described in Chapters II, III, and IV is quite natural and lies in the basis of whole Finslerian geometry. Path integral (1.1) defines length of curves on Finslerian manifold, while matrix g_{ij} in (1.2) define components of metric tensor of Finslerian metric $\mathbf{g}$. Finslerian metric is extended tensor tensor field of type $(0,2)$. Its components

$$g_{jk} = \frac{1}{2}\,\tilde{\nabla}_j\tilde{\nabla}_k\mathcal{H} \tag{1.3}$$

depend on velocity vector, they are homogeneous functions of degree 0 respective to $\mathbf{v}$. Applying covariant derivative $\tilde{\nabla}_i$ to (1.3), we get tensor field of type $(0,3)$:

$$C_{ijk} = \frac{1}{2}\,\tilde{\nabla}_i g_{jk} = \frac{1}{4}\,\tilde{\nabla}_i\tilde{\nabla}_j\tilde{\nabla}_k\mathcal{H}. \tag{1.4}$$

Tensor field $\mathbf{C}$ with components (1.4) is called **Cartan tensor**, its symmetric in all three indices. In Finslerian geometry Cartan tensor is non-zero, therefore $\tilde{\nabla}\mathbf{g} \neq 0$. This means that covariant differentiation is not in concordance with metric. We should take into account this fact in raising and lowering indices of tensor fields.

Let's use Euler's theorem on homogeneous functions (see [87]) to components of metric tensor g_{ij}, which are homogeneous functions of degree 0. As a result we get

$$\sum_{k=1}^{n}\frac{\partial g_{ij}}{\partial v^k}\,v^k = 0.$$

This relationship is used to write as a property of Cartan tensor, which partially compensate non-concordance of Finslerian metric and covariant differentiation $\tilde{\nabla}$:

$$\sum_{k=1}^{n} C_{ijk}\,v^k = 0. \tag{1.5}$$

Applying Euler's theorem on homogeneous functions to the function $\mathcal{H} = \mathcal{F}^2$, which is homogeneous of degree 2 respective to $\mathbf{v}$, we get

$$\sum_{i=1}^{n}\frac{\partial\mathcal{H}}{\partial v^i}\,v^i = 2\,\mathcal{H}. \tag{1.6}$$

Left and right hand sides of (1.6) are homogeneous functions of degree 2 in v. Applying Euler's theorem once more, we get

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2\mathcal{H}}{\partial v^i\,\partial v^j}\,v^i\,v^j + \sum_{i=1}^{n}\frac{\partial\mathcal{H}}{\partial v^i}\,v^i = 4\,\mathcal{H}.$$

Taking into account (1.6) and (1.2), we can write this relationship as

$$\sum_{i=1}^{n}\sum_{j=1}^{n} g_{ij}\,v^i\,v^j = \mathcal{H}. \tag{1.7}$$

In other words, the value of quadratic form given by metric tensor **g**, when applied to vector **v**, is equal to square of the function $\mathcal{F}$. Therefore we can write path integral (1.1) for Finslerian length of curve in the form, which is exactly the same as in case of such integral in Riemannian geometry:

$$I=\int\limits_{t_1}^{t_2}\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n}g_{ij}\,v^i\,v^j}\,dt. \tag{1.8}$$

§ 2. Dynamic connection of Cartan.

Connection of Levi-Civita (or metric connection), which plays central role in Riemannian geometry, is defined as symmetric connection such that $\nabla\mathbf{g}=0$. In Finslerian geometry similar condition $\nabla\mathbf{g}=0$ defines **dynamic connection of Cartan**, which was introduced in [89] (see also [90], [91], and [61]). Let's write this condition $\nabla\mathbf{g}=0$ in local coordinates M:

$$\nabla_i g_{jk}=\frac{\partial g_{jk}}{\partial x^i}-\sum_{s=1}^{n}\Gamma^s_{ij}\,g_{sk}-\sum_{s=1}^{n}\Gamma^s_{ik}\,g_{js}-\sum_{s=1}^{n}\sum_{r=1}^{n}\Gamma^s_{ir}\,\frac{\partial g_{jk}}{\partial v^s}\,v^r=0.$$

Here components of connection Γ^k_{ij} depend not only on coordinates $x^1,\,\ldots\,,\,x^n$, but on components of velocity vector **v**. This is complete agreement with what was said in § 4 of Chapter III. Let's introduce the following notation:

$$\Gamma_{kij}=\sum_{s=1}^{n}g_{ks}\,\Gamma^s_{ij}. \tag{2.1}$$

This simplifies the relationship written above:

$$\frac{\partial g_{jk}}{\partial x^i}-\Gamma_{kij}-\Gamma_{jik}-\sum_{s=1}^{n}\sum_{r=1}^{n}\Gamma^s_{ir}\,\frac{\partial g_{jk}}{\partial v^s}\,v^r=0. \tag{2.2}$$

Let's make cyclic rearrangement of indices $i\to j\to k\to i$ in formula (2.2):

$$\frac{\partial g_{ki}}{\partial x^j}-\Gamma_{ijk}-\Gamma_{kji}-\sum_{s=1}^{n}\sum_{r=1}^{n}\Gamma^s_{jr}\,\frac{\partial g_{ki}}{\partial v^s}\,v^r=0. \tag{2.3}$$

$$\frac{\partial g_{ij}}{\partial x^k}-\Gamma_{jki}-\Gamma_{ikj}-\sum_{s=1}^{n}\sum_{r=1}^{n}\Gamma^s_{kr}\,\frac{\partial g_{ij}}{\partial v^s}\,v^r=0. \tag{2.4}$$

Let's add (2.2) and (2.3), then subtract (2.4) from the sum obtained. Thereby we take into account symmetry of Cartan connection, i. e. $\Gamma_{kij}=\Gamma_{kji}$. Then we get

$$\begin{gathered}\frac{\partial g_{jk}}{\partial x^i}+\frac{\partial g_{ki}}{\partial x^j}-\frac{\partial g_{ij}}{\partial x^k}-2\,\Gamma_{kij}=\\ =\sum_{s=1}^{n}\sum_{r=1}^{n}\left(\Gamma^s_{ir}\,\frac{\partial g_{jk}}{\partial v^s}+\Gamma^s_{jr}\,\frac{\partial g_{ki}}{\partial v^s}-\Gamma^s_{kr}\,\frac{\partial g_{ij}}{\partial v^s}\right)v^r.\end{gathered} \tag{2.5}$$

For the sake of convenience in calculations let's introduce the following notations:

$$\gamma_{kij} = \frac{1}{2}\left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ki}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right), \tag{2.6}$$

$$\gamma^k_{ij} = \frac{1}{2}\sum_{s=1}^{n} g^{ks}\left(\frac{\partial g_{js}}{\partial x^i} + \frac{\partial g_{si}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^s}\right), \tag{2.7}$$

Formulas for γ_{kij} and γ^k_{ij} coincide with well-known formulas for components metric connection in Riemannian geometry (see (1.2) in Chapter V). However, in Finslerian geometry they do not define connection components.

Taking into account notations (2.6) and (2.7), we transform the relationship (2.5) for components of Cartan connection as follows:

$$\Gamma_{kij} = \gamma_{kij} - \sum_{s=1}^{n}\sum_{r=1}^{n}\left(\Gamma^s_{ir}\, C_{sjk} + \Gamma^s_{jr}\, C_{ski} - \Gamma^s_{kr}\, C_{sij}\right) v^r. \tag{2.8}$$

Here we also took into account the relationship (1.4), that determines components of Cartan tensor. The equalities (2.8) are algebraic equations for components of Cartan connection Γ_{kij}. They should be resolved. Let's multiply (2.8) by v^i and v^j, then contract it with respect to indices i and j. If we take into account (1.5), we get

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\Gamma_{kij}\, v^i\, v^j = \sum_{i=1}^{n}\sum_{j=1}^{n}\gamma_{kij}\, v^i\, v^j. \tag{2.9}$$

The relationship (2.9) prompts the following notations:

$$\begin{aligned} G_k &= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\gamma_{kij}\, v^i\, v^j, \\ G^k &= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\gamma^k_{ij}\, v^i\, v^j. \end{aligned} \tag{2.10}$$

Now we multiply (2.8) and contract with respect to index j. Thereby we take into account (1.5) and notations (2.10):

$$\sum_{j=1}^{n}\Gamma_{kij}\, v^j = \sum_{j=1}^{n}\gamma_{kij}\, v^j - \sum_{s=1}^{n} 2\, G^s\, C_{ski}. \tag{2.11}$$

Upon raising index k in formula (2.11) we obtain

$$\sum_{r=1}^{n}\Gamma^s_{ir}\, v^r = \sum_{r=1}^{n}\gamma^s_{ir}\, v^r - \sum_{q=1}^{n} 2\, G^q\, C^s_{qi}. \tag{2.12}$$

Further let's note that in right hand side of (2.8) we have the same sums as in left hand side of formula (2.12). Therefore the relationship (2.8) can be rewritten as:

$$\begin{aligned}\Gamma_{kij} = \gamma_{kij} - \sum_{s=1}^{n}\sum_{r=1}^{n}\left(\gamma^{s}_{ir}\,C_{sjk} + \gamma^{s}_{jr}\,C_{ski} - \gamma^{s}_{kr}\,C_{sij}\right)v^{r} +\\ + \sum_{s=1}^{n}\sum_{q=1}^{n} 2\left(G^{q}\,C^{s}_{qi}\,C_{sjk} + G^{q}\,C^{s}_{qj}\,C_{ski} - G^{q}\,C^{s}_{qk}\,C_{sij}\right).\end{aligned} \tag{2.13}$$

This is required explicit formula for components of Cartan connection in Finslerian geometry. With the aim of simplifying formula (2.13) let's make some further transformations in it. Let's calculate derivatives of G^{s} in (2.10):

$$\frac{\partial G^{s}}{\partial v^{i}} = \frac{1}{2}\sum_{r=1}^{n}\sum_{j=1}^{n}\frac{\partial\gamma^{s}_{rj}}{\partial v^{i}}\,v^{r}\,v^{j} + \sum_{r=1}^{n}\gamma^{s}_{ir}\,v^{r}. \tag{2.14}$$

Partial derivatives $\partial\gamma^{s}_{rj}/\partial v^{i}$ in (2.14) are calculated by direct differentiation of formula (2.7) determining quantities γ^{s}_{rj}:

$$\begin{aligned}\frac{\partial\gamma^{s}_{rj}}{\partial v^{i}} = \frac{1}{2}\sum_{a=1}^{n}\frac{\partial g^{sa}}{\partial v^{i}}\left(\frac{\partial g_{ja}}{\partial x^{r}} + \frac{\partial g_{ar}}{\partial x^{j}} - \frac{\partial g_{rj}}{\partial x^{a}}\right) +\\ + \frac{1}{2}\sum_{a=1}^{n} g^{sa}\left(\frac{\partial^{2} g_{ja}}{\partial x^{r}\,\partial v^{i}} + \frac{\partial^{2} g_{ar}}{\partial x^{j}\,\partial v^{i}} - \frac{\partial^{2} g_{rj}}{\partial x^{a}\,\partial v^{i}}\right).\end{aligned} \tag{2.15}$$

Matrices with components g^{ij} and g_{ij} are inverse to each other. Therefore we can calculate partial derivatives $\partial g^{sa}/\partial v^{i}$:

$$\frac{\partial g^{sa}}{\partial v^{i}} = -\sum_{k=1}^{n}\sum_{m=1}^{n} g^{sk}\,\frac{\partial g_{km}}{\partial v^{i}}\,g^{ma} = -\sum_{k=1}^{n}\sum_{m=1}^{n} 2\,g^{sk}\,C_{ikm}\,g^{ma}.$$

Let's substitute this expression into (2.15) and let's take into account (1.4):

$$\begin{aligned}\frac{\partial\gamma^{s}_{rj}}{\partial v^{i}} = -\sum_{k=1}^{n}\sum_{m=1}^{n} 2\,g^{sk}\,C_{ikm}\gamma^{m}_{rj} +\\ + \sum_{a=1}^{n} g^{sa}\left(\frac{\partial C_{ija}}{\partial x^{r}} + \frac{\partial C_{iar}}{\partial x^{j}} - \frac{\partial C_{irj}}{\partial x^{a}}\right).\end{aligned} \tag{2.16}$$

By substituting (2.16) into the formula (2.14) we contract derivatives $\partial C_{ija}/\partial x^{r}$, $\partial C_{iar}/\partial x^{j}$, and $\partial C_{irj}/\partial x^{a}$ with v^{j} and v^{r} respective to r and j. All such contractions yield zero due to (1.5). Therefore (2.14) can be written as follows:

$$\frac{\partial G^{s}}{\partial v^{i}} = -\sum_{m=1}^{n} 2\,C^{s}_{im}\,G^{m} + \sum_{r=1}^{n}\gamma^{s}_{ir}\,v^{r}. \tag{2.17}$$

Let's compare formulas (2.17) and (2.12). As a result of comparison we derive

$$\sum_{r=1}^{n}\Gamma^{s}_{ir}\,v^{r}=\frac{\partial G^{s}}{\partial v^{i}}. \tag{2.18}$$

Substituting (2.18) into (2.8), we get another formula for components of Cartan connection. It is simpler than formula (2.13) above:

$$\Gamma_{kij}=\gamma_{kij}-\sum_{s=1}^{n}\left(\frac{\partial G^{s}}{\partial v^{i}}\,C_{sjk}+\frac{\partial G^{s}}{\partial v^{j}}\,C_{ski}-\frac{\partial G^{s}}{\partial v^{k}}\,C_{sij}\right). \tag{2.19}$$

From (2.13) or (2.19) one can derive that components of Cartan connection are homogeneous functions of degree 0 with respect to components of velocity vector $\mathbf{v}$ in fibers of tangent bundle TM. These formulas in Finslerian geometry are well-known. The above derivation of them can be found in [61].

§ 3. Projectors defined by velocity vector and some identities for curvature tensors.

Let's consider the velocity vector $\mathbf{v}$. It is interpreted as extended vector field on the manifold M. Let's consider its covariant derivatives in Finslerian geometry. By direct calculations according to formulas (7.3) and (7.4) from Chapter III we get:

$$\nabla_{k}v^{i}=0, \qquad \tilde{\nabla}_{k}v^{i}=\delta^{i}_{k}. \tag{3.1}$$

These relationships coincide with relationships (5.5) in Chapter V. They are fulfilled for all extended connections on M, including Cartan connection.

Let's apply lowering index procedure to the vector field of velocity $\mathbf{v}$, and obtain covector field of velocity. In order to do it we use Finslerian metric:

$$v_{i}=\sum_{s=1}^{n}g_{is}(x^{1},\ldots,x^{n},\mathbf{v})\,v^{s}. \tag{3.2}$$

Now let's apply covariant derivatives to this covector field:

$$\nabla_{k}v_{i}=0, \qquad \tilde{\nabla}_{k}v_{i}=g_{ik}. \tag{3.3}$$

First relationship (3.3) is a consequence of the relationship $\nabla_{k}g_{is}=0$ that express concordance of Finslerian metric and Cartan connection. Second is less trivial:

$$\tilde{\nabla}_{k}v_{i}=\sum_{s=1}^{n}g_{is}\tilde{\nabla}_{k}v^{s}+\sum_{s=1}^{n}\tilde{\nabla}_{k}g_{is}\,v^{s}=g_{ik}+\sum_{s=1}^{n}2\,C_{iks}\,v^{s}.$$

Indeed, last term in the above expression vanishes due to the identity (1.5) for Cartan tensor. As a result we prove second relationship in (3.3).

Denote by v modulus of velocity vector: $v = |\mathbf{v}|$. This is extended scalar field on M, square of this field is determined by formula

$$|\mathbf{v}|^2 = (v\,|\,v) = \sum_{i=1}^{n}\sum_{j=1}^{n} g_{ij}(x^1, \ldots, x^n, \mathbf{v})\, v^i\, v^j = \sum_{i=1}^{n} v_i\, v^i.$$

Formulas (3.1) and (3.3) allow us to calculate covariant derivatives of $v = |\mathbf{v}|$:

$$\nabla_k v = 0, \qquad \tilde{\nabla}_k v = N_k. \tag{3.4}$$

Here N_k are covariant components of unitary vectorial field $\mathbf{N}$ directed along $\mathbf{v}$:

$$N^i = \frac{v^i}{|\mathbf{v}|}, \qquad N_i = \frac{v_i}{|\mathbf{v}|}. \tag{3.5}$$

Let's calculate covariant derivatives of the fields (3.5). In order to do it we use above formulas (3.1), (3.3), and (3.4):

$$\nabla_k N^i = 0, \qquad \nabla_k N_i = 0. \tag{3.6}$$

In calculating covariant derivatives $\tilde{\nabla}_k N^i$ tensor field $\mathbf{P}$ arises. Its values are interpreted as projectors to hyperplane perpendicular to velocity vector $\mathbf{v}$:

$$\tilde{\nabla}_k N^i = \frac{1}{|\mathbf{v}|}\left(\delta^i_k - N_k\, N^i\right) = \frac{1}{|\mathbf{v}|}\, P^i_k, \tag{3.7}$$

$$\tilde{\nabla}_k N_i = \frac{P_{ik}}{|\mathbf{v}|} = \sum_{s=1}^{n} \frac{g_{is}\, P^s_k}{|\mathbf{v}|}. \tag{3.8}$$

Indeed, if $\mathbf{X}$ is some arbitrary vector field, and $\mathbf{P}$ is operator field with components

$$P^i_k = \delta^i_k - N_k\, N^i, \tag{3.9}$$

then for the field $\mathbf{Y} = \mathbf{P}\mathbf{Y}$ we have the orthogonality

$$(\mathbf{Y}\,|\,\mathbf{v}) = \sum_{i=1}^{n}\sum_{j=1}^{n} Y^i\, g_{ij}(x^1, \ldots, x^n, \mathbf{v})\, v^j = 0. \tag{3.10}$$

Scalar product $(\mathbf{Y}\,|\,\mathbf{v})$ in (3.10) is linear with respect to $\mathbf{Y}$, however, it is not linear with respect to $\mathbf{v}$. This is due to special role of velocity vector: it defines the point of tangent bundle TM and it forms extended vector field simultaneously.

Further let's calculate covariant derivatives for projector field $\mathbf{P}$. From the relationships (3.6) and (3.9) we derive that

$$\nabla_k P^j_i = 0. \tag{3.11}$$

Similarly from the relationships (3.6) and (3.9) we get

$$\tilde{\nabla}_k P^j_i = -\frac{N_i\,P^j_k + P_{ik}\,N^j}{|\mathbf{v}|}. \tag{3.12}$$

Formulas (3.11) and (3.12) are identical to corresponding formulas in § 5 of Chapter V. The same is true for formulas (3.3), (3.4), (3.6), (3.7), and (3.8). This coincidence is because we choose Cartan connection, which is concordant with Finslerian metric, and since we have identity (1.5) for Cartan tensor.

Theorem 3.1. *Tensor of dynamic curvature* $\mathbf{D}$*, which is defined by Cartan connection* (2.19) *according to formula* (6.5) *from Chapter III, satisfies the relationship*

$$(\mathbf{v}\,|\,\mathbf{D}(\mathbf{X},\mathbf{Y})\mathbf{v}) = \sum_{m=1}^{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} D^m_{kij}(x^1,\ldots,x^n,\mathbf{v})\,X^i\,Y^j\,v^k\,v_m = 0,$$

where $\mathbf{X}$ *and* $\mathbf{Y}$ *are two arbitrary vector fields on Finslerian manifold* M*.*

Proof. Let's use one of the commutational relationships (7.8) from Chapter III:

$$[\nabla_i,\,\tilde{\nabla}_j]\varphi = -\sum_{m=1}^{n}\sum_{k=1}^{n} v^k\,D^m_{kij}\,\tilde{\nabla}_m\varphi. \tag{3.13}$$

Here φ is arbitrary scalar field from extended algebra of tensor fields. Let's substitute $\varphi = |\mathbf{v}|^2$ into (3.13). In left hand side we get

$$[\nabla_i,\,\tilde{\nabla}_j]\varphi = \nabla_i\tilde{\nabla}_j\varphi - \tilde{\nabla}_j\nabla_i\varphi = \nabla_i(2\,v\,N_j) = 0.$$

Similar calculations with $\varphi = |\mathbf{v}|^2$ for right hand side of formula (3.13) yield

$$\sum_{m=1}^{n}\sum_{k=1}^{n} v^k\,D^m_{kij}\,\tilde{\nabla}_m\varphi = \sum_{m=1}^{n}\sum_{k=1}^{n} 2\,D^m_{kij}\,v^k\,v_m.$$

Equating these two expressions for left and right hand sides of (3.13), we get

$$\sum_{m=1}^{n}\sum_{k=1}^{n} D^m_{kij}\,v^k\,v_m = 0.$$

Now, for to complete proof of theorem one should multiply this equality by $X^i\,Y^j$, and should accomplish summation in indices i and j. □

Theorem 3.2. *Tensor of Riemannian curvature* $\mathbf{R}$*, defined by Cartan connection* (2.19) *according to formula* (6.8) *from Chapter III, satisfies the relationship*

$$(\mathbf{v}\,|\,\mathbf{R}(\mathbf{X},\mathbf{Y})\mathbf{v}) = \sum_{m=1}^{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} R^m_{kij}(x^1,\ldots,x^n,\mathbf{v})\,X^i\,Y^j\,v^k\,v_m = 0,$$

where $\mathbf{X}$ *and* $\mathbf{Y}$ *are two arbitrary vector fields on Finslerian manifold* M.

PROOF. Proof of this theorem is similar to that of previous one. However, instead of (7.8) here we use formula (7.10) from Chapter III. Here it is written as

$$[\nabla_i,\,\nabla_j]\varphi=-\sum_{m=1}^{n}\sum_{k=1}^{n}v^m\,R^k_{mij}\,\tilde{\nabla}_k\varphi,\tag{3.14}$$

since torsion field for Cartan connection is equal to zero. Let's substitute $\varphi=|\mathbf{v}|^2$ into (3.14). In left hand side we have

$$[\nabla_i,\,\nabla_j]\varphi=\nabla_i\nabla_j\varphi-\nabla_j\nabla_i\varphi=0,$$

this follows from (3.4). In right hand side of (3.14) we get

$$\sum_{m=1}^{n}\sum_{k=1}^{n}v^m\,R^k_{mij}\,\tilde{\nabla}_k\varphi=\sum_{m=1}^{n}\sum_{k=1}^{n}2\,R^k_{mij}\,v^m\,v_k.$$

Equating these two expressions for left and right hand sides of (3.14), we get

$$\sum_{m=1}^{n}\sum_{k=1}^{n}R^m_{kij}\,v^k\,v_m=0.$$

Now, for to complete proof of theorem one should multiply this equality by $X^i\,Y^j$, and should accomplish summation in indices i and j. □

Cartan connection (2.19) is symmetric: $\Gamma^m_{ij}=\Gamma^m_{ji}$. Therefore tensor of dynamic curvature $\mathbf{D}$ is also symmetric. This is expressed by the equality

$$D^m_{kij}=D^m_{ikj}.\tag{3.15}$$

Due to (3.15) we can transpose vectors $\mathbf{X}$ and $\mathbf{v}$ in formula from theorem 3.1:

$$(\mathbf{v}\,|\,\mathbf{D}(\mathbf{v},\mathbf{Y})\mathbf{X})=0.\tag{3.16}$$

§ 4. Hypersurfaces in Finslerian space and their normal vector.

Suppose that S is a hypersurface in Finslerian manifold M, and let $u^1,\,\ldots\,,\,u^{n-1}$ be local coordinates on S. In local coordinates $x^1,\,\ldots\,,\,x^n$ on M this hypersurface can be represented by parametric equations:

$$\begin{aligned}&x^1=x^1(u^1,\ldots,u^{n-1}),\\&\ldots\ldots\ldots\ldots\ldots\ldots\\&x^n=x^n(u^1,\ldots,u^{n-1}).\end{aligned}\tag{4.1}$$

Tangent hyperplane to S is defined by corresponding coordinate tangent vectors $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_{n-1}$. In local coordinates $x^1, \ldots, x^n$ they are expanded as follows:

$$\boldsymbol{\tau}_k = \sum_{i=1}^{n} \tau^i_k \frac{\partial}{\partial x^i}. \tag{4.2}$$

Definition 4.1. Let S be hypersurface in Finslerian manifold M, and suppose that $p \in S$. Vector $\mathbf{n}$ from tangent space $T_p(M)$ is called **normal vector** to S, if it is perpendicular to vectors (4.2) with respect to Finslerian metric (1.2):

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \tau^i_k \, g_{ij}(x^1, \ldots, x^n, \mathbf{n}) \, n^j = 0. \tag{4.3}$$

Problem of existence of normal vector in the sense of definition 4.1 in Finslerian manifold is nontrivial, since the equations (4.3) are non-linear with respect to components of vector $\mathbf{n}$. In order to solve this problem we consider the map $\mathbf{n} \to \boldsymbol{\xi}(\mathbf{n})$, that maps vector $\mathbf{n}$ to the covector $\boldsymbol{\xi}$ by means lowering index procedure:

$$\xi_i = \sum_{j=1}^{n} g_{ij}(x^1, \ldots, x^n, \mathbf{n}) \, n^j. \tag{4.4}$$

According to the equations (4.3) the contraction of $\boldsymbol{\xi}$ with coordinate tangent vectors $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_{n-1}$ from (4.2) should yield zero result:

$$\sum_{i=1}^{n} \tau^i_k \; \xi_i = 0. \tag{4.5}$$

We see that (4.5) is a system of homogenized linear equations with matrix of rank $n-1$ respective to $\xi_1, \ldots, \xi_n$. They define $\boldsymbol{\xi}$ uniquely up to a numeric multiple. Therefore problem of existence of normal vector reduces to inverting map $\mathbf{n} \to \boldsymbol{\xi}(\mathbf{n})$ defined by (4.4). By differentiating (4.4) we calculate Jacoby matrix of this map:

$$\begin{gathered} J_{ij} = \frac{\partial \xi_i}{\partial n^j} = g_{ij} + \sum_{k=1}^{n} \frac{\partial g_{ik}}{\partial n^j} \, n^k = g_{ij}(x^1, \ldots, x^n, \mathbf{n}) + \\ + \sum_{k=1}^{n} 2\, C_{jik}(x^1, \ldots, x^n, \mathbf{n}) \, n^k = g_{ij}(x^1, \ldots, x^n, \mathbf{n}). \end{gathered}$$

The result of contracting $\mathbf{n}$ with Cartan tensor $\mathbf{C}$ here is zero due to (1.5). The equality $J_{ij} = g_{ij}$ means that $\det J \neq 0$ since matrix of metric tensor is positively definite (see axiom $\Phi 3$ in definition of Finslerian manifold).

Thus, the map $\mathbf{n} \to \boldsymbol{\xi}(\mathbf{n})$ in (4.4) is locally bijective. It is homogeneous of degree 1: $\boldsymbol{\xi}(k \cdot \mathbf{n}) = k \cdot \boldsymbol{\xi}(\mathbf{n})$. If we fix some Euclidean metric in vector space $T_p(M)$ and some other metric in covector space $T^*_p(M)$, then we can treat $\mathbf{n} \to \boldsymbol{\xi}(\mathbf{n})$ as a map from one unit sphere S^{n-1} to other unit sphere S^{n-1}. Due to local bijectivity this map is open, hence the image of S^{n-1} is an open set in other sphere. But, at the same time, this image is closed as an image of bicompact space S^{n-1} under the map into Hausdorf space S^{n-1} (see [92] or [93]) Hence, since sphere S^{n-1} is connected topological space, map $\mathbf{n} \to \boldsymbol{\xi}(\mathbf{n})$ is surjective. It is non-ramified covering due to local bijectiveness. Then remember, that non-ramified covering of multidimensional sphere S^{n-1} is one-sheeted, because sphere S^{n-1} is simply connected space for $n \geqslant 3$ (see [94]). This means that $\mathbf{n} \to \boldsymbol{\xi}(\mathbf{n})$ is a bijection of spheres (diffeomorphism). This result form a theorem.

THEOREM 4.1. *Each smooth hypersurface S in Finslerian manifold M of the dimension $n \geqslant 3$ possess unitary normal vector $\mathbf{n}$ in the sense of definition 4.1, this vector is fixed uniquely up to a choice of orientation on S.*

This result is true in two-dimensional case $n = 2$ as well, i. e. we have the following theorem.

THEOREM 4.2. *Each smooth hypersurface S in Finslerian manifold M possess unitary normal vector $\mathbf{n}$ in the sense of definition 4.1, this vector is fixed uniquely up to a choice of orientation on S.*

In two dimensional case $n = 2$ let's fix a point $p \in M$ and consider the map $\mathbf{n} \to \boldsymbol{\xi}(\mathbf{n})$ from $T_p(M)$ to $T^*_p(M)$ given by formula (4.4). Apart from Finslerian metric in tangent space $T_p(P)$ we fix Euclidean metric $\tilde{g}$ with unit matrix in some local coordinates $x^1, \ldots, x^n$:

$$\tilde{g}_{ij} = \tilde{g}^{ij} = \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix}. \tag{4.6}$$

Metric (4.6) defines raising index procedure. Let's apply it to covector field $\boldsymbol{\xi}$. Then formula (4.4) can be replaced by the following one:

$$\xi^i = \sum_{k=1}^{2} \sum_{j=1}^{2} \tilde{g}^{ik}\, g_{kj}(x^1, x^2, \mathbf{n})\, n^j. \tag{4.7}$$

Written as (4.7), $\mathbf{n} \to \boldsymbol{\xi}(\mathbf{n})$ is a map from $T_p(M)$ to $T_p(M)$. Let's take $\mathbf{n}$ to be unitary vector in metric (4.6). Then $\mathbf{n}$ and $\boldsymbol{\xi}(\mathbf{n})$ are written as:

$$\mathbf{n} = \begin{Vmatrix} \cos(\varphi) \\ \\ \sin(\varphi \end{Vmatrix}, \qquad \boldsymbol{\xi}(\mathbf{n}) = |\boldsymbol{\xi}| \cdot \begin{Vmatrix} \cos(\tilde{\varphi}) \\ \\ \sin(\tilde{\varphi}) \end{Vmatrix}. \tag{4.8}$$

Map $\mathbf{n} \to \boldsymbol{\xi}(\mathbf{n})$ defines non-ramified covering of unit circle by some other unit circle. In terms of angles φ and $\tilde{\varphi}$ it is given by strictly monotonous smooth function $\tilde{\varphi}(\varphi)$. The difference $\tilde{\varphi}(\varphi + 2\pi) - \tilde{\varphi}(\varphi)$ is an integer multiple of 2π:

$$\tilde{\varphi}(\varphi + 2\pi) - \tilde{\varphi}(\varphi) = 2\pi\, m. \tag{4.9}$$

Integer number $m \neq 0$ in (4.9) is positive, if function $\tilde{\varphi}(\varphi)$ is increasing function; it is negative, if $\tilde{\varphi}(\varphi)$ is decreasing function. Modulus of this number is equal to the number of sheets in non-ramified covering $S^1 \to S^1$.

Suppose that $m \neq 1$. Let $\psi(\varphi) = \tilde{\varphi}(\varphi) - \varphi$. Then for the function $\psi(\varphi)$ from the relationship (4.9) we derive that

$$\psi(\varphi + 2\pi) - \psi(\varphi) = 2\pi(m - 1) \neq 0.$$

This means that function has indefinitely large values of one sign as $\varphi \to +\infty$, and it has indefinitely large values of opposite sign as $\varphi \to -\infty$. Hence, since it is continuous, there exist an angle φ_0 such that $\psi(\varphi_0) = \pi/2$. For such angle $\varphi = \varphi_0$ we have $\tilde{\varphi} = \varphi + \pi/2$, i. e. vectors $\mathbf{n}$ and $\boldsymbol{\xi}(\mathbf{n})$ from (4.8) are perpendicular to each other in Euclidean metric (4.6):

$$\tilde{\mathbf{g}}(\mathbf{n}, \boldsymbol{\xi}) = \sum_{i=1}^{2}\sum_{j=1}^{2} \tilde{g}_{ij}\, n^i\, \xi^j = \sum_{i=1}^{2}\sum_{j=1}^{2} g_{kj}(x^1, x^2, \mathbf{n})\, n^k\, n^j = 0.$$

This relationship contradicts to the fact that matrix $g_{kj}(x^1, x^2, \mathbf{n})$ in (1.2) is positively definite (see axiom Φ3 from the definition of Finslerian manifold). This contradiction proves that $m = 1$, i. e. $\mathbf{n} \to \boldsymbol{\xi}(\mathbf{n})$ is one-sheeted covering $S^1 \to S^1$. Thus, theorem 4.2 is proved.

Existence of normal vectors allows us to generalize the theory of dynamical systems admitting the normal shift for the case of Finslerian manifolds.

§ 5. Geodesic normal shift.

Let's consider some Newtonian dynamical system on Finslerian manifold M. In local coordinates $x^1, \ldots, x^n$ on M it is given by ordinary differential equations

$$\dot{x}^i = v^i, \qquad \nabla_t v^i = F^i(x^1, \ldots, x^n, v^1, \ldots, v^n). \tag{5.1}$$

Here $F^1, \ldots, F^n$ are components of force vector. These equations (5.1) differ from corresponding equations in Riemannian geometry only in covariant derivatives $\nabla_t v^i$, here they are calculated respective to extended Cartan connection:

$$\nabla_t v^i = \dot{v}^i + \sum_{j=1}^{n}\sum_{k=1}^{n} \dot{x}^j\, \Gamma^i_{jk}(x^1, \ldots, x^n, v^1, \ldots, v^n)\, v^k. \tag{5.2}$$

Derivative $\nabla_t \mathbf{v}$ with components given by (5.2) is a covariant derivative of velocity vector $\mathbf{v}$ with respect to parameter t along the curve (compare formula (5.2) with formulas (3.10) and (3.13) in Chapter IV).

The simplest Newtonian dynamical system of the form (5.1) is a geodesic flow of Finslerian metric (1.2). It corresponds to purely zero force field in $\mathbf{F} = 0$:

$$\dot{x}^i = v^i, \qquad \nabla_t v^i = 0. \tag{5.3}$$

Suppose that S is some hypersurface from transformation class in M. In local coordinates $u^1, \ldots, u^{n-1}$ on S it is given by parametric equations (4.1). Let's fix on S (or possibly on some smaller piece of S)) a smooth field of unitary normal vectors $\mathbf{n} = \mathbf{n}(u^1, \ldots, u^{n-1})$. Vector $\mathbf{n}$ possess unit length in Finslerian metric, it is perpendicular to coordinate tangent vectors $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_{n-1}$ in the sense of definition 4.1. We shall use S for to define data that initiate a shift along trajectories of geodesic flow (5.3). Let's write them as follows:

$$\begin{aligned} &x^k(t)\Big|_{t=0} = x^k(u^1, \ldots, u^{n-1}),\\ &v^k(t)\Big|_{t=0} = n^k(u^1, \ldots, u^{n-1}). \end{aligned} \tag{5.4}$$

Solution of Cauchy problem (5.4) for the equations (5.3) is given by functions

$$\begin{aligned} &x^1 = x^1(u^1, \ldots, u^{n-1}, t),\\ &\dots\dots\dots\dots\dots\dots\\ &x^n = x^n(u^1, \ldots, u^{n-1}, t). \end{aligned} \tag{5.5}$$

These functions define geodesic shift $f_t\colon\ S \to S_t$ of hypersurface S. Here t is a parameter of shift, variables $u^1, \ldots, u^{n-1}$ are local coordinates on S for $t = 0$, they can be used as local coordinates on shifted hypersurfaces S_t for sufficiently small $t \neq 0$ as well. If we denote $t = u^n$, then complete set of variables $u^1, \ldots, u^n$ form the set of local coordinates on M in some open neighborhood of S. These coordinates are called **semigeodesic coordinates**.

Theorem 5.1. *Components of metric tensor for Finslerian metric* $\mathbf{g}$ *in semigeodesic coordinates* $u^1, \ldots, u^n$ *satisfy the following conditions:*

$$g_{kn}(u^1, \ldots, u^n, 0, \ldots, 0, v^n) = 0 \quad \text{for} \quad k \neq n, \tag{5.6}$$

$$g_{nn}(u^1, \ldots, u^n, 0, \ldots, 0, v^n) = 1 \quad \text{for} \quad k = n. \tag{5.7}$$

Proof. Due to the choice of semigeodesic coordinates by itself geodesic lines, defined by the equations (5.3) and initial data (5.4) on S, in semigeodesic coordinates are written in the following form:

$$u^1(t) = \text{const}, \ \ldots, \ u^{n-1}(t) = \text{const}, \qquad u^n(t) = t. \tag{5.8}$$

Differentiating (5.8) in t we find components of velocity vector on geodesic lines:

$$v^1(t) = \ldots = v^{n-1}(t) = 0, \qquad v^n(t) = 1. \tag{5.9}$$

The equations (5.3), on account of (5.2), are written as

$$\dot{v}^i+\sum_{j=1}^{n}\sum_{k=1}^{n}\Gamma^i_{jk}(u^1,\ldots,u^n,v^1,\ldots,v^n)\,v^j\,v^k=0. \tag{5.10}$$

Let's substitute (5.8) and (5.9) into the equations (5.10), then for Γ^i_{jk} we get

$$\Gamma^k_{nn}(u^1,\ldots,u^{n-1},t,0,\ldots,0,1)=0\quad\text{for}\quad k\leqslant n. \tag{5.11}$$

In formula (5.11) we substitute parameter t by u^n. Moreover, we take into account that components of Cartan connection Γ^i_{jk} are homogeneous functions of degree 0 with respect to components of velocity vector $\mathbf{v}$. This yields

$$\Gamma^k_{nn}(u^1,\ldots,u^n,0,\ldots,0,v^n)=0\quad\text{for}\quad k\leqslant n. \tag{5.12}$$

Now let's write the concordance condition for Cartan connection and Finslerian metric $\mathbf{g}$ in semigeodesic coordinates. It looks like

$$\nabla_n g_{nn}=\frac{\partial g_{nn}}{\partial u^n}-\sum_{k=1}^{n}2\,\Gamma^k_{nn}\,g_{kn}-\sum_{k=1}^{n}\Gamma^k_{nn}\,\frac{\partial g_{nn}}{\partial v^k}\,v^n=0.$$

Let's recall arguments of Γ^k_{nn} from (5.12). Then from $\Gamma^k_{nn}=0$ we derive

$$\frac{\partial g_{nn}(u^1,\ldots,u^n,0,\ldots,0,v^n)}{\partial u^n}=0. \tag{5.13}$$

Let's write (5.4) in semigeodesic coordinates:

$$u^k(t)\,\Big|_{t=0}=u^k,\qquad\qquad v^k(t)\,\Big|_{t=0}=n^k. \tag{5.14}$$

Remember that $u^n=0$ for $t=0$ on S. Further, combining (5.14) and (5.9), we calculate the components of unit vector $\mathbf{n}$ on initial hypersurface S:

$$\begin{aligned}&n^k=n^k(u^1,\ldots,u^{n-1},0)=0\quad\text{for}\quad k\neq n,\\&n^k=n^k(u^1,\ldots,u^{n-1},0)=1\quad\text{for}\quad k=n.\end{aligned} \tag{5.15}$$

Now let's write the condition that $|\mathbf{n}|=1$ on S:

$$|\mathbf{n}|^2=\sum_{i=1}^{n}\sum_{i=1}^{n}g_{ij}(u^1,\ldots,u^{n-1},0,\mathbf{n})\,n^i\,n^j=1.$$

Combining this equality with formulas (5.15) for components of $\mathbf{n}$, we get

$$g_{nn}(u^1,\ldots,u^{n-1},0,0,\ldots,0,1)=1. \tag{5.16}$$

Components of metric tensor for Finslerian metric are homogeneous functions of degree 0 respective to $\mathbf{v}$. Therefore (5.16) yields

$$g_{nn}(u^1,\ldots,u^{n-1},0,0,\ldots,0,v^n)=1. \tag{5.17}$$

Let's compare (5.17) and (5.13). This comparison shows that $u^n=0$ in (5.17) can be replaced by arbitrary value of the variable u^n, i. e. we get the relationships coinciding with the relationships (5.7) in the statement of theorem:

$$g_{nn}(u^1,\ldots,u^n,0,\ldots,0,v^n)=1.$$

In order to prove the relationships (5.6) from theorem 5.1 let's apply the condition of concordance for Finslerian metric and Cartan connection. We write $\nabla_k g_{nn}=0$ expressing covariant derivative ∇_k through partial derivatives:

$$\nabla_i g_{nn}=\frac{\partial g_{nn}}{\partial u^i}-\sum_{k=1}^{n}2\,\Gamma^k_{in}\,g_{kn}-\sum_{s=1}^{n}\sum_{r=1}^{n}\Gamma^s_{ir}\,\frac{\partial g_{nn}}{\partial v^s}\,v^r=0.$$

Let's substitute $v^1=\ldots=v^{n-1}=0$ and $v^n=1$ into this formula. Then sum in r reduces to one summand with $r=0$, while partial derivative $\partial g_{nn}/\partial u^i$ in the above equation vanishes due to the relationship (5.7), which we have already proved:

$$\sum_{k=1}^{n}2\,\Gamma^k_{in}\,g_{kn}+\sum_{s=1}^{n}\Gamma^s_{in}\,\frac{\partial g_{nn}}{\partial v^s}\,v^n=0. \tag{5.18}$$

In calculating $\partial g_{nn}/\partial v^s$ we use formula (1.4) and symmetry of Cartan tensor:

$$\frac{\partial g_{nn}}{\partial v^s}=2\,C_{snn}=2\,C_{nsn}. \tag{5.19}$$

Now let's write the relationship (1.5) substituting arguments from (5.12) in it. The sum in k thereby reduces to one term with $k=n$:

$$0=\sum_{k=1}^{n}C_{nsk}\,v^k=C_{nsn}(u^1,\ldots,u^n,0,\ldots,0,v^n)\,v^n. \tag{5.20}$$

The relationships (5.19) and (5.20) show that if we substitute $v^1=\ldots=v^{n-1}=0$ and $v^n=1$ into (5.18), sum in s vanishes:

$$\begin{gathered}\sum_{k=1}^{n}\Gamma^k_{in}(u^1,\ldots,u^n,0,\ldots,0,v^n)\,\times\\ \times\;g_{kn}(u^1,\ldots,u^n,0,\ldots,0,v^n)=0.\end{gathered} \tag{5.21}$$

We take into account this relationship and use the condition of concordance of Γ and

g for the third time. We write this condition $\nabla_n g_{ni} = 0$ in semigeodesic coordinates expressing ∇_n through partial derivatives:

$$\frac{\partial g_{ni}}{\partial u^n} - \sum_{k=1}^{n} \Gamma^k_{nn}\, g_{ki} - \sum_{k=1}^{n} \Gamma^k_{ni}\, g_{nk} - \sum_{k=1}^{n}\sum_{r=1}^{n} \Gamma^k_{nr}\, \frac{\partial g_{ni}}{\partial v^k}\, v^r = 0.$$

Second sum with respect to index k in left hand side of resulting equality is the same sum as in (5.21). It vanishes if we substitute $v^1 = \ldots = v^{n-1} = 0$ and $v^n = 1$. First sum in k vanishes due to (5.12), while sum in r reduces to one summand with $r = n$. This summand vanishes due to relationships analogous to (5.19) and (5.20). Indeed, from the identity (1.5) we derive

$$\frac{\partial g_{ni}}{\partial v^k}\, v^n = 2\, C_{ikn}(u^1, \ldots , u^n, 0, \ldots , 0, v^n)\, v^n = \sum_{s=1}^{n} 2\, C_{iks}\, v^s = 0.$$

Due to all above reductions the relationship $\nabla_n g_{ni} = 0$ yields

$$\frac{\partial g_{ni}(u^1, \ldots , u^n, 0, \ldots , 0, v^n)}{\partial u^n} = 0. \tag{5.22}$$

Let's consider coordinate tangent vectors (4.2) on the hypersurface S. In semigeodesic coordinates this hypersurface is described by the equation $u^n = 0$, while coordinate tangent vectors $\boldsymbol{\tau}_1, \ldots , \boldsymbol{\tau}_{n-1}$ have unitary components:

$$\tau^k_i = \frac{\partial u^k}{\partial u^i} = \delta^k_i = \begin{cases} 1 & \text{for } k = i, \\ 0 & \text{for } k \neq i. \end{cases} \tag{5.23}$$

Let's write (4.3) in semigeodesic coordinates and let's take into account (5.15) and (5.23). For the components of metric this yields

$$g_{ni}(u^1, \ldots , u^{n-1}, 0, 0, \ldots , 0, v^n) = 0 \quad \text{for} \quad i \neq n. \tag{5.24}$$

Comparing (5.24) and (5.22), we find that substituting $u^n = 0$ by nonzero value of this variable do not break the relationships (5.24). Therefore we have

$$g_{ni}(u^1, \ldots , u^n, 0, \ldots , 0, v^n) = 0 \quad \text{for} \quad i \neq n.$$

They coincide with the equalities (5.6) in the statement of theorem 5.1 that was to be proved. Proof is over. □

Comparing proof of the theorem 5.1 with proof of analogous proposition in § 1 of Chapter V, we conclude that they are based on the same idea. Additional obstacles due to the dependence of g_{ij} and Γ^k_{ij} upon **v** are surmounted at the expense of (1.4) and (1.5).

Theorem 5.1 describes geodesic shift $f_t\colon S \to S_t$ defined by the equations (5.3) and initial data (5.4). The relationships (5.6) show that this normal shift, i. e. vectors tangent to trajectories are perpendicular to hypersurfaces S_t in the sense of definition 4.1. We should stress that this is true for **any choice** of hypersurface S.

§ 6. Newtonian dynamical systems admitting the normal shift.

In this section we generalize the theory of Newtonian dynamical systems admitting the normal shift to the background of Finslerian geometry. Let's start with basic definitions. According to § 3 for the hypersurfaces of Finslerian manifold the concept of normal vector is well defined. Therefore we can state initial data

$$x^k(t)\Big|_{t=0} = x^k(p), \qquad\qquad v^k(t)\Big|_{t=0} = \nu(p)\cdot n^k(p) \tag{6.1}$$

in considering Newtonian dynamical systems (5.1). Here p is a point of hypersurface S and $\nu(p)$ is some function that defines modulus of initial velocity on trajectories of dynamical system (5.1) coming out from S in the direction of its normal vector. Suppose that $f_t\colon\ S \to S_t$ is a shift of hypersurface S defined by dynamical system (5.1) and initial data (6.1). It is called the **normal shift**, if trajectories of shift are directed along normal vectors for all hypersurfaces obtained by the shift $f_t\colon\ S \to S_t$. Dynamical system (5.1) with force field $\mathbf{F}$ is called **admitting the normal shift**, if it is able to accomplish normal shift of any hypersurface. More precisely, there is the definition including the normalization condition

$$\nu(p_0) = \nu_0 \tag{6.2}$$

with arbitrary number $\nu_0 \neq 0$ at arbitrary point p_0 on S.

Definition 6.1. Newtonian dynamical system (5.1) on Finslerian manifold M is called the system **admitting the normal shift**, if for any hypersurface S in M, for any point $p_0 \in S$, and for any real number $\nu_0 \neq 0$ there is a piece S' of hypersurface S belonging to transformation class and containing p_0, and there exists a function $\nu(p)$ belonging to transformation class on S' and normalized by (6.2) such that the shift $f_t : S' \to S'_t$, defined by this function in initial data (6.1), is a normal shift along the trajectories of dynamical system.

Let $u^1, \ldots, u^{n-1}$ be local coordinates on S in the neighborhood of some marked point p_0. Shift $f_t : S' \to S'_t$ transfers these coordinates to all hypersurfaces S_t. Therefore all these hypersurfaces can be represented in parametric form

$$\begin{aligned} x^1 &= x^1(u^1, \ldots, u^{n-1}, t),\\ &\ldots\ldots\ldots\ldots\ldots\ldots\\ x^n &= x^n(u^1, \ldots, u^{n-1}, t), \end{aligned} \tag{6.3}$$

Differentiating these functions (6.3) with respect to $u^1, \ldots, u^{n-1}$, we define tangent vectors $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_{n-1}$ to S with components

$$\tau^i_k = \frac{\partial x^i}{\partial u^k},$$

while differentiating them with respect to t, we get the components of velocity vector on trajectories of shift: $v^i = \dot{x}^i = \partial x^i/\partial t$. Functions of deviation are defined as scalar products of vectors $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_{n-1}$ with vector $\mathbf{v}$:

$$\varphi_k = (\boldsymbol{\tau}_k \,|\, \mathbf{v}) = \sum_{i=1}^{n}\sum_{j=1}^{n} \tau^i_k \, g_{ij}(x^1, \ldots, x^n, \mathbf{v})\, v^j. \tag{6.4}$$

Normality of shift $f_t : S \to S_t$ now is formulated as vanishing condition for all functions of deviation (6.4) simultaneously:

$$\varphi_k(t) = 0 \quad \text{for} \quad k = 1, \ldots, n-1 \tag{6.5}$$

(compare with the relationship (4.3) that defines normal vector to hypersurfaces in Finslerian geometry). From (6.5) for $t = 0$ we get

$$\varphi_k(t)\Big|_{t=0} = 0, \qquad\qquad \dot{\varphi}_k(t)\Big|_{t=0} = 0 \tag{6.6}$$

The equalities of first series in (6.6) are fulfilled unconditionally due to initial data (6.1) defining the trajectories of shift. Second series of equalities in (6.6) isn't a consequence of (6.1). On a base of them we formulate **additional normality condition** for Newtonian dynamical systems.

Definition 6.2. Suppose that Finslerian manifold M of the dimension $n \geqslant 2$ is equipped with some Newtonian dynamical system, which is used to arrange the shift of hypersurfaces along its trajectories according to the initial data (6.1). We say that this dynamical system satisfies **additional normality** condition, if for any hypersurface S in M, for any point p_0 on S, and for any real number $\nu_0 \neq 0$ there exists some smaller piece S' of hypersurface S belonging to transformation class and containing the point p_0, and there exists some function $\nu(p)$ from transformation class on S' normalized by the condition (6.2) and such that, when substituted to (6.1), it defines the shift $f_t\colon S' \to S'_t$ such that

$$\dot{\varphi}_k(t)\Big|_{t=0} = 0, \quad \text{where} \quad k = 1, \ldots, n-1. \tag{6.7}$$

Here $\varphi_k = (\boldsymbol{\tau}_k \,|\, \mathbf{v})$ are the functions of deviation defined by some choice of local coordinates $u^1, \ldots, u^{n-1}$ on S and corresponding vectors of variation $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_{n-1}$.

As well as in Riemannian geometry, here the definition 6.2 formulates a condition, which is additional to the condition of **weak normality**. The latter is formulated as follows.

Definition 6.3. We say that Newtonian dynamical system on Finslerian manifold M of the dimension $n \geqslant 2$ satisfies **weak normality** condition, if for each its trajectory there exists some ordinary differential equation

$$\ddot{\varphi} = \mathcal{A}(t)\, \dot{\varphi} + \mathcal{B}(t)\, \varphi \tag{6.8}$$

such that any function of deviation $\varphi(t)$ corresponding to any choice of variation vector $\boldsymbol{\tau}$ on that trajectory is the solution of this equation.

Let's recall that variation vector $\boldsymbol{\tau}$ appears in considering some family of trajectories parameterized by some variable u. If in local coordinates $x^1, \ldots, x^n$ in M trajectories are given by functions $x^i(t, u)$, then components of $\boldsymbol{\tau}$ are calculated as derivatives $\partial x^i/\partial u$. Vectors $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_{n-1}$ that arise in considering shift $f_t\colon S \to S_t$ are special cases of this general construction. In case of Newtonian dynamical systems on Finslerian manifold M arbitrary vector of variation $\boldsymbol{\tau}$ for its trajectories satisfies ordinary differential equation of the form

$$\begin{aligned}\nabla_{tt}\boldsymbol{\tau} = C(\nabla_t\boldsymbol{\tau}\otimes\tilde{\nabla}\mathbf{F}) + C(\boldsymbol{\tau}\otimes\nabla\mathbf{F})-\\ -\mathbf{R}(\boldsymbol{\tau},\mathbf{v})\mathbf{v} - \mathbf{D}(\boldsymbol{\tau},\mathbf{F})\mathbf{v} + \mathbf{D}(\mathbf{v},\nabla_t\boldsymbol{\tau})\mathbf{v}.\end{aligned}\tag{6.9}$$

Derivation of these equations (6.9) does not differ from derivation of the equations (3.10) in Chapter V. Thereby we use formulas (5.14) and (5.15) in Chapter IV. In local coordinates the equation (6.9) is written as

$$\begin{aligned}\nabla_{tt}\tau^k = \sum_{m=1}^{n}\nabla_t\tau^m\,\tilde{\nabla}_m F^k + \sum_{m=1}^{n}\tau^m\,\nabla_m F^k-\\ -\sum_{m=1}^{n}\sum_{i=1}^{n}\sum_{j=1}^{n}\bigl(R^k_{mij}\,\tau^i\,v^j + D^k_{mij}\bigl(\tau^i\,F^j - v^i\,\nabla_t\tau^j\bigr)\bigr)v^m.\end{aligned}\tag{6.10}$$

Each vector of variation $\boldsymbol{\tau}$ determines the function of deviation φ, which is formed as scalar product of two vectors $\boldsymbol{\tau}$ and $\mathbf{v}$:

$$\varphi = (\boldsymbol{\tau}\,|\,\mathbf{v}).\tag{6.11}$$

Scalar product (6.11) is linear with respect to $\boldsymbol{\tau}$, but it isn't linear with respect to its second argument $\mathbf{v}$, since metric tensor by itself depends on $\mathbf{v}$:

$$\varphi = \sum_{i=1}^{n}\sum_{j=1}^{n}\tau^i\,g_{ij}(x^1,\ldots,x^n,\mathbf{v})\,v^j.\tag{6.12}$$

Formula (6.12) is in a complete agreement with formula (6.4) above, and with definition of normal vector to hypersurfaces in Finslerian manifolds (see definition 4.1). In differentiating with (6.12) with respect to t here we get

$$\dot{\varphi} = (\mathbf{F}\,|\,\boldsymbol{\tau}) + (\mathbf{v}\,|\,\nabla_t\boldsymbol{\tau}) + \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\tilde{\nabla}_k g_{ij}\,\tau^i\,v^j\,F^k.$$

This formula contain additional term that differs it from corresponding formula (4.7) in Chapter V. But due to (1.4) and (1.5) this term appears to vanish. Therefore

$$\dot{\varphi} = (\boldsymbol{\tau}\,|\,\mathbf{F}) + (\nabla_t\boldsymbol{\tau}\,|\,\mathbf{v}).\tag{6.13}$$

In calculating second derivative of deviation function $\ddot{\varphi}$ we also get additional terms:

$$\begin{aligned}\ddot{\varphi}&=(\boldsymbol{\tau}\,|\,\nabla_t\mathbf{F})+2\,(\nabla_t\boldsymbol{\tau}\,|\,\mathbf{F})+(\nabla_{tt}\boldsymbol{\tau}\,|\,\mathbf{v})+\\&+\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\tilde{\nabla}_k g_{ij}\left(\tau^i\,F^j+\nabla_t\tau^i\,v^j\right)F^k,\end{aligned}\tag{6.14}$$

Such terms differ (6.14) from corresponding formula (4.8) in Chapter V. Some of them vanish de to the equalities (1.4) and (1.5). Therefore (6.14) is rewritten as

$$\ddot{\varphi}=(\boldsymbol{\tau}\,|\,\nabla_t\mathbf{F})+2\,(\nabla_t\boldsymbol{\tau}\,|\,\mathbf{F})+(\nabla_{tt}\boldsymbol{\tau}\,|\,\mathbf{v})\,+2\,\mathbf{C}(\boldsymbol{\tau},\mathbf{F},\mathbf{F}).\tag{6.15}$$

Here $\mathbf{C}(\boldsymbol{\tau},\mathbf{F},\mathbf{F})$ is a value of completely symmetric trilinear form defined by Cartan tensor $\mathbf{C}$.

In order to calculate second derivative of variation vector $\nabla_{tt}\boldsymbol{\tau}$ in (6.15) we use formula (6.9), while to calculate $\nabla_t\mathbf{F}$ we apply formula (4.10) from Chapter V. This formula is a special case of formula (4.3) from Chapter IV, it is true irrespective to the metric, in Riemannian geometry and in Finslerian geometry as well. Taking into account these formulas we transform (6.15) to

$$\begin{aligned}\ddot{\varphi}&=(C(\mathbf{v}\otimes\nabla\mathbf{F})\,|\,\boldsymbol{\tau})+(C(\mathbf{F}\otimes\tilde{\nabla}\mathbf{F})\,|\,\boldsymbol{\tau})+2\,(\mathbf{F}\,|\,\nabla_t\boldsymbol{\tau})+\\&+(\mathbf{v}\,|\,C(\boldsymbol{\tau}\otimes\nabla\mathbf{F}))+(\mathbf{v}\,|\,C(\nabla_t\boldsymbol{\tau}\otimes\tilde{\nabla}\mathbf{F}))-(\mathbf{v}\,|\,\mathbf{R}(\boldsymbol{\tau},\mathbf{v})\mathbf{v})-\\&-(\mathbf{v}\,|\,\mathbf{D}(\boldsymbol{\tau},\mathbf{F})\mathbf{v})+(\mathbf{v}\,|\,\mathbf{D}(\mathbf{v},\nabla_t\boldsymbol{\tau})\mathbf{v})+2\,\mathbf{C}(\boldsymbol{\tau},\mathbf{F},\mathbf{F}).\end{aligned}$$

This formula differs from corresponding formula (4.10) in Chapter V by last three terms. Two terms $(\mathbf{v}\,|\,\mathbf{D}(\boldsymbol{\tau},\mathbf{F})\mathbf{v})$ and $(\mathbf{v}\,|\,\mathbf{D}(\mathbf{v},\nabla_t\boldsymbol{\tau})\mathbf{v})$ in it are zero due to theorem 3.1. Term $(\mathbf{v}\,|\,\mathbf{R}(\boldsymbol{\tau},\mathbf{v})\mathbf{v})$ is zero due to theorem 3.2. Therefore

$$\begin{aligned}\ddot{\varphi}&=(C(\mathbf{v}\otimes\nabla\mathbf{F})\,|\,\boldsymbol{\tau})+(C(\mathbf{F}\otimes\tilde{\nabla}\mathbf{F})\,|\,\boldsymbol{\tau})+\\&+2\,(\mathbf{F}\,|\,\nabla_t\boldsymbol{\tau})+(\mathbf{v}\,|\,C(\boldsymbol{\tau}\otimes\nabla\mathbf{F}))+\\&+(\mathbf{v}\,|\,C(\nabla_t\boldsymbol{\tau}\otimes\tilde{\nabla}\mathbf{F}))+2\,\mathbf{C}(\boldsymbol{\tau},\mathbf{F},\mathbf{F}).\end{aligned}\tag{6.16}$$

When written in local coordinates, formula (6.16) looks like

$$\begin{aligned}\ddot{\varphi}&=\sum_{i=1}^{n}\sum_{j=1}^{n}\left(2\,g_{ij}\,F^j+\sum_{k=1}^{n}v^j\,g_{jk}\tilde{\nabla}_iF^k\right)\nabla_t\tau^i+\\&+\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\Bigl(v^k\left(\nabla_kF^j\,g_{ij}+\nabla_iF^j\,g_{jk}\right)+\\&\qquad\qquad+F^k\,\tilde{\nabla}_kF^j\,g_{ij}+2\,C_{ijk}\,F^j\,F^k\Bigr)\,\tau^i.\end{aligned}$$

Contracting with metric tensor we lower indices (this correspond to transfer from vectors to covectors). Such operation commutates with covariant differentiation ∇. But in Finslerian geometry it doesn't commutate with second covariant differentiation $\tilde{\nabla}$, since $\tilde{\nabla}\mathbf{g} \neq 0$. Therefore

$$\sum_{k=1}^{n} F^k\,\tilde{\nabla}_k F_i = \sum_{j=1}^{n}\sum_{k=1}^{n}\Bigl(F^k\,\tilde{\nabla}_k F^j\,g_{ij} + 2\,F^k\,C_{ijk}\,F^j\Bigr),$$

$$\sum_{j=1}^{n} v^j\,\tilde{\nabla}_i F_j = \sum_{j=1}^{n}\sum_{k=1}^{n}\Bigl(v^j\,g_{jk}\,\tilde{\nabla}_i F^k + 2\,F^k\,C_{ijk}\,v^j\Bigr).$$

Taking into account these formulas and taking into account the equality (1.5) for Cartan tensor, we transform formula for $\ddot{\varphi}$ to

$$\begin{aligned}\ddot{\varphi} &= \sum_{i=1}^{n}\left(2\,F_i + \sum_{j=1}^{n} v^j\,\tilde{\nabla}_i F_j\right)\nabla_t\tau^i +\\ &+\sum_{i=1}^{n}\left(\sum_{j=1}^{n} v^j\,(\nabla_j F_i + \nabla_i F_j) + \sum_{j=1}^{n} F^j\,\tilde{\nabla}_j F_i\right)\tau^i.\end{aligned} \tag{6.17}$$

This form of formula (6.15) is identical to corresponding formula (4.16) in Chapter V. Further we introduce covector fields $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ with components

$$\begin{aligned}\alpha_i &= 2\,F_i + \sum_{j=1}^{n} v^j\,\tilde{\nabla}_i F_j,\\ \beta_i &= \sum_{j=1}^{n} v^j\,(\nabla_j F_i + \nabla_i F_j) + \sum_{j=1}^{n} F^j\,\tilde{\nabla}_j F_i.\end{aligned} \tag{6.18}$$

Then formulas for derivatives $\dot{\varphi}$ and $\ddot{\varphi}$ are transformed to

$$\dot{\varphi} = (\mathbf{F}\,|\,\mathbf{P}\,\boldsymbol{\tau}) + \frac{(\mathbf{F}\,|\,\mathbf{N})}{v}\,(\mathbf{v}\,|\,\boldsymbol{\tau}) + (\mathbf{v}\,|\,\nabla_t\boldsymbol{\tau}). \tag{6.19}$$

$$\ddot{\varphi} = \boldsymbol{\alpha}(\mathbf{P}\nabla_t\boldsymbol{\tau}) + \boldsymbol{\beta}(\mathbf{P}\boldsymbol{\tau}) + \frac{\boldsymbol{\alpha}(\mathbf{N})}{v}\,(\mathbf{v}\,|\,\nabla_t\boldsymbol{\tau}) + \frac{\boldsymbol{\beta}(\mathbf{N})}{v}\,(\mathbf{v}\,|\,\boldsymbol{\tau}). \tag{6.20}$$

Further calculations, which are used to derive **weak normality** equations, do not differ from corresponding calculations in Riemannian geometry (see § 6 in Chapter V). We reproduce them here omitting details. Substituting (6.11), (6.19), and (6.20) into the differential equation (6.8), we get

$$\begin{aligned}&\left(\frac{\boldsymbol{\alpha}(\mathbf{N})}{v} - \mathcal{A}\right)(\mathbf{v}\,|\,\nabla_t\boldsymbol{\tau}) + \boldsymbol{\alpha}(\mathbf{P}\nabla_t\boldsymbol{\tau}) + \boldsymbol{\beta}(\mathbf{P}\boldsymbol{\tau}) -\\ &-\mathcal{A}\,(\mathbf{F}\,|\,\mathbf{P}\,\boldsymbol{\tau}) + \left(\frac{\boldsymbol{\beta}(\mathbf{N})}{v} - \mathcal{A}\,\frac{(\mathbf{F}\,|\,\mathbf{N})}{v} - \mathcal{B}\right)(\mathbf{v}\,|\,\boldsymbol{\tau}) = 0.\end{aligned} \tag{6.21}$$

From (6.21) one can derive four equations, two of them determine coefficients $\mathcal{A}$ and $\mathcal{B}$ in ordinary differential equation (6.8):

$$\mathcal{A} = \frac{\boldsymbol{\alpha}(\mathbf{N})}{v}, \qquad \mathcal{B} = \frac{\boldsymbol{\beta}(\mathbf{N})}{v} - \mathcal{A}\,\frac{(\mathbf{F}\,|\,\mathbf{N})}{v}. \tag{6.22}$$

Two other equations have the following form:

$$\boldsymbol{\alpha}(\mathbf{P}\nabla_t\boldsymbol{\tau}) = 0, \qquad \boldsymbol{\beta}(\mathbf{P}\boldsymbol{\tau}) = \mathcal{A}\,(\mathbf{F}\,|\,\mathbf{P}\,\boldsymbol{\tau}). \tag{6.23}$$

If we take into account that $\boldsymbol{\tau}$ and $\nabla_t\boldsymbol{\tau}$ are two arbitrary vectors, then from (6.23) we derive two weak normality equations. In local coordinates they are written as

$$\sum_{i=1}^{n}\left(v^{-1}\,F_i + \sum_{j=1}^{n}\tilde{\nabla}_i\left(N^j\,F_j\right)\right)P^i_k = 0, \tag{6.24}$$

$$\begin{aligned}&\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\nabla_iF_j + \nabla_jF_i - 2\,v^{-1}\,F_i\,F_j\right)N^j\,P^i_k\,+\\ &+\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\frac{F^j\,\tilde{\nabla}_jF_i}{v} - \sum_{r=1}^{n}\frac{N^r\,N^j\,\tilde{\nabla}_jF_r}{v}\,F_i\right)P^i_k = 0.\end{aligned} \tag{6.25}$$

The equations (6.24) and (6.25) are identical to weak normality condition (6.16) and (6.17) in Chapter V. Here we have theorem analogous to theorem 6.2 in Chapter V.

THEOREM 6.1. *Newtonian dynamical system on Finslerian manifold M of the dimension $n \geqslant 2$ satisfies weak normality condition if and only if its force field satisfies the equations* (6.24) *and* (6.25) *at all points $q = (p, \mathbf{v})$ of tangent bundle TM, where $\mathbf{v} \neq 0$.*

Analysis of additional normality condition in Finslerian geometry is more complicated than analysis of weak normality condition (though this is true in Riemannian geometry too). Let's begin with applying formula (6.13) for to transform the equation (6.7). Thereby we take into account the relationship (7.8) from Chapter V:

$$\frac{\partial\nu}{\partial u^k} = -\nu^{-1}\,(\boldsymbol{\tau}_k\,|\,\mathbf{F}) - \nu\,(\nabla_{u^k}\mathbf{n}\,|\,\mathbf{n}). \tag{6.26}$$

Here ν is a function on hypersurface S that determines modulus of velocity vector at initial instant of time, $\mathbf{n}$ is a normal vector to hypersurface S, $u^1, \ldots, u^{n-1}$ are local coordinates on S and $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_{n-1}$ are corresponding coordinate tangent vectors (vectors of variations). Covariant derivative $\nabla_{u^k}\mathbf{n}$ is a derivative of vector $\mathbf{n}$ with respect to parameter $\mathbf{u}^k$ along coordinate lines on hypersurface S (see § 5 in Chapter IV). Components of metric tensor and components of Cartan connection in Finslerian geometry depend on velocity vector $\mathbf{v}$. Therefore in order to calculate covariant derivative $\nabla_{u^k}\mathbf{n}$ and in order to calculate scalar products in (6.26) we should fix some lift of curves from M to TM. Due to the origin of the equations

(6.26) such lift is defined by velocity vector on trajectories of shift $f_t\colon\,S \to S_t$, on initial hypersurface S this lift is determined by normal vector $\mathbf{n}$, since $\mathbf{v} = \nu\,\mathbf{n}$.

In two-dimensional case $n = 2$ the dimension of hypersurface S is equal to unity. The number of equations (6.26) is also equal to unity, while partial derivative $\partial\nu/\partial u^1$ turns to ordinary derivative. Therefore we have theorem similar to theorem 9.1 in Chapter V.

THEOREM 6.2. *In two-dimensional case* $\dim M = 2$ *additional normality condition is fulfilled unconditionally for arbitrary Newtonian dynamical system on Finslerian manifold* M*.*

Analysis of multidimensional case $\dim M \geqslant 3$ requires the use of theory of hypersurfaces in Finslerian manifolds.

§ 7. Induced Riemannian metric and derivational formulas of Weingarten.

Let S be hypersurface in Finslerian manifold of the dimension $\dim M \geqslant 3$ and let $u^1, \ldots, u^{n-1}$ be local coordinates on S. Local coordinates $u^1, \ldots, u^{n-1}$ define coordinate network on S, coordinate tangent vectors $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_{n-1}$ being tangent to the lines of this network. In § 5 above we showed that there exists a normal vector $\mathbf{n}$ perpendicular to vectors $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_{n-1}$. It is determined uniquely up to a change of sign $\mathbf{n} \to \pm\,\mathbf{n}$, this arbitrariness is eliminated by fixing orientation on S.

Completing the set of vectors $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_{n-1}$ by unitary normal vector $\mathbf{n}$, we get moving frame of hypersurface of S. Vectors of such frame are defined on the lines of coordinate network of S. Therefore we can differentiate them with respect to parameters $u^1, \ldots, u^{n-1}$ along coordinate lines. In Chapter IV we defined **two types** of covariant differentiations with respect to parameter along the curves. **First type** of differentiation is determined by canonical lift of curves from M to TM. Here parameter t is interpreted as time variable, tangent vector to curve is interpreted as velocity vector. **Second type** of covariant differentiation corresponds to deformations of curves, when, apart from main parameter t, we have parameter of deformation u. This is the very type of covariant differentiation naturally arises in the equation (6.26). In considering separate hypersurface S, which is not included into the scheme of normal shift, parameter t doesn't arise. But here we have normal vector $\mathbf{n}$, which determines the direction of velocity vector in case if we want to initiate normal shift:

$$\mathbf{v} = \nu \cdot \mathbf{n}. \tag{7.1}$$

Components of metric tensor of Finslerian metric $\mathbf{g}$ and components of Cartan connection are homogeneous functions of degree 0 respective to the vector $\mathbf{v}$. Therefore from (7.1) we get

$$g_{ij}(x^1, \ldots, x^n, \mathbf{v}) = g_{ij}(x^1, \ldots, x^n, \mathbf{n}). \tag{7.2}$$

$$\Gamma^k_{ij}(x^1, \ldots, x^n, \mathbf{v}) = \Gamma^k_{ij}(x^1, \ldots, x^n, \mathbf{n}). \tag{7.3}$$

Therefore we can replace canonical lift of curves by **normal lift** in calculating covariant derivatives ∇_u for the families of curves on S.

Let $\mathbf{X}$ be smooth tensor field of the type (r, s) defined at the points of hypersurface S. This means that to each point $p \in S$ one puts into correspondence some tensor $\mathbf{X}(p) \in T^r_s(p, M)$. Suppose that we have some parametric curve with parameter u on S, and let $\boldsymbol{\tau}$ be a tangent vector to this curve. Restriction of the field $\mathbf{X}$ to this curve we denote by $\mathbf{X}(u)$. Applying differentiation ∇_u to $\mathbf{X}(u)$ we get tensor field $\nabla_u\mathbf{X}(u)$of the type (r, s) on the curve. Its components are calculated by formula

$$\begin{aligned}\nabla_u X^{i_1\dots\,i_r}_{j_1\dots\,j_s} &= \frac{dX^{i_1\dots\,i_r}_{j_1\dots\,j_s}}{du}+\\ &+\sum^{r}_{k=1}\sum^{n}_{m=1}\sum^{n}_{a_k=1}\tau^m\,\Gamma^{i_k}_{m\,a_k}\,X^{i_1\dots\,a_k\dots\,i_r}_{j_1\dots\dots\dots\,j_s}-\\ &-\sum^{s}_{k=1}\sum^{n}_{m=1}\sum^{n}_{b_k=1}\tau^m\,\Gamma^{b_k}_{m\,j_k}X^{i_1\dots\dots\dots\,i_r}_{j_1\dots\,b_k\dots\,j_s},\end{aligned}\tag{7.4}$$

where $\Gamma^k_{ij} = \Gamma^k_{ij}(x^1, \dots, x^n, \mathbf{n})$, and $\tau^1, \dots, \tau^n$ are components of tangent vector $\boldsymbol{\tau}$ corresponding to parameter u.

Let $\mathbf{X} = \mathbf{X}(q)$ be a tensor field of type (r, s) from extended algebra of tensor fields on M. If to each point $p \in S$ we put into correspondence point $q = (p, \mathbf{n})$ from tangent bundle TM, then we get **normal lift** of hypersurface S to TM. This allows us to define the restriction of extended tensor field $\mathbf{X}$ to S. The derivative $\nabla_u\mathbf{X}$ for such restriction can be calculated by formula

$$\nabla_u\mathbf{X} = C(\tau \otimes \nabla\mathbf{X}) + C(\nabla_u\mathbf{n} \otimes \tilde{\nabla}\mathbf{X}).\tag{7.5}$$

Formula (7.5) is proved by direct calculation in local coordinates on a base of formula (7.4) and formulas (7.3) and (7.4) from Chapter III.

Suppose that on hypersurface S some local coordinates $u^1, \dots, u^{n-1}$ are chosen. Let's apply differentiation ∇_{u^k} to coordinate tangent vector field $\boldsymbol{\tau}_r$. As a result we get vector field on S, it can be expanded in the base of vectors $\boldsymbol{\tau}_1, \dots, \boldsymbol{\tau}_{n-1}$ and $\mathbf{n}$:

$$\nabla_{u^k}\boldsymbol{\tau}_r = \sum^{n-1}_{m=1}\theta^m_{kr}\,\boldsymbol{\tau}_m + b_{kr}\,\mathbf{n},\tag{7.6}$$

Formula (7.6) is analog of derivational formula (7.9), which we used in Chapter V. Let's find geometrical interpretation for the quantities θ^m_{kr} and b_{kr} in right hand side of (7.6). Changing local coordinates $u^1, \dots, u^{n-1}$ by $\tilde{u}^1, \dots, \tilde{u}^{n-1}$ and taking into account corresponding change of base vectors $\boldsymbol{\tau}_1, \dots, \boldsymbol{\tau}_{n-1}$ by $\tilde{\boldsymbol{\tau}}_1, \dots, \tilde{\boldsymbol{\tau}}_{n-1}$, one easily finds that θ^m_{kr} are components of affine connection, and b_{kr} are components of tensor field in S. From symmetry of Γ^k_{ij} in (7.4) we derive symmetry of θ^m_{kr} and b_{kr}:

$$\theta^m_{kr} = \theta^m_{rk}, \qquad\qquad b_{kr} = b_{rk}.\tag{7.7}$$

Finslerian metric $\mathbf{g}$ is an extended tensor field of type $(0,2)$ on M. Using the construction of normal lift of S to TM, we restrict $\mathbf{g}$ to S; and by means of such restriction we define scalar product of coordinate vector fields on S:

$$\rho_{kr} = (\boldsymbol{\tau}_k \,|\, \boldsymbol{\tau}_r) = \sum_{i=1}^{n}\sum_{j=1}^{n} g_{ij}(x^1, \ldots, x^n, \mathbf{n})\,\tau^i_k\,\tau^j_r. \tag{7.8}$$

Quantities ρ_{kr} in (7.8) are components of Riemannian metric $\boldsymbol{\rho}$ in inner geometry of the manifold S. The condition $|\mathbf{n}| = 1$ is written in terms of scalar product:

$$(\mathbf{n} \,|\, \mathbf{n}) = \sum_{i=1}^{n}\sum_{j=1}^{n} g_{ij}(x^1, \ldots, x^n, \mathbf{n})\,n^i\,n^j = 1. \tag{7.9}$$

Let's differentiate the relationship (7.9) with respect to u^k:

$$\begin{gathered}\frac{\partial(\mathbf{n}\,|\,\mathbf{n})}{\partial u^k} = \nabla_{u^k}(\mathbf{n}\,|\,\mathbf{n}) = \nabla_{u^k} C(\mathbf{n}\otimes\mathbf{n}\otimes\mathbf{g}) =\\ = (\nabla_{u^k}\mathbf{n}\,|\,\mathbf{n}) + (\mathbf{n}\,|\,\nabla_{u^k}\mathbf{n}) + C(\mathbf{n}\otimes\mathbf{n}\otimes\nabla_{u^k}\mathbf{g}) = 0.\end{gathered}$$

In order to calculate $\nabla_{u^k}\mathbf{g}$ in this equality we use formula (7.5) and take into account that $\nabla\mathbf{g} = 0$ and $\tilde{\nabla}\mathbf{g} = \mathbf{C}$ (see formula (1.4) above). As a result we get

$$2\,(\nabla_{u^k}\mathbf{n}\,|\,\mathbf{n}) + 2\,\mathbf{C}(\mathbf{n}, \mathbf{n}, \nabla_{u^k}\mathbf{n}) = 0. \tag{7.10}$$

Second term in left hand side of (7.10) is a result of contracting Cartan tensor with two copies of vector n and with vector $\nabla_{u^k}\mathbf{n}$. It is zero due to the relationship (1.5):

$$\mathbf{C}(\mathbf{n}, \mathbf{n}, \nabla_{u^k}\mathbf{n}) = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{s=1}^{n} C_{ijs}(x^1, \ldots, x^n, \mathbf{n})\,n^i\,n^j\,\nabla_{u^k} n^s = 0.$$

Then from (7.10) we derive $(\nabla_{u^k}\mathbf{n}\,|\,\mathbf{n}) = 0$. This means that vector $\nabla_{u^k}\mathbf{n}$ is perpendicular to $\mathbf{n}$, i. e. it is in tangent hyperplane to S. Therefore it is given by some linear combination of tangent vectors $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_{n-1}$:

$$\nabla_{u^k}\mathbf{n} = \sum_{m=1}^{n-1} d^m_k\,\boldsymbol{\tau}_m. \tag{7.11}$$

Coefficients d^m_k in linear combination (7.11) are components of tensor field of type $(1,1)$ in inner geometry of manifold M. In order to calculate these coefficients let's write the equality $(\boldsymbol{\tau}_r\,|\,\mathbf{n}) = 0$, which means $\boldsymbol{\tau}_r \perp \mathbf{n}$, and let's differentiate this equality with respect to the parameter u^k:

$$\frac{\partial(\boldsymbol{\tau}_r\,|\,\mathbf{n})}{\partial u^k} = \nabla_{u^k}(\boldsymbol{\tau}_r\,|\,\mathbf{n}) = \nabla_{u^k} C(\boldsymbol{\tau}_r\otimes\mathbf{n}\otimes\mathbf{g}) =$$

$$= (\nabla_{u^k}\boldsymbol{\tau}_r\,|\,\mathbf{n}) + (\boldsymbol{\tau}_r\,|\,\nabla_{u^k}\mathbf{n}) + C(\boldsymbol{\tau}_r\otimes\mathbf{n}\otimes\nabla_{u^k}\mathbf{g}) =$$

$$= (\nabla_{u^k}\boldsymbol{\tau}_r\,|\,\mathbf{n}) + (\boldsymbol{\tau}_r\,|\,\nabla_{u^k}\mathbf{n}) + 2\,\mathbf{C}(\boldsymbol{\tau}_r,\mathbf{n},\nabla_{u^k}\mathbf{n}) = 0.$$

Last term in left hand side of this equality is zero due to the identity (1.5):

$$\mathbf{C}(\boldsymbol{\tau}_r,\mathbf{n},\nabla_{u^k}\mathbf{n}) = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{s=1}^{n} C_{ijs}(x^1,\ldots,x^n,\mathbf{n})\,n^i\,\tau^j_r\,\nabla_{u^k}n^s = 0.$$

Therefore $(\nabla_{u^k}\boldsymbol{\tau}_r\,|\,\mathbf{n}) + (\boldsymbol{\tau}_r\,|\,\nabla_{u^k}\mathbf{n}) = 0$. Now by means of derivational formulas (7.6) and (7.11) for $\nabla_{u^k}\boldsymbol{\tau}_r$ and $\nabla_{u^k}\mathbf{n}$ we get

$$b_{kr} = -\sum_{m=1}^{n-1}\rho_{rm}\,d^m_k. \tag{7.12}$$

Formula (7.12) can be inverted. Then coefficients d^m_k from (7.11) appear to be expressed through components of second fundamental form:

$$d^m_k = -b^m_k = -\sum_{r=1}^{n-1}\rho^{mr}\,b_{rk}. \tag{7.13}$$

Finally derivational formulas (7.6) and (7.11) are written as

$$\nabla_{u^k}\boldsymbol{\tau}_r = \sum_{m=1}^{n-1}\theta^m_{kr}\,\boldsymbol{\tau}_m + b_{kr}\,\mathbf{n}, \tag{7.14}$$

$$\nabla_{u^k}\mathbf{n} = -\sum_{m=1}^{n-1} b^m_k\,\boldsymbol{\tau}_m. \tag{7.15}$$

Now let's study how Riemannian metric with components (7.8) and symmetric affine connection θ^m_{kr} in (7.14) are bound with each other. Let's calculate derivatives

$$\frac{\partial\rho_{kr}}{\partial u^s} = \nabla_{u^s}(\boldsymbol{\tau}_k\,|\,\boldsymbol{\tau}_r) = \nabla_{u^s}C(\boldsymbol{\tau}_k\otimes\boldsymbol{\tau}_r\otimes\mathbf{g}) =$$

$$= (\nabla_{u^s}\boldsymbol{\tau}_k\,|\,\boldsymbol{\tau}_r) + (\boldsymbol{\tau}_k\,|\,\nabla_{u^s}\boldsymbol{\tau}_r) + C(\boldsymbol{\tau}_k\otimes\boldsymbol{\tau}_r\otimes\nabla_{u^s}\mathbf{g}).$$

We use derivational formula (7.14) for to calculate $\nabla_{u^s}\boldsymbol{\tau}_k$ and $\nabla_{u^s}\boldsymbol{\tau}_r$, and we use formula (7.5) to transform $\nabla_{u^s}\mathbf{g}$. Then we obtain

$$\frac{\partial\rho_{kr}}{\partial u^s} - \sum_{m=1}^{n-1}\theta^m_{sk}\rho_{mr} - \sum_{m=1}^{n-1}\theta^m_{sr}\rho_{km} = -\sum_{m=1}^{n-1} 2\,c_{krm}\,b^m_s. \tag{7.16}$$

The quantities c_{krm} in (7.16) are obtained by contracting Cartan tensor $\mathbf{C}$ with three coordinate tangent vectors $\boldsymbol{\tau}_k$, $\boldsymbol{\tau}_r$, and $\boldsymbol{\tau}_m$:

$$c_{krm} = \mathbf{C}(\boldsymbol{\tau}_k, \boldsymbol{\tau}_r, \boldsymbol{\tau}_m) = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{s=1}^{n} C_{ijs}(x^1, \ldots, x^n, \mathbf{n})\,\tau^i_k\,\tau^j_r\,\tau^s_m.$$

They are components of tensor field $\mathbf{c}$ of type $(0,3)$ in inner geometry of submanifold S. The relationship (7.16) in coordinate free form is written as

$$\nabla\boldsymbol{\rho} = -2\,C(\mathbf{c}\otimes\mathbf{B}). \tag{7.17}$$

In general case right hand side of (7.17) is nonzero, i. e. Riemannian metric $\boldsymbol{\rho}$ from (7.8) and connection θ from (7.14) are not concordant. Tensor $\boldsymbol{\beta}$ with components

$$\beta_{ijs} = \sum_{m=1}^{n-1} c_{ijm}\,b^m_s$$

is a measure of such disconcordance. We can calculate θ^k_{ij} explicitly:

$$\begin{aligned}\theta^k_{ij} = \frac{1}{2}\sum_{s=1}^{n-1}\rho^{ks}\left(\frac{\partial\rho_{sj}}{\partial u^i} + \frac{\partial\rho_{is}}{\partial u^j} - \frac{\partial\rho_{ij}}{\partial u^s}\right) + \\ + \sum_{s=1}^{n-1}\rho^{ks}\left(\beta_{jsi} + \beta_{sij} - \beta_{ijs}\right).\end{aligned} \tag{7.18}$$

Formula (7.18) is an analog of well known formula for components of metric connection. It generalizes formula (7.11) from Chapter V to the present situation.

§ 8. The additional normality equations.

Now let's apply the result of § 7 for further analysis of the equations (6.26) derived from additional normality condition. Scalar product $(\nabla_{u^k}\mathbf{n}\,|\,\mathbf{n})$ in right hand side of (6.26) vanishes due to derivational formulas (7.15). Therefore

$$\frac{\partial\nu}{\partial u^k} = -\nu^{-1}\,(\boldsymbol{\tau}_k\,|\,\mathbf{F}). \tag{8.1}$$

Vector field $\mathbf{F}$ can be represented by covector field $\mathbf{F}$ obtained by lowering index procedure. Here is the formula for components of such field:

$$F_i = \sum_{j=1}^{n} g_{ij}(x^1, \ldots, x^n, \mathbf{v})\,F^j.$$

Using covector field $\mathbf{F}$, we can eliminate explicit entry of metric in right hand side of the equation (8.1). Now this equation is written as

$$\frac{\partial\nu}{\partial u^k} = -\nu^{-1}\,C(\boldsymbol{\tau}_k\otimes\mathbf{F}). \tag{8.2}$$

In multidimensional case $\dim M \geqslant 3$ the equations (8.2) form complete system of Pfaff equations with respect to scalar function $\nu(u^1, \ldots, u^{n-1})$. Brief sketch of this theory is given in § 8 of Chapter V. Normalization condition (6.2) with arbitrary constant ν_0 means that system of Pfaff equations (8.2) is compatible in the sense of definition 8.1 from Chapter V. Hence for (8.2) compatibility equations are fulfilled (see equations (8.7) in Chapter V). It order to write these equations let's calculate second order derivatives of ν by virtue of the equations (8.2):

$$\frac{\partial^2\nu}{\partial u^k\,\partial u^r} = -\nabla_{u^r} C(\boldsymbol{\tau}_k\otimes\mathbf{F}) = \frac{C(\boldsymbol{\tau}_k\otimes\mathbf{F})}{\nu^2}\,\frac{\partial\nu}{\partial u^r} -$$

$$-\,\nu^{-1}\,C(\nabla_{u^r}\boldsymbol{\tau}_k\otimes\mathbf{F}) - \nu^{-1}\,C(\boldsymbol{\tau}_k\otimes\nabla_{u^r}\mathbf{F}) =$$

$$= -\frac{C(\boldsymbol{\tau}_k\otimes\mathbf{F})\,C(\boldsymbol{\tau}_r\otimes\mathbf{F})}{\nu^3} - \frac{C(\nabla_{u^r}\boldsymbol{\tau}_k\otimes\mathbf{F})}{\nu} - \frac{C(\boldsymbol{\tau}_k\otimes\nabla_{u^r}\mathbf{F})}{\nu}.$$

Let's transpose indices r and k. Then we have

$$\frac{\partial^2\nu}{\partial u^r\,\partial u^k} = -\nabla_{u^k} C(\boldsymbol{\tau}_r\otimes\mathbf{F}) = \frac{C(\boldsymbol{\tau}_r\otimes\mathbf{F})}{\nu^2}\,\frac{\partial\nu}{\partial u^k} -$$

$$-\,\nu^{-1}\,C(\nabla_{u^k}\boldsymbol{\tau}_r\otimes\mathbf{F}) - \nu^{-1}\,C(\boldsymbol{\tau}_r\otimes\nabla_{u^k}\mathbf{F}) =$$

$$= -\frac{C(\boldsymbol{\tau}_r\otimes\mathbf{F})\,C(\boldsymbol{\tau}_k\otimes\mathbf{F})}{\nu^3} - \frac{C(\nabla_{u^k}\boldsymbol{\tau}_r\otimes\mathbf{F})}{\nu} - \frac{C(\boldsymbol{\tau}_r\otimes\nabla_{u^k}\mathbf{F})}{\nu}.$$

Let's subtract one of these equations from another. Thereby we take into account the relationship $\nabla_{u^k}\boldsymbol{\tau}_r = \nabla_{u^r}\boldsymbol{\tau}_k$. This equality holds, since for to calculate covariant both derivatives ∇_{u^k} and ∇_{u^r} we use the same lift of curves given by the velocity vector on shift trajectories (or, what is equivalent, by normal vector on initial hypersurface S). Then we get

$$C(\boldsymbol{\tau}_k\otimes\nabla_{u^r}\mathbf{F}) = C(\boldsymbol{\tau}_r\otimes\nabla_{u^k}\mathbf{F}). \tag{8.3}$$

Compatibility equations (8.7) from Chapter V for the system of Pfaff equations (8.2) mean that the relationships (8.3) are fulfilled identically with respect to n variables $u^1, \ldots, u^{n-1}$ and ν. Its important to note that the value of function ν in (8.3) is understood as independent variable. For covariant derivatives $\nabla_{u^k}\mathbf{F}$ and $\nabla_{u^r}\mathbf{F}$ in (8.3) we use formula (8.18) from Chapter V, which, in turn, is derived from formula (5.19) in Chapter IV: $\nabla_{u^k}\mathbf{F} = C(\nabla_{u^k}\mathbf{v}\otimes\tilde{\nabla}\mathbf{F}) + C(\boldsymbol{\tau}_k\otimes\nabla\mathbf{F})$. In order to calculate covariant derivative $\nabla_{u^k}\mathbf{v}$ on S we use formula $\mathbf{v} = \nu\cdot\mathbf{n}$, which holds on initial hypersurface S. And for covariant derivative $\nabla_{u^k}\mathbf{n}$ we use derivational formula (7.15). As a result for $\nabla_{u^k}\mathbf{v}$ we get the relationship

$$\nabla_{u^k}\mathbf{v} = -\frac{C(\boldsymbol{\tau}_k\otimes\mathbf{F})}{\nu}\cdot\mathbf{n} - \nu\cdot\sum_{m=1}^{n-1} b^m_k\,\boldsymbol{\tau}_m$$

coinciding with (8.19) in Chapter V. From (8.3) we obtain

$$\frac{C(\boldsymbol{\tau}_k\otimes\mathbf{F})\,C(\mathbf{n}\otimes\tilde{\nabla}\mathbf{F}\otimes\boldsymbol{\tau}_r)}{\nu}-\frac{C(\boldsymbol{\tau}_r\otimes\mathbf{F})\,C(\mathbf{n}\otimes\tilde{\nabla}\mathbf{F}\otimes\boldsymbol{\tau}_k)}{\nu}+C(\boldsymbol{\tau}_r\otimes\nabla\mathbf{F}\otimes\boldsymbol{\tau}_k)-$$
$$-\,C(\boldsymbol{\tau}_k\otimes\nabla\mathbf{F}\otimes\boldsymbol{\tau}_r)=\sum_{m=1}^{n-1}\nu\left(b^m_r\,C(\boldsymbol{\tau}_m\otimes\tilde{\nabla}\mathbf{F}\otimes\boldsymbol{\tau}_k)-b^m_k\,C(\boldsymbol{\tau}_m\otimes\tilde{\nabla}\mathbf{F}\otimes\boldsymbol{\tau}_r)\right).$$

Remember that here, as in (8.2), by $\mathbf{F}$ we denote covector field obtained from vector field $\mathbf{F}$ as a result of lowering index by means of Finslerian metric $\mathbf{g}$. In coordinate form this equation takes the form

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\tau^i_k\,\tau^j_r\left(\sum_{m=1}^{n}n^m\,\frac{F_i\,\tilde{\nabla}_mF_j-F_j\,\tilde{\nabla}_mF_i}{\nu}+\nabla_jF_i-\nabla_iF_j\right)=$$
$$=\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{m=1}^{n-1}\nu\left(b^m_r\,\tau^j_m\tilde{\nabla}_jF_i\,\tau^i_k-b^m_k\,\tau^j_m\tilde{\nabla}_jF_i\,\tau^i_r\right).\tag{8.4}$$

which coincides with corresponding equation (8.21) in Chapter V. In what follows the equations (8.4) should be subdivided into two parts. For this goal we introduce two operators $\mathbf{P}$ and $\mathbf{H}$ with components

$$P^i_\varepsilon=\sum_{s=1}^{n}\sum_{a=1}^{n-1}\sum_{k=1}^{n-1}g_{\varepsilon s}\,\tau^s_a\,\rho^{ak}\,\tau^i_k,\tag{8.5}$$

$$H^j_\varepsilon=\sum_{s=1}^{n}\sum_{a=1}^{n-1}\sum_{k=1}^{n-1}\sum_{m=1}^{n-1}g_{\varepsilon s}\,\tau^s_a\,\rho^{ak}\,b^m_k\,\tau^j_m.\tag{8.6}$$

Here ρ^{ak} is determined by induced Riemannian metric $\boldsymbol{\rho}$ on S (see § 7 above), b^m_k are components of second fundamental form from (7.15). Formulas (8.5) and (8.6) are analogous to formulas (9.6) and (9.11) in Chapter V. The only difference is that quantities $g_{\varepsilon s}$ depend on normal vector

$$g_{\varepsilon s}=g_{\varepsilon s}(x^1,\ldots,x^n,\mathbf{v})=g_{\varepsilon s}(x^1,\ldots,x^n,\mathbf{n}).$$

This is the feature of Finslerian metric. But this feature doesn't affect further transformation of the equations (8.4). First they are brought to the form

$$\sum_{i=1}^{n}\sum_{j=1}^{n}P^i_\varepsilon\,P^j_\sigma\left(\sum_{m=1}^{n}n^m\,\frac{F_i\,\tilde{\nabla}_mF_j-F_j\,\tilde{\nabla}_mF_i}{\nu^2}\;+\right.$$
$$\left.+\;\frac{\nabla_jF_i-\nabla_iF_j}{\nu}\right)=\sum_{i=1}^{n}\sum_{j=1}^{n}(P^i_\varepsilon\,H^j_\sigma-P^i_\sigma\,H^j_\varepsilon)\,\tilde{\nabla}_jF_i\tag{8.7}$$

coinciding with formula (9.12) from Chapter V. Then some properties of the operators **P** and **H** defined by (8.5) and (8.6) are used. We use the following ones:

(1) operators **P** and **H** are symmetric with respect to the metric with components $g_{ij} = g_{ij}(x^1, \ldots, x^n, \mathbf{n})$, they are bound by the relationship

$$\mathbf{P} \circ \mathbf{H} = \mathbf{H} \circ \mathbf{P} = \mathbf{H}, \tag{8.8}$$

(2) operator **P** is an orthogonal projector to the hyperplane tangent to S.

Operators **P** and **H** with the properties (1) and (2) above are determined by hypersurface S. But this hypersurface by itself in definitions 6.1 and 6.2 is quite arbitrary. It can be passing through any preassigned point p on M, and it can have any preassigned direction of normal vector $\mathbf{n}$. Moreover, it can have any preassigned symmetric matrix as a matrix of second fundamental form at this point. The latter is expressed by theorem which is analogous to theorem 9.4 in Chapter V.

Theorem 8.1 (**on the second fundamental form**). *Let $\mathbf{v}_0$ be nonzero vector at some point p_0 on Finslerian manifold M, and let $\mathbf{P}$ be the operator of orthogonal projection onto the hyperplane α perpendicular to $\mathbf{v}_0$. Then for any linear operator $\mathbf{H}$ in $T_{p_0}(M)$ symmetric in metric $\mathbf{g}(p_0, \mathbf{v}_0)$ on M and satisfying the relationships* (8.8) *one can find hypersurface S passing through the point p_0 and perpendicular to $\mathbf{v}_0$ such that matrix elements of the operator $\mathbf{H}$ are determined by second fundamental form of S according to the formula* (8.6).

Theorem 8.1 allows us to identify **P** with projector field of orthogonal projectors to hyperplane α perpendicular to the vector of velocity. Operator **H** is identified with arbitrary operator field from extended algebra of tensor fields symmetric with respect to Finslerian metric **g** and bound with **P** by (8.8). This arbitrariness of **H** allows us to subdivide (8.7) into two parts. Resulting equations have the form:

$$\begin{aligned}&\sum_{i=1}^{n}\sum_{j=1}^{n} P^i_\varepsilon\, P^j_\sigma \left(\sum_{m=1}^{n} N^m\, \frac{F_i\, \tilde{\nabla}_m F_j}{v} - \nabla_i F_j\right) =\\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} P^i_\varepsilon\, P^j_\sigma \left(\sum_{m=1}^{n} N^m\, \frac{F_j\, \tilde{\nabla}_m F_i}{v} - \nabla_j F_i\right),\end{aligned} \tag{8.9}$$

$$\sum_{m=1}^{n}\sum_{i=1}^{n}\sum_{j=1}^{n} g^{\sigma m}\, P^j_m\, \tilde{\nabla}_j F_i\, P^i_\varepsilon = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{P^{ij}\, \tilde{\nabla}_j F_i}{n-1}\, P^\sigma_\varepsilon. \tag{8.10}$$

These are the required **additional normality** equations for the force field of Newtonian dynamical system. Their derivation above is based on the same arguments as the derivation of the equations (9.20) and (9.42) in § 9 of Chapter V. The difference between Riemannian and Finslerian cases appears to be not essential. Therefore we have a theorem similar to the theorem 9.5 in Chapter V.

Theorem 8.5. *Newtonian dynamical system on Finslerian manifold of the dimension $n \geqslant 3$ satisfies additional normality condition if and only if its force field satisfies the equations* (8.9) *and* (8.10) *at all points $q = (p, \mathbf{v})$ of tangent bundle TM, except for those, where $\mathbf{v} = 0$.*

Proof of the auxiliary theorem 9.4 on second fundamental form is given in § 10 of Chapter V. This proof can be easily modified for the case of Finslerian manifolds. Here we shall not give the proof of theorem 8.1 in details. We shall sketch it and mark out the differences occurring due to difference between Riemannian and Finslerian cases. In the first step theorem 8.1 is reduced to the following one that coincides with theorem 10.1 from Chapter V almost literally.

Theorem 8.3. *Suppose that in tangent space $T_{p_0}(M)$ at some point p_0 on Finslerian manifold of the dimension $n \geqslant 3$ we choose some nonzero vector $\mathbf{v}_0$, mark hyperplane α perpendicular to $\mathbf{v}_0$, mark some base $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_{n-1}$ in α, and choose some symmetric $(n-1) \times (n-1)$ matrix b. Then there exists some hypersurface S passing through the point p_0 tangent to α and there exist some local coordinates $u^1, \ldots, u^{n-1}$ on S such that vectors $\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_{n-1}$ are coordinate tangent vectors to S at the point p_0, while b coincides with the matrix of second fundamental form for S at this point.*

Lemma 8.1. *For any point p_0 on Finslerian manifold M and for any vector $\mathbf{v}_0 \neq 0$ at this point there exist local coordinates $x^1, \ldots, x^n$ such that values of all components of Cartan connection $\Gamma^k_{ij}(x^1, \ldots, x^1, v^1_0, \ldots, v^n_0)$ are zero at the point $q_0 = (p_0, \mathbf{v}_0)$ of tangent bundle TM.*

Lemma 8.1 differs from corresponding lemma 10.1 in Chapter V, since Cartan connection is extended one. Its components depend not only on the point of M, but on velocity vector $\mathbf{v}$ at this point. Therefore formula (10.6) from Chapter V here in Finslerian geometry looks like

$$x^k = \tilde{x}^k + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \tilde{\Gamma}^k_{ij}(p_0, \mathbf{v}_0)\, \tilde{x}^i\, \tilde{x}^j. \tag{8.11}$$

In other details proof of lemma 8.1 is a copy of the proof of corresponding lemma 10.1 in Chapter V.

Proof of lemma 8.1 is a second step in proving theorem on second fundamental form. Proof of theorem 8.3 is a last step. It doesn't differ from proof of corresponding theorem 10.1 in Chapter V.

§ 9. Complete and strong normality conditions.

Theorems 6.1 and 8.2 reduce **weak normality** condition from definition 6.3 and **additional normality** condition from definition 6.2 to differential equations (6.24), (6.25), (8.9), and (8.10) for the force field $\mathbf{F}$ of Newtonian dynamical system. These equations form complete system of **normality equations**. Therefore the couple of two conditions (of weak normality and of additional normality) is called **complete normality** condition. While the condition given by definition 6.1 is called **strong**

normality condition. In Finslerian geometry we have a theorem establishing equivalence of last two conditions.

THEOREM 9.1. *Complete and strong normality conditions for Newtonian dynamical systems on Finslerian manifolds are equivalent to each other.*

This is analog of theorem 12.1 in Chapter V. Direct assertion of this theorem follows from the simple fact that the solution of homogeneous linear ordinary differential equation $\ddot{\varphi} = \mathcal{A}(t)\,\dot{\varphi} + \mathcal{B}(t)$ with zero initial data (6.6) is identically zero: $\varphi = 0$. Proof of inverse assertion is more complicated. But it reproduces, almost literally, all arguments that we used in the proof of theorem 12.1 in Chapter V.

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