ON THE GEOMETRY OF A DISLOCATED MEDIUM.

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Abstract. Purely real space versions of the differential equations describing the kinematics of a dislocated crystalline medium are considered. The differential geometric structures associated with them are revealed.

1. Introduction.

In [1] a phenomenological approach to the nonlinear theory of plasticity in glasses, pitches, and soft polymers was suggested (see also [2]). Currently, we have no direct microscopic support for the results of [1] since microscopic mechanisms of plasticity in amorphous materials are not yet completely understood, especially if one needs an exact quantitative description (see papers [3] and [4] where some approaches are developed, but this is by no means the ultimate theory). Having no direct way, one should maneuver choosing a detour, a roundabout course to the goal. For the theory of plasticity this course goes through the theory of dislocations (see [5–7]). The matter is that dislocations provide a microscopic mechanism explaining the plasticity of crystals. Relying on the integrity of the nature, one can expect that the plasticity phenomenon in crystals and in amorphous materials are described similarly.

The paper [5] is a review of the basics. There the nonlinear elastic and plastic deformation tensors $\hat{\mathbf{G}}$ and $\check{\mathbf{G}}$ for a crystalline medium are defined, and the following differential equations for them are derived:

$$\begin{gathered}\frac{\partial\hat{G}_{kq}}{\partial t}+\sum_{r=1}^{3}v^r\,\nabla_r\hat{G}_{kq}=-\sum_{r=1}^{3}\nabla_k v^r\,\hat{G}_{rq}-\sum_{r=1}^{3}\hat{G}_{kr}\,\nabla_q v^r+\\+\sum_{r=1}^{3}\theta^r_k\,\hat{G}_{rq}+\sum_{r=1}^{3}\hat{G}_{kr}\,\theta^r_q,\end{gathered}\tag{1.1}$$

$$\frac{\partial\check{G}^k_i}{\partial t}+\sum_{r=1}^{3}v^r\,\nabla_r\check{G}^k_i=\sum_{r=1}^{3}\left(\check{G}^r_i\,\nabla_r v^k-\nabla_i v^r\,\check{G}^k_r\right)-\sum_{r=1}^{3}\theta^k_r\,\check{G}^r_i.\tag{1.2}$$

Here v^1, v^2, v^3 are the components of the velocity vector $\mathbf{v}$ of a point of the medium. By ∇ in (1.1) and (1.2) we denote the covariant differentiation. In Cartesian coordinates x^1, x^2, x^3 the covariant derivative ∇_i coincides with $\partial/\partial x^i$. However, for

2000 *Mathematics Subject Classification.* 74A05, 74C15, 74C20, 74D10.

the sake of generality below we use curvilinear coordinates y^1, y^2, y^3. In this case ∇_i is written through the Christoffel symbols (see [8]) of the standard Euclidean metric in the space $\mathbb{E}$ (the real space, where the dynamics of any medium occurs):

$$\Gamma^k_{ij} = \sum_{r=1}^{3} \frac{g^{kr}}{2} \left(\frac{\partial g_{rj}}{\partial y^i} + \frac{\partial g_{ir}}{\partial y^j} - \frac{\partial g_{ij}}{\partial y^k} \right). \tag{1.3}$$

It is remarkable that the same differential equations (1.1) and (1.2) govern the evolution of $\hat{\mathbf{G}}$ and $\check{\mathbf{G}}$ in amorphous media (see [1] and [2]). The difference is that the tensorial parameter $\boldsymbol{\theta}$ for amorphous materials is introduced empirically (see [1]), while for crystalline materials we have the formula

$$\theta^{\,r}_q = -\sum_{i=1}^{3} \hat{S}^r_i \, j^{\,i}_q + \sum_{i=1}^{3} \sum_{p=1}^{3} v^p \, \hat{S}^r_i \, (\nabla_p \hat{T}^i_q - \nabla_q \hat{T}^i_p) \tag{1.4}$$

expressing $\boldsymbol{\theta}$ through other tensorial parameters of a medium: $\hat{\mathbf{T}}$ and $\mathbf{j}$ (see [6] and [7] for more details); $\hat{\mathbf{T}}$ is called the *incompatible distorsion tensor* and $\mathbf{j}$ is the tensor of the *density of the Burgers vector flow*. Unlike $\boldsymbol{\theta}$, both $\hat{\mathbf{T}}$ and $\mathbf{j}$ are double space tensorial quantities: their upper index i in (1.4) is associated with the Burgers space. The tensor field $\hat{\mathbf{S}}$ in (1.4) is expressed through $\hat{\mathbf{T}}$ as the inverse matrix: $\hat{\mathbf{S}} = \hat{\mathbf{T}}^{-1}$. Its lower index i is associated with the Burgers space.

The Burgers space $\mathbb{B}$ is introduced in [5] as a container for Burgers vectors. It is a copy of the real space $\mathbb{E}$ filled with the infinite defect-free crystalline grid of that material which we have in the real space. The copy of the Euclidean metric $\mathbf{g}$ in the Burgers space is denoted by $\overset{\star}{\mathbf{g}}$. The Burgers space $\mathbb{B}$ is usually equipped with Cartesian coordinates x^1, x^2, x^3. Therefore, we have

$$\overset{\star}{g}_{ij} = \text{const}, \qquad\qquad \overset{\star}{g}{}^{ij} = \text{const}\,.$$

Suppose that $\mathbf{X}$ is a double space tensor field in $\mathbb{E}$, and assume that i and j are its indices associated with the Burgers space:

$$X^{\dots\, i \,\dots\dots\dots}_{\dots\dots\dots\, j\, \dots} = X^{\dots\, i \,\dots\dots\dots}_{\dots\dots\dots\, j\, \dots}(y^1, y^2, y^3). \tag{1.5}$$

Applying $\hat{\mathbf{S}}$ and $\hat{\mathbf{T}}$ to the components (1.5) of the tensor field $\mathbf{X}$, one can convert i and j into the real space indices p and q respectively:

$$X^{\dots\, p \,\dots\dots\dots}_{\dots\dots\dots\, j\, \dots} = \sum_{i=1}^{3} X^{\dots\, i \,\dots\dots\dots}_{\dots\dots\dots\, j\, \dots} \; \hat{S}^p_i, \tag{1.6}$$

$$X^{\dots\, i \,\dots\dots\dots}_{\dots\dots\dots\, q\, \dots} = \sum_{j=1}^{3} X^{\dots\, i \,\dots\dots\dots}_{\dots\dots\dots\, j\, \dots} \; \hat{T}^j_q. \tag{1.7}$$

The elastic deformation tensor $\hat{\mathbf{G}}$ is produced from $\overset{\star}{\mathbf{g}}$ according to the recipe (1.7):

$$\hat{G}_{pq} = \sum_{i=1}^{3} \sum_{j=1}^{3} \overset{\star}{g}_{ij} \; \hat{T}^i_p \; \hat{T}^j_q. \tag{1.8}$$

The purely real space tensors $\mathbf{J}$ and $\hat{\mathbf{Z}}$ are produced according to the recipe (1.6):

$$J^p_q = \sum_{i=1}^{3} \hat{S}^p_i \; j^{\,i}_{\,q}, \qquad\qquad \hat{Z}^{\,r}_{pq} = \sum_{i=1}^{3} \hat{S}^r_i \; \nabla_p \hat{T}^i_q. \tag{1.9}$$

The purely real space tensors $\mathbf{R}$ and $\hat{\mathbf{R}}$ are defined similarly (see [7]):

$$R^p_q = \sum_{i=1}^{3} \hat{S}^p_i \; \rho^{\,i}_{\,q}, \qquad\qquad \hat{R}^{\,r}_{pq} = \sum_{i=1}^{3} \hat{S}^r_i \,(\nabla_p \hat{T}^i_q - \nabla_q \hat{T}^i_p). \tag{1.10}$$

Here $\rho^{\,i}_{\,q}$ are the components of the dual space tensor field $\boldsymbol{\rho}$, this tensor field is defined as the *density of Burgers vector* for the dislocations in a crystal (see [5]). The formulas (1.9) and (1.10) express our intension to write a substantial part of the theory in terms of purely real space tensor fields. This goal was declared in [7]. The other goal of the present paper is to reveal geometric structures hidden underneath the theory of dislocations.

2. The elastic deformation metric and associated connection with torsion.

Let's consider the elastic deformation tensor $\hat{\mathbf{G}}$. It is defined by formula (1.8). Since $\det \hat{\mathbf{T}} \neq 0$, this formula determines a Riemannian metric in $\mathbb{E}$ other than the basic Euclidean metric $\mathbf{g}$. Let's call it the *elastic deformation metric* and denote it by $\hat{\mathbf{G}}$. Then remember the following differential equation derived in [7]:

$$\nabla_p \hat{G}_{qk} = \sum_{r=1}^{3} \hat{Z}^{\,r}_{pq}\, \hat{G}_{rk} + \sum_{r=1}^{3} \hat{Z}^{\,r}_{pk}\, \hat{G}_{qr}. \tag{2.1}$$

Expressing ∇_p in (2.1) through the partial derivative $\partial/\partial y^p$, we get

$$\frac{\partial \hat{G}_{qk}}{\partial y^p} - \sum_{r=1}^{3} \Gamma^r_{pq}\, \hat{G}_{rk} - \sum_{r=1}^{3} \Gamma^r_{pk}\, \hat{G}_{qr} = \sum_{r=1}^{3} \hat{Z}^{\,r}_{pq}\, \hat{G}_{rk} + \sum_{r=1}^{3} \hat{Z}^{\,r}_{pk}\, \hat{G}_{qr}. \tag{2.2}$$

Here the Christoffel symbols Γ^r_{pq} and Γ^r_{pk} are determined by the formula (1.3). Comparing the left and right hand sides of (2.2), we see that it is convenient to introduce the other set of Christoffel symbols:

$$\hat{\Gamma}^k_{ij} = \Gamma^k_{ij} + \hat{Z}^k_{ij}. \tag{2.3}$$

The Christoffel symbols (2.3) define the other connection in the real space $\mathbb{E}$ (different from the standard Euclidean connection (1.3)) and the other covariant differentiation $\hat{\nabla}$. In terms of $\hat{\nabla}$ the equation (2.1) is written as

$$\hat{\nabla}_p \hat{G}_{qk} = 0. \tag{2.4}$$

The equations like (2.4) are known as concordance conditions for metrics and connections. For instance, $\mathbf{g}$ and ∇ are concordant and (1.3) is derived from the concordance condition $\nabla_p\, g_{qk} = 0$ (see [9]).

Unlike Γ^k_{ij}, the newly introduced Christoffel symbols (2.3) are not symmetric, i. e. $\hat{\Gamma}^k_{ij} \neq \hat{\Gamma}^k_{j\,i}$. Therefore, they define a nonzero torsion:

$$\hat{T}^k_{ij} = \hat{\Gamma}^k_{ij} - \hat{\Gamma}^k_{j\,i}. \tag{2.5}$$

For the *torsion tensor* with the components (2.5) we use the same symbol $\hat{\mathbf{T}}$ as for the distorsion tensor above[1]. However, one should remember that they are two different things: the distorsion tensor $\hat{\mathbf{T}}$ is a double space tensor with two indices, while the torsion tensor $\hat{\mathbf{T}}$ is a purely real space tensor with three indices. Substituting (2.3) into (2.5), for the torsion tensor $\hat{\mathbf{T}}$ we derive:

$$\hat{T}^k_{ij} = \hat{Z}^k_{ij} - \hat{Z}^k_{j\,i} = \hat{R}^k_{ij}. \tag{2.6}$$

Now let's remember that the tensor fields $\mathbf{R}$ and $\hat{\mathbf{R}}$ with the components (1.10) are related to each other by the following equality derived in [7]:

$$\hat{R}^k_{ij} = \sum_{s=1}^{3}\sum_{r=1}^{3} \omega_{sij}\; g^{sr}\, R^k_r. \tag{2.7}$$

Here ω_{sij} are the components of the *volume tensor* $\boldsymbol{\omega}$ (see [8] and [9] for more details). Substituting (2.7) into (2.6), we obtain the equality

$$\hat{T}^k_{ij} = \sum_{s=1}^{3}\sum_{r=1}^{3} \omega_{sij}\; g^{sr}\, R^k_r. \tag{2.8}$$

Theorem 2.1. *The torsion tensor of the connection* (2.3) *is determined by the density of Burgers vector of a dislocated medium through the equality* (2.8).

Apart from the torsion tensor, any connection possesses another tensorial parameter — the *curvature tensor*. It is given by the formula

$$\hat{R}^p_{q\,ij} = \frac{\partial \hat{\Gamma}^p_{j\,q}}{\partial y^i} - \frac{\partial \hat{\Gamma}^p_{i\,q}}{\partial y^j} + \sum_{m=1}^{3} \hat{\Gamma}^m_{j\,q}\,\hat{\Gamma}^p_{i\,m} - \sum_{m=1}^{3} \hat{\Gamma}^m_{i\,q}\,\hat{\Gamma}^p_{j\,m}. \tag{2.9}$$

The standard Euclidean connection (1.3) is a flat connection, this means that its curvature tensor is identically equal to zero:

$$R^p_{q\,ij} = \frac{\partial \Gamma^p_{j\,q}}{\partial y^i} - \frac{\partial \Gamma^p_{i\,q}}{\partial y^j} + \sum_{m=1}^{3} \Gamma^m_{j\,q}\,\Gamma^p_{i\,m} - \sum_{m=1}^{3} \Gamma^m_{i\,q}\,\Gamma^p_{j\,m} = 0. \tag{2.10}$$

Substituting (2.3) into (2.9) and taking into account (2.10), we derive

$$\hat{R}^p_{q\,ij} = \nabla_i \hat{Z}^p_{j\,q} - \nabla_j \hat{Z}^p_{i\,q} + \sum_{m=1}^{3} \hat{Z}^m_{j\,q}\;\hat{Z}^p_{i\,m} - \sum_{m=1}^{3} \hat{Z}^m_{i\,q}\;\hat{Z}^p_{j\,m}. \tag{2.11}$$

[1] The matter is that in geometry the symbol «T» is a typical notation for a torsion.

In order to calculate $\nabla_i \hat{Z}^p_{j\,q}$ and $\nabla_j \hat{Z}^p_{i\,q}$ in (2.11) we use the second formula (1.9):

$$\nabla_i \hat{Z}^p_{j\,q} = \sum_{m=1}^{3} \nabla_i(\hat{S}^p_m\ \nabla_j \hat{T}^m_q) = \sum_{m=1}^{3} \hat{S}^p_m\ \nabla_i \nabla_j \hat{T}^m_q + \sum_{m=1}^{3} \nabla_i \hat{S}^p_m\ \nabla_j \hat{T}^m_q. \tag{2.12}$$

The matrix $\hat{\mathbf{S}}$ is inverse to $\hat{\mathbf{T}}$. Therefore, we have the equality

$$\nabla_i \hat{S}^p_m = -\sum_{n=1}^{3}\sum_{k=1}^{3} \hat{S}^p_n\ \nabla_i \hat{T}^n_k\ \hat{S}^k_m. \tag{2.13}$$

Now from (2.12) and (2.13) for $\nabla_i \hat{Z}^p_{j\,q}$ we derive

$$\nabla_i \hat{Z}^p_{j\,q} = \sum_{m=1}^{3} \hat{S}^p_m\ \nabla_i \nabla_j \hat{T}^m_q - \sum_{m=1}^{3} \hat{Z}^p_{i\,m}\, \hat{Z}^m_{j\,q}, \tag{2.14}$$

and then for $\nabla_j \hat{Z}^p_{i\,q}$ we write by analogy

$$\nabla_j \hat{Z}^p_{i\,q} = \sum_{m=1}^{3} \hat{S}^p_m\ \nabla_j \nabla_i \hat{T}^m_q - \sum_{m=1}^{3} \hat{Z}^p_{j\,m}\, \hat{Z}^m_{i\,q}. \tag{2.15}$$

Due to (2.14) and (2.15) the above expression (2.11) is transformed to

$$\hat{R}^p_{q\,ij} = \sum_{m=1}^{3} \hat{S}^p_m\,(\nabla_i \nabla_j \hat{T}^m_q - \nabla_j \nabla_i \hat{T}^m_q).$$

Due to the symmetry $\Gamma^k_{ij} = \Gamma^k_{j\,i}$ and due to the flatness equality (2.10) the covariant derivatives ∇_i and ∇_j are commutative: $\nabla_i \nabla_j = \nabla_j \nabla_i$. Hence, we have

$$\hat{R}^p_{q\,ij} = 0. \tag{2.16}$$

Theorem 2.2. *Any dislocated crystalline medium is described by a Riemannian metric $\hat{\mathbf{G}}$ (the elastic deformation metric) and by a non-symmetric flat connection $\hat{\mathbf{\Gamma}}$ being concordant with the metric $\hat{\mathbf{G}}$.*

The proof is obvious. Indeed, the formula (1.8) provides a metric and (2.3) provides a connection. Due to (2.7) this connection is non-symmetric. Due to (2.4) this connection is concordant with the metric (1.8) and due to (2.16) it is flat, i. e. its curvature tensor is zero.

The result of the theorem 2.2 is not new. As reported in [10], Kondo, Bilby, Bullough, and Smith (see [11] and [12]) in 1950s recognized that dislocations should be described in terms of the differential geometry. However, their results are not widely known to physicists and engineers.

3. Reconstructing the distorsion.

According to the strategy declared in [7], we are going to replace double space tensors by purely real space tensorial parameters of a medium and write the complete set of the differential equations in terms of these parameters. Suppose for a while that this work is done and suppose that the elastic deformation metric $\hat{\mathbf{G}}$ and the density of the Burgers vector in its real space form $\mathbf{R}$ (see (1.10)) are evaluated for some particular medium in some particular case. Then the torsion tensor $\hat{\mathbf{T}}$ is also known (see formulas (2.7) and (2.8)). The following theorem says that the connection $\hat{\mathbf{\Gamma}}$ can be derived from $\hat{\mathbf{G}}$ and $\mathbf{R}$.

Theorem 3.1. *For any Riemannian metric $\hat{\mathbf{G}}$ and for any tensorial field $\hat{\mathbf{T}}$ of the type $(1,3)$ there exists a unique connection $\hat{\mathbf{\Gamma}}$ concordant with this metric and having $\hat{\mathbf{T}}$ as its torsion tensor.*

This theorem is a well-known geometric result. In symmetric case, i.e. if $\hat{\mathbf{T}}$ is zero, the connection $\hat{\mathbf{\Gamma}}$ is called the *standard metric connection* or the *Levi-Civita connection* of the metric $\hat{\mathbf{G}}$ (see [9]).

Proof. The theorem 3.1 is proved by deriving the explicit formula for the components of $\hat{\mathbf{\Gamma}}$. Let's denote by $\widetilde{\mathbf{\Gamma}}$ the symmetric part of $\hat{\mathbf{\Gamma}}$ and denote by $\widetilde{\mathbf{G}}$ the inverse matrix for the matrix of the metric tensor $\hat{\mathbf{G}}$:

$$\widetilde{\Gamma}^k_{ij} = \frac{\hat{\Gamma}^k_{ij} + \hat{\Gamma}^k_{ji}}{2}, \qquad\qquad \widetilde{G}^{ij} = [\hat{G}^{-1}]^{ij}. \tag{3.1}$$

Traditionally, in differential geometry $\widetilde{\mathbf{G}}$ is called the *inverse* or the *dual* metric tensor for $\hat{\mathbf{G}}$ and is denoted by the same symbol $\hat{\mathbf{G}}$. Here we cannot use such notations since, apart from $\hat{\mathbf{G}}$, we have the basic Euclidean metric $\mathbf{g}$, hence, $\hat{G}^{ij}$ are implicitly determined by the standard index raising procedure:

$$\hat{G}^{ij} = \sum_{i=1}^{3}\sum_{j=1}^{3} g^{ip}\, g^{jq}\, \hat{G}_{pq} \neq \widetilde{G}^{ij}.$$

From (2.5) and from the first equality (3.1) we derive

$$\hat{\Gamma}^k_{ij} = \widetilde{\Gamma}^k_{ij} + \frac{1}{2}\,\hat{T}^k_{ij}. \tag{3.2}$$

Now let's write the concordance condition (2.4) explicitly using (3.2):

$$\frac{\partial \hat{G}_{qk}}{\partial y^p} - \sum_{r=1}^{3} \widetilde{\Gamma}^r_{pq}\, \hat{G}_{rk} - \sum_{r=1}^{3} \widetilde{\Gamma}^r_{pk}\, \hat{G}_{qr} = \frac{1}{2}\sum_{r=1}^{3} \hat{T}^r_{pq}\, \hat{G}_{rk} + \frac{1}{2}\sum_{r=1}^{3} \hat{T}^r_{pk}\, \hat{G}_{qr}. \tag{3.3}$$

Looking at (3.3), we see that it is convenient to denote

$$\widetilde{\Gamma}_{kij} = \sum_{r=1}^{3} \widetilde{\Gamma}^r_{ij}\, \hat{G}_{rk}, \qquad\qquad \widetilde{T}_{kij} = \sum_{r=1}^{3} \hat{T}^r_{ij}\, \hat{G}_{rk}. \tag{3.4}$$

Since $\hat{\mathbf{G}}$ is a non-degenerate matrix, the transformations (3.4) are invertible:

$$\widetilde{\Gamma}^k_{pq} = \sum_{r=1}^{3} \widetilde{\Gamma}_{rpq}\, \widetilde{G}^{rk}, \qquad\qquad \hat{T}^k_{pq} = \sum_{r=1}^{3} \widetilde{T}_{rpq}\, \widetilde{G}^{rk}. \tag{3.5}$$

Applying (3.4) to (3.3), we can rewrite (3.3) in the following form:

$$\widetilde{\Gamma}_{kpq}+\widetilde{\Gamma}_{qpk}=\frac{\partial\hat{G}_{qk}}{\partial y^p}-\frac{1}{2}\,\widetilde{T}_{kpq}-\frac{1}{2}\,\widetilde{T}_{qpk}. \tag{3.6}$$

Performing two cyclic transpositions of indices $p\to q\to k\to p$ in (3.6), we produce the other two equalities from the equality (3.6):

$$\widetilde{\Gamma}_{pqk}+\widetilde{\Gamma}_{kqp}=\frac{\partial\hat{G}_{kp}}{\partial y^q}-\frac{1}{2}\,\widetilde{T}_{pqk}-\frac{1}{2}\,\widetilde{T}_{kqp}, \tag{3.7}$$

$$\widetilde{\Gamma}_{qkp}+\widetilde{\Gamma}_{pkq}=\frac{\partial\hat{G}_{pq}}{\partial y^k}-\frac{1}{2}\,\widetilde{T}_{qkp}-\frac{1}{2}\,\widetilde{T}_{pkq}. \tag{3.8}$$

Now let's add (3.6) and (3.7), then subtract (3.8) from the sum taking into account the symmetry of $\widetilde{\mathbf{\Gamma}}$ and the skew-symmetry of $\widetilde{\mathbf{T}}$:

$$2\,\widetilde{\Gamma}_{kpq}=\left(\frac{\partial\hat{G}_{qk}}{\partial y^p}+\frac{\partial\hat{G}_{kp}}{\partial y^q}-\frac{\partial\hat{G}_{pq}}{\partial y^k}\right)-\widetilde{T}_{pqk}-\widetilde{T}_{qpk}.$$

From this equality, applying (3.5), we derive the following explicit formula for $\widetilde{\Gamma}^k_{pq}$:

$$\widetilde{\Gamma}^k_{pq}=\sum_{r=1}^{3}\frac{\widetilde{G}^{kr}}{2}\left(\frac{\partial\hat{G}_{qr}}{\partial y^p}+\frac{\partial\hat{G}_{rp}}{\partial y^q}-\frac{\partial\hat{G}_{pq}}{\partial y^r}-\widetilde{T}_{pqr}-\widetilde{T}_{qpr}\right). \tag{3.9}$$

And finally, substituting (3.9) into the formula (3.2), we get

$$\begin{aligned}\hat{\Gamma}^k_{ij}=&\sum_{r=1}^{3}\frac{\widetilde{G}^{kr}}{2}\left(\frac{\partial\hat{G}_{jr}}{\partial y^i}+\frac{\partial\hat{G}_{ri}}{\partial y^j}-\frac{\partial\hat{G}_{ij}}{\partial y^r}\right)-\\&-\sum_{r=1}^{3}\sum_{s=1}^{3}\frac{\hat{G}_{is}\,\hat{T}^s_{jr}+\hat{G}_{js}\,\hat{T}^s_{ir}}{2}\,\widetilde{G}^{kr}+\frac{1}{2}\,\hat{T}^k_{ij}\,.\end{aligned} \tag{3.10}$$

This is the explicit formula for the components of the connection declared in the theorem 3.1. Thus, its existence and uniqueness is proved. □

A remark. Because of the formulas (2.3) and (2.8) the equality (3.10) is equivalent to the equality (3.13) from [7]. Thus, the theorem 3.1 yields a geometric interpretation for the formula (3.13) derived in the previous paper [7].

Now let's return to the equations (1.9). Since $\hat{\mathbf{S}}$ and $\hat{\mathbf{T}}$ are inverse to each other ($\hat{\mathbf{S}}=\hat{\mathbf{T}}^{-1}$), the second equation (1.9) is rewritten in the following form:

$$\nabla_p\hat{T}^i_q=\sum_{r=1}^{3}\hat{Z}^{\,r}_{pq}\,\hat{T}^i_r. \tag{3.11}$$

More explicitly this formula is written as

$$\frac{\partial\hat{T}^i_q}{\partial y^p}-\sum_{r=1}^{3}\Gamma^r_{pq}\,\hat{T}^i_r=\sum_{r=1}^{3}\hat{Z}^{\,r}_{pq}\,\hat{T}^i_r. \tag{3.12}$$

Applying (2.3) to (3.12), we can bring the equation (3.12) to the following one:

$$\frac{\partial \hat{T}^i_q}{\partial y^p}=\sum_{r=1}^{3}\hat{\Gamma}^r_{pq}\,\hat{T}^i_r. \tag{3.13}$$

And finally, there is the most simple form of the equality (3.13), where $\hat{\nabla}_p$ is used:

$$\hat{\nabla}_p\hat{T}^i_q=0. \tag{3.14}$$

Note that the upper index i does not affect the expansion of the covariant derivatives ∇_p and $\hat{\nabla}_p$ in (3.11) and (3.14). This is because the distorsion tensor $\hat{\mathbf{T}}$ is a double space tensor and its upper index i is associated with the Burgers space $\mathbb{B}$.

The equations (3.13) form the so-called *complete system of Pfaff equations*. The section 8 in Chapter V of the thesis [13] and the Appendix A in [14] can be used as a brief introduction to the theory of Pfaff systems. All complete Pfaff systems are overdetermined systems of partial differential equations. The compatibility equations form the basic feature of Pfaff systems. In order to derive them for (3.13) one should calculate the second order partial derivatives using (3.13):

$$\frac{\partial \hat{T}^i_r}{\partial y^p\,\partial y^q}=\sum_{s=1}^{3}\frac{\partial\hat{\Gamma}^s_{qr}}{\partial y^p}\,\hat{T}^i_s+\sum_{s=1}^{3}\hat{\Gamma}^s_{qr}\,\frac{\partial \hat{T}^i_s}{\partial y^p}=\sum_{s=1}^{3}\frac{\partial\hat{\Gamma}^s_{qr}}{\partial y^p}\,\hat{T}^i_s+\sum_{s=1}^{3}\sum_{m=1}^{3}\hat{\Gamma}^s_{qr}\,\hat{\Gamma}^m_{ps}\,\hat{T}^i_m,$$

$$\frac{\partial \hat{T}^i_r}{\partial y^p\,\partial y^q}=\sum_{s=1}^{3}\left(\frac{\partial\hat{\Gamma}^s_{qr}}{\partial y^p}+\sum_{m=1}^{3}\hat{\Gamma}^m_{qr}\,\hat{\Gamma}^s_{pm}\right)\hat{T}^i_s. \tag{3.15}$$

Exchanging the indices p and q in (3.15), we derive

$$\frac{\partial \hat{T}^i_r}{\partial y^q\,\partial y^p}=\sum_{s=1}^{3}\left(\frac{\partial\hat{\Gamma}^s_{pr}}{\partial y^q}+\sum_{m=1}^{3}\hat{\Gamma}^m_{pr}\,\hat{\Gamma}^s_{qm}\right)\hat{T}^i_s. \tag{3.16}$$

Subtracting (3.16) from (3.15) and taking into account (2.9), we get

$$\sum_{s=1}^{3}\hat{R}^s_{rpq}\,\hat{T}^i_s=0. \tag{3.17}$$

The equality (3.17) should be fulfilled for any solution of the Pfaff system (3.13). However, in our case we have the stronger result given by the theorem 2.2 and the equality (2.16). Regarding the Pfaff equations the equality (2.16) is called the *compatibility condition* of the Pfaff system.

Let's fix some point P_0 within the continuous medium. Without loss of generality we can assume that its curvilinear coordinates are equal to zero: $y^i(P_0)=0$. Then we choose some constant matrix and denote its components by $\hat{T}^i_s(0)$. The equality

$$\hat{T}^i_s(0,0,0)=\hat{T}^i_s(0).$$

can be understood as the initial value condition for the solution $\hat{T}^i_s(y^1,y^2,y^3)$ of the Pfaff equations (3.13). We prefer to write it in the following form:

$$\hat{T}^i_s\Big|_{P=P_0} = \hat{T}^i_s(0). \tag{3.18}$$

Theorem 3.2. *The initial value problem (3.18) for the system of Pfaff equations (3.13) has a unique local solution*[1] *for any predefined matrix $\hat{T}^i_s(0)$ if and only if the compatibility condition (2.16) is fulfilled.*

The theorem 3.2 is a standard fact of the theory of Pfaff equations. We do not give the proof of this theorem here. The idea for its proof can be found in the section 8 of the Chapter V in [13].

In practice, we need only non-degenerate solutions of the Pfaff equations (3.13), i. e. $\det \hat{\mathbf{T}} \neq 0$. Suppose we have two different solutions of the equations (3.13), we denote them $\hat{\mathbf{T}}[1]$ and $\hat{\mathbf{T}}[2]$. Then their initial values at the point P_0 are related to each other by means of some non-degenerate constant matrix $\mathbf{O}$:

$$\hat{T}^i_s[2](0) = \sum_{j=1}^{3} \hat{T}^j_s[1](0)\ O^i_j. \tag{3.19}$$

Since (3.13) are linear equations and since the theorem 3.2 provides the uniqueness of the solution, the equality (3.19) is fulfilled identically at all points:

$$\hat{T}^i_s[2] = \sum_{j=1}^{3} \hat{T}^j_s[1]\ O^i_j. \tag{3.20}$$

If both solutions $\hat{\mathbf{T}}[1]$ and $\hat{\mathbf{T}}[2]$ correspond to the same elastic deformation tensor $\hat{\mathbf{G}}$, then from (1.8) and (3.20) we derive

$$\overset{\star}{g}_{rs} = \sum_{i=1}^{3}\sum_{j=1}^{3} \overset{\star}{g}_{ij}\ O^i_r\ O^j_s. \tag{3.21}$$

This equality (3.21) means that $\mathbf{O}$ is a constant orthogonal matrix with respect to Euclidean metric in the Burgers space $\mathbb{B}$. Such a matrix corresponds to a global rotation with or without reflection in $\mathbb{B}$. Explaining the concept of the Burgers space in [7], we said that it can be understood as an isometric copy of the real space $\mathbb{E}$. From this point of view the global rotations and reflections are inessential transformations in $\mathbb{B}$. Therefore, up to this inessential uncertainty in defining Burgers vectors, now we have the one-to-one correspondence:

$$\hat{\mathbf{T}} \rightleftarrows \hat{\mathbf{G}}, \mathbf{R}. \tag{3.22}$$

The double space tensorial field of distorsion $\hat{\mathbf{T}}$ defines the pair of purely real space tensorial field: the elastic deformation tensor $\hat{\mathbf{G}}$ and the tensor $\mathbf{R}$, which is the real space representation of the Burgers vector density $\boldsymbol{\rho}$ (see formula (1.10) above). For the sake of brevity, from now on, we shall call $\mathbf{R}$ the *Burgers vector density* as

[1] This means the solution in some neighborhood of the initial point P_0.

well. The tensor $\hat{\mathbf{G}}$ is derived from the distorsion $\hat{\mathbf{T}}$ by means of the formula (1.8). The formula for tensor $\mathbf{R}$ is more complicated:

$$R^r_k = \sum_{i=1}^{3}\sum_{s=1}^{3}\sum_{p=1}^{3}\sum_{q=1}^{3} \hat{S}^r_i\, g_{sk}\ \omega^{spq}\ \nabla_p \hat{T}^i_q. \tag{3.23}$$

The formulas (1.8) and (3.23) correspond to the upper right arrow in (3.22). The lower left arrow in (3.22) goes through formula (2.8), through the theorem 3.1 provided by the formula (3.10), and through solving the system of Pfaff equations (3.13). As a result we recover the incompatible distorsion tensor $\hat{\mathbf{T}}$ from $\hat{\mathbf{G}}$ and $\mathbf{R}$.

A remark. Being produced from the single field $\hat{\mathbf{T}}$, the tensor field $\hat{\mathbf{G}}$ and $\mathbf{R}$ are not absolutely independent. They are related to each other through the zero-curvature condition (2.16).

4. Time evolution and consistence of the kinematic equations in whole.

The time evolution of the elastic deformation tensor $\hat{\mathbf{G}}$ is given by the equation (1.1). The time evolution for the tensor $\mathbf{R}$ is given by the equation (4.11) from [7]:

$$\begin{gathered}\frac{\partial R^k_q}{\partial t} - \sum_{p=1}^{3} J^{\,k}_p\, R^p_q - \sum_{m=1}^{3} \nabla_m v^k\, R^m_q - \sum_{m=1}^{3}\sum_{p=1}^{3} v^p\, \hat{Z}^k_{mp}\, R^m_q\, + \\ + \sum_{m=1}^{3}\sum_{r=1}^{3}\sum_{s=1}^{3} g_{qm}\,\omega^{mrs}\,\nabla_r J^{\,k}_s + \sum_{m=1}^{3}\sum_{r=1}^{3}\sum_{s=1}^{3}\sum_{p=1}^{3} g_{qm}\,\omega^{mrs}\,\hat{Z}^k_{rp}\, J^{\,p}_s = 0.\end{gathered} \tag{4.1}$$

Now let's remember the second relationship (1.9) and apply it to (3.23). Then

$$R^p_q = \sum_{m=1}^{3}\sum_{r=1}^{3}\sum_{s=1}^{3} g_{qm}\ \omega^{mrs}\ \hat{Z}^p_{rs}. \tag{4.2}$$

Substituting (4.2) into the second term of (4.1) and rearranging the terms, we get

$$\begin{gathered}\frac{\partial R^k_q}{\partial t} - \sum_{m=1}^{3}\left(\nabla_m v^k + \sum_{p=1}^{3} \hat{Z}^k_{mp}\, v^p\right) R^m_q\, + \\ + \sum_{m=1}^{3}\sum_{r=1}^{3}\sum_{s=1}^{3} g_{qm}\,\omega^{mrs}\left(\nabla_r J^{\,k}_s + \sum_{p=1}^{3} \hat{Z}^k_{rp}\, J^{\,p}_s - \hat{Z}^p_{rs}\, J^{\,k}_p\right) = 0.\end{gathered} \tag{4.3}$$

Then we remember the relationship (2.3) and replace ∇ by $\hat{\nabla}$ in (4.3):

$$\frac{\partial R^k_q}{\partial t} - \sum_{m=1}^{3} \hat{\nabla}_m v^k\, R^m_q + \sum_{m=1}^{3}\sum_{r=1}^{3}\sum_{s=1}^{3} g_{qm}\,\omega^{mrs}\,\hat{\nabla}_r J^{\,k}_s = 0. \tag{4.4}$$

In a similar way let's replace ∇ by $\hat{\nabla}$ in the evolution equation (1.1) for the elastic deformation tensor $\hat{\mathbf{G}}$. Applying (2.3) to (1.1) and using (2.4), we derive

$$\begin{gathered}\frac{\partial\hat{G}_{kq}}{\partial t}+\sum_{r=1}^{3}\hat{\nabla}_k v^r\,\hat{G}_{rq}+\sum_{r=1}^{3}\sum_{p=1}^{3}v^r\left(\hat{Z}^p_{rk}-\hat{Z}^p_{k\,r}\right)\hat{G}_{pq}+\sum_{r=1}^{3}\hat{\nabla}_q v^r\,\hat{G}_{kr}+\\+\sum_{r=1}^{3}\sum_{p=1}^{3}v^r\left(\hat{Z}^p_{rq}-\hat{Z}^p_{qr}\right)\hat{G}_{kp}=\sum_{r=1}^{3}\theta^r_k\,\hat{G}_{rq}+\sum_{r=1}^{3}\hat{G}_{kr}\,\theta^r_q.\end{gathered}\tag{4.5}$$

Now let's compare (1.9) with (1.4). As a result we write (1.4) as

$$\theta^r_q=-J^r_q+\sum_{p=1}^{3}v^p\left(\hat{Z}^r_{p\,q}-\hat{Z}^r_{qp}\right).\tag{4.6}$$

Applying (4.6) to (4.5), we transform (4.5) to the following equation:

$$\begin{gathered}\frac{\partial\hat{G}_{kq}}{\partial t}+\sum_{r=1}^{3}\hat{\nabla}_k v^r\,\hat{G}_{rq}+\sum_{r=1}^{3}\hat{\nabla}_q v^r\,\hat{G}_{kr}+\\+\sum_{r=1}^{3}J^r_k\,\hat{G}_{rq}+\sum_{r=1}^{3}\hat{G}_{kr}\,J^r_q=0.\end{gathered}\tag{4.7}$$

The differential equations (4.4) and (4.7) describe the time evolution of the tensor fields $\hat{\mathbf{G}}$ and $\mathbf{R}$. The next step is to show that this time evolution is compatible with the zero-curvature condition (2.16). The following commutation relationships are derived from the definition of covariant derivatives by direct calculations:

$$[\partial_t,\hat{\nabla}_p]\,X^k=\sum_{s=1}^{3}\frac{\partial\hat{\Gamma}^k_{ps}}{\partial t}\,X^s,\tag{4.8}$$

$$[\partial_t,\hat{\nabla}_p]\,X_k=-\sum_{s=1}^{3}\frac{\partial\hat{\Gamma}^s_{pk}}{\partial t}\,X_s.\tag{4.9}$$

Here $\partial_t=\partial/\partial t$, while X^k and X_k stand for the components of arbitrary vectorial and covectorial fields respectively. In order to calculate the time derivatives of the connection components in (4.8) and (4.9) we use the equality (2.4) rewritten as

$$\frac{\partial\hat{G}_{qk}}{\partial y^p}-\sum_{s=1}^{3}\hat{\Gamma}^s_{p\,q}\,\hat{G}_{sk}-\sum_{s=1}^{3}\hat{\Gamma}^s_{p\,k}\,\hat{G}_{qs}=0.\tag{4.10}$$

For the sake of brevity in the further calculations we denote

$$\frac{\partial\hat{\Gamma}^k_{rs}}{\partial t}=\dot{\hat{\Gamma}}^k_{rs},\qquad\qquad\frac{\partial\hat{G}rs}{\partial t}=\dot{\hat{G}}_{rs},\qquad\qquad\frac{\partial\hat{T}^k_{rs}}{\partial t}=\dot{\hat{T}}^k_{rs},$$

where $\hat{T}^k_{rs}$ are the components of the torsion tensor $\hat{\mathbf{T}}$ related to $\mathbf{R}$ through (2.8). Now differentiating (4.10) with respect to t, we find

$$\sum_{s=1}^{3}\dot{\hat{\Gamma}}^s_{p\,q}\,\hat{G}_{sk}+\sum_{s=1}^{3}\dot{\hat{\Gamma}}^s_{p\,k}\,\hat{G}_{qs}=\hat{\nabla}_p(\dot{\hat{G}}_{qk}).\tag{4.11}$$

The equality (4.11) is similar to (3.3). Therefore, we shall treat it similarly. Let's apply (3.1) and (3.2) to (4.11). As a result we get

$$\begin{gathered}\sum_{s=1}^{3}\dot{\hat{\Gamma}}^{s}_{p\,q}\,\hat{G}_{sk}+\sum_{s=1}^{3}\dot{\hat{\Gamma}}^{s}_{p\,k}\,\hat{G}_{qs}=\\ =\hat{\nabla}_p(\dot{\hat{G}}_{qk})-\frac{1}{2}\sum_{s=1}^{3}\dot{\hat{T}}^{s}_{p\,q}\,\hat{G}_{sk}-\frac{1}{2}\sum_{s=1}^{3}\dot{\hat{T}}^{s}_{p\,k}\,\hat{G}_{qs}.\end{gathered}\tag{4.12}$$

Now we introduce the following notations similar to (3.4):

$$\check{\Gamma}_{k\,ij}=\sum_{r=1}^{3}\dot{\hat{\Gamma}}^{r}_{ij}\,\hat{G}_{rk},\qquad\qquad \check{T}_{k\,ij}=\sum_{r=1}^{3}\dot{\hat{T}}^{r}_{ij}\,\hat{G}_{rk}.\tag{4.13}$$

Applying (4.13) to (4.12), we strengthen the resemblance of (4.12) and (3.3):

$$\check{\Gamma}_{kpq}+\check{\Gamma}_{qpk}=\hat{\nabla}_p(\dot{\hat{G}}_{qk})-\frac{1}{2}\,\check{T}_{kpq}-\frac{1}{2}\,\check{T}_{qpk}.\tag{4.14}$$

Note that (4.14) looks pretty like the equality (3.6). Therefore, we can use the same arguments as in proving the theorem 3.1 and derive the following formula:

$$\begin{gathered}\dot{\hat{\Gamma}}^{k}_{ij}=\sum_{r=1}^{3}\frac{\widetilde{G}^{kr}}{2}\Bigl(\hat{\nabla}_i(\dot{\hat{G}}_{jr})+\hat{\nabla}_j(\dot{\hat{G}}_{ri})-\hat{\nabla}_r(\dot{\hat{G}}_{ij})\Bigr)-\\ -\sum_{r=1}^{3}\sum_{s=1}^{3}\frac{\hat{G}_{is}\,\dot{\hat{T}}^{s}_{jr}+\hat{G}_{js}\,\dot{\hat{T}}^{s}_{ir}}{2}\,\widetilde{G}^{kr}+\frac{1}{2}\,\dot{\hat{T}}^{k}_{ij}\,.\end{gathered}\tag{4.15}$$

The covariant derivatives $\hat{\nabla}_i(\dot{\hat{G}}_{jr})$, $\hat{\nabla}_j(\dot{\hat{G}}_{ri})$, and $\hat{\nabla}_r(\dot{\hat{G}}_{ij})$ are given by the formula (4.7). Applying this formula, we get

$$\begin{gathered}\hat{\nabla}_i(\dot{\hat{G}}_{jr})+\hat{\nabla}_j(\dot{\hat{G}}_{ri})+\hat{\nabla}_r(\dot{\hat{G}}_{ij})=-\sum_{m=1}^{3}\Bigl(\hat{\nabla}_i\hat{\nabla}_j\,v^m+\\ +\hat{\nabla}_j\hat{\nabla}_i\,v^m+\hat{\nabla}_iJ^{\,m}_j+\hat{\nabla}_jJ^{\,m}_i\Bigr)\,\hat{G}_{mr}+\sum_{m=1}^{3}\Bigl([\hat{\nabla}_r,\,\hat{\nabla}_i]v^m+\hat{\nabla}_rJ^{\,m}_i-\\ -\hat{\nabla}_iJ^{\,m}_r\Bigr)\,\hat{G}_{mj}+\sum_{m=1}^{3}\Bigl([\hat{\nabla}_r,\,\hat{\nabla}_j]v^m+\hat{\nabla}_rJ^{\,m}_j-\hat{\nabla}_jJ^{\,m}_r\Bigr)\,\hat{G}_{mi}.\end{gathered}\tag{4.16}$$

As for the time derivatives $\dot{\hat{T}}^{s}_{jr}$, $\dot{\hat{T}}^{s}_{ir}$, and $\dot{\hat{T}}^{k}_{ij}$, we use the formula (2.8) differentiating it with respect to t. As a result, applying (4.4), we obtain

$$\dot{\hat{T}}^{k}_{ij}=\sum_{r=1}^{3}\sum_{s=1}^{3}\omega_{sij}\,\,g^{sr}\left(\sum_{n=1}^{3}\hat{\nabla}_nv^k\,R^n_r-\sum_{n=1}^{3}\sum_{\alpha=1}^{3}\sum_{\beta=1}^{3}g_{rn}\,\,\omega^{n\alpha\beta}\,\,\hat{\nabla}_\alpha J^{\,k}_\beta\right).$$

The above huge formula can be simplified to the following one:

$$\dot{\hat{T}}^k_{ij} = \hat{\nabla}_j J^k_i - \hat{\nabla}_i J^k_j + \sum_{n=1}^{3} \hat{\nabla}_n v^k \, \hat{T}^n_{ij}. \tag{4.17}$$

From (4.17) we easily derive the following two equalities:

$$\sum_{s=1}^{3} \hat{G}_{is} \, \dot{\hat{T}}^s_{jr} = \sum_{m=1}^{3} \hat{G}_{im} \left(\hat{\nabla}_r J^m_j - \hat{\nabla}_j J^m_r + \sum_{n=1}^{3} \hat{\nabla}_n v^m \, \hat{T}^n_{jr} \right), \tag{4.18}$$

$$\sum_{s=1}^{3} \hat{G}_{js} \, \dot{\hat{T}}^s_{ir} = \sum_{m=1}^{3} \hat{G}_{jm} \left(\hat{\nabla}_r J^m_i - \hat{\nabla}_i J^m_r + \sum_{n=1}^{3} \hat{\nabla}_n v^m \, \hat{T}^n_{ir} \right). \tag{4.19}$$

Now we apply (4.16), (4.17), (4.18), and (4.19) to (4.15). As a result we obtain

$$\begin{aligned} \dot{\hat{\Gamma}}^k_{ij} = &-\frac{\hat{\nabla}_i \hat{\nabla}_j \, v^k + \hat{\nabla}_j \hat{\nabla}_i \, v^k}{2} - \hat{\nabla}_i J^k_j + \frac{1}{2} \sum_{n=1}^{3} \hat{\nabla}_n v^k \, \hat{T}^n_{ij} + \\ &+ \sum_{m=1}^{3} \sum_{r=1}^{3} \frac{\widetilde{G}^{kr}}{2} \left([\hat{\nabla}_r, \, \hat{\nabla}_i] v^m - \sum_{n=1}^{3} \hat{\nabla}_n v^m \, \hat{T}^n_{ir} \right) \hat{G}_{mj} + \\ &+ \sum_{m=1}^{3} \sum_{r=1}^{3} \frac{\widetilde{G}^{kr}}{2} \left([\hat{\nabla}_r, \, \hat{\nabla}_j] v^m - \sum_{n=1}^{3} \hat{\nabla}_n v^m \, \hat{T}^n_{jr} \right) \hat{G}_{mi}. \end{aligned} \tag{4.20}$$

The commutators $[\hat{\nabla}_r, \, \hat{\nabla}_i]$ and $[\hat{\nabla}_r, \, \hat{\nabla}_j]$ in the above formula (4.20) can be calculated on the base of the well-known differential-geometric formula

$$[\hat{\nabla}_i, \, \hat{\nabla}_j] X^k = \sum_{n=1}^{3} \hat{R}^k_{nij} \; X^n - \sum_{n=1}^{3} \hat{T}^n_{ij} \; \hat{\nabla}_n X^k. \tag{4.21}$$

Here X^k and X^n stand for the components of an arbitrary vector field, while $\hat{R}^k_{nij}$ are the components of the curvature tensor given by the formula (2.9). Choosing $\mathbf{X} = \mathbf{v}$ in (4.21) and substituting (4.21) into (4.20), we derive

$$\begin{aligned} \dot{\hat{\Gamma}}^k_{ij} = &-\frac{\hat{\nabla}_i \hat{\nabla}_j \, v^k + \hat{\nabla}_j \hat{\nabla}_i \, v^k}{2} - \hat{\nabla}_i J^k_j + \frac{1}{2} \sum_{n=1}^{3} \hat{\nabla}_n v^k \, \hat{T}^n_{ij} + \\ &+ \sum_{m=1}^{3} \sum_{r=1}^{3} \sum_{n=1}^{3} \left(\frac{\widetilde{G}^{kr} \, \hat{R}^m_{nri} \, v^n \, \hat{G}_{mj}}{2} + \frac{\widetilde{G}^{kr} \, \hat{R}^m_{nrj} \, v^n \, \hat{G}_{mi}}{2} \right). \end{aligned} \tag{4.22}$$

Now we are ready to calculate the time derivative of the curvature tensor $\hat{\mathbf{R}}$ due to the time evolution of tensor fields $\hat{\mathbf{G}}$ and $\mathbf{R}$ given by the equations (4.4) and (4.7). Differentiating (2.9) with respect to t, we get

$$\frac{\partial \hat{R}^k_{q\,ij}}{\partial t} = \hat{\nabla}_i \dot{\hat{\Gamma}}^k_{j\,q} - \hat{\nabla}_j \dot{\hat{\Gamma}}^k_{i\,q} + \sum_{m=1}^{3} \hat{T}^m_{ij} \, \dot{\hat{\Gamma}}^k_{m\,q}. \tag{4.23}$$

Before substituting (4.22) into (4.23) we transform it using the identity (4.21) again:

$$\hat{\nabla}_j \hat{\nabla}_i\, v^k = \hat{\nabla}_i \hat{\nabla}_j\, v^k - \sum_{n=1}^{3} \hat{R}^k_{nij}\, v^n + \sum_{n=1}^{3} \hat{T}^n_{ij}\, \hat{\nabla}_n v^k.$$

Due to this identity, the equality (4.22) now looks like

$$\begin{gathered}\dot{\hat{\Gamma}}{}^k_{ij} = -\hat{\nabla}_i \hat{\nabla}_j\, v^k - \hat{\nabla}_i J^k_j + \frac{1}{2} \sum_{n=1}^{3} \hat{R}^k_{nij}\, v^n + \\ + \sum_{m=1}^{3} \sum_{r=1}^{3} \sum_{n=1}^{3} \left(\frac{\widetilde{G}^{kr}\, \hat{R}^m_{nri}\, v^n\, \hat{G}_{mj}}{2} + \frac{\widetilde{G}^{kr}\, \hat{R}^m_{nrj}\, v^n\, \hat{G}_{mi}}{2} \right).\end{gathered}$$

Continuing our calculations, below we shall omit the terms containing the components of the curvature tensor and their spatial derivatives denoting them by dots:

$$\dot{\hat{\Gamma}}{}^k_{ij} = -\hat{\nabla}_i \hat{\nabla}_j\, v^k - \hat{\nabla}_i J^k_j + \ldots\ . \tag{4.24}$$

Then, substituting (4.24) into (4.23), we derive:

$$\begin{gathered}\frac{\partial \hat{R}^k_{q\,ij}}{\partial t} = -\hat{\nabla}_i \hat{\nabla}_j \hat{\nabla}_q\, v^k - \hat{\nabla}_i \hat{\nabla}_j J^k_q + \hat{\nabla}_j \hat{\nabla}_i \hat{\nabla}_q\, v^k + \\ + \hat{\nabla}_j \hat{\nabla}_i J^k_q - \sum_{m=1}^{3} \hat{T}^m_{ij}\, \hat{\nabla}_m \hat{\nabla}_q\, v^k - \sum_{m=1}^{3} \hat{T}^m_{ij}\, \hat{\nabla}_m J^k_q + \ldots\ .\end{gathered} \tag{4.25}$$

In order to simplify this formula let's write (4.25) as follows:

$$\begin{aligned}\frac{\partial \hat{R}^k_{q\,ij}}{\partial t} = &- [\hat{\nabla}_i,\, \hat{\nabla}_j] \left(\hat{\nabla}_q\, v^k + J^k_q \right) - \\ &- \sum_{m=1}^{3} \hat{T}^m_{ij}\, \nabla_m \left(\hat{\nabla}_q\, v^k + J^k_q \right) + \ldots\ .\end{aligned} \tag{4.26}$$

The further transformation of (4.26) is based on the other well-known formula of the differential geometry, which is similar to (4.21):

$$[\hat{\nabla}_i,\, \hat{\nabla}_j] X^k_q = \sum_{n=1}^{3} \hat{R}^k_{nij}\ X^n_q - \sum_{n=1}^{3} \hat{R}^n_{qij}\ X^k_n - \sum_{n=1}^{3} \hat{T}^n_{ij}\ \hat{\nabla}_n X^k_q. \tag{4.27}$$

Here X^k_q stand for the components of an arbitrary tensorial field of the type $(1,1)$. Substituting $X^k_q = \hat{\nabla}_q\, v^k + J^k_q$ into (4.27), we see that (4.26) can be written as

$$\frac{\partial \hat{R}^k_{q\,ij}}{\partial t} = -\sum_{n=1}^{3} \hat{R}^k_{nij} \left(\hat{\nabla}_q\, v^n + J^n_q \right) + \sum_{n=1}^{3} \hat{R}^n_{qij} \left(\hat{\nabla}_n\, v^k + J^k_n \right) + \ldots\ . \tag{4.28}$$

Looking at (4.28), we see that two terms explicitly written in the right hand side of this formula contain the components of the curvature tensor $\hat{\mathbf{R}}$. According to the above our convention, they should also be denoted by dots. This means that all terms in the right hand side of (4.28) would vanish provided the zero-curvature condition (2.16) is fulfilled.

Theorem 4.1. *The time evolution of the tensor fields* $\hat{\mathbf{G}}$ *and* $\mathbf{R}$ *given by the equations* (4.4) *and* (4.7) *preserves the zero-curvature condition* (2.16)*, i. e. the curvature tensor* $\hat{\mathbf{R}}$ *is permanently equal to zero if* $\hat{\mathbf{R}} = 0$ *at some initial instant of time.*

Passing from $\hat{\mathbf{\Gamma}}$ back to the standard Euclidean connection Γ in the real space $\mathbb{E}$, and hence, from $\hat{\nabla}$ back to ∇, we can formulate the theorem 4.1 as follows.

Theorem 4.2. *The time evolution of the tensor fields* $\hat{\mathbf{G}}$ *and* $\mathbf{R}$ *given by the equations* (1.1) *and* (4.1) *preserves the zero-curvature condition* (2.16)*, i. e. the curvature tensor* $\hat{\mathbf{R}}$ *is permanently equal to zero if* $\hat{\mathbf{R}} = 0$ *at some initial instant of time.*

5. Zero-divergency condition.

The zero-curvature condition (2.16) is not the only condition the tensor fields $\hat{\mathbf{G}}$ and $\mathbf{R}$ should obey. Another condition was derived in [7] from the equality $\operatorname{div}\boldsymbol{\rho} = 0$. Therefore, we call it the *zero-divergency condition*:

$$\sum_{p=1}^{3}\sum_{q=1}^{3} g^{p\,q}\,\nabla_{p} R^{k}_{q} + \sum_{m=1}^{3}\sum_{p=1}^{3}\sum_{q=1}^{3} g^{pq}\,R^{m}_{q}\,\hat{Z}^{k}_{pm} = 0 \tag{5.1}$$

(see (4.8) in [7]). The following theorem shows that (5.1) is not an independent condition for $\hat{\mathbf{G}}$ and $\mathbf{R}$.

Theorem 5.1. *The zero-divergency condition* (5.1) *can be derived from the zero-curvature condition* (2.16)*.*

Formulating the theorem 5.1, we assume that the tensor fields $\hat{\mathbf{G}}$ and $\mathbf{R}$ are given and that the following conditions are fulfilled:

(1) $\mathbf{R}$ define the torsion tensor $\hat{\mathbf{T}}$ according to the equality (2.8);
(2) $\hat{\mathbf{G}}$ and $\mathbf{R}$ define the non-symmetric connection $\hat{\mathbf{\Gamma}}$ with the torsion $\hat{\mathbf{T}}$ according to the theorem 3.1;
(3) the curvature tensor $\hat{\mathbf{R}}$ of the connection $\hat{\mathbf{\Gamma}}$ is equal to zero;
(4) the tensor field $\hat{\mathbf{Z}}$ is determined as the difference of the connection $\hat{\mathbf{\Gamma}}$ and the standard symmetric Euclidean connection $\mathbf{\Gamma}$ according to the formula (2.3).

From the above four conditions (1)–(4) one easily derives the equality

$$R^{k}_{q} = \sum_{n=1}^{3}\sum_{r=1}^{3}\sum_{s=1}^{3} g_{qn}\;\omega^{nrs}\;\hat{Z}^{k}_{rs}. \tag{5.2}$$

and the formula (2.11) for the components of the curvature tensor $\hat{\mathbf{R}}$. The zero-curvature condition $\hat{\mathbf{R}} = 0$ then is equivalent to the symmetry condition

$$U^{k}_{sp\,r} = U^{k}_{s\,rp}, \tag{5.3}$$

where the quantities $U^k_{s\,p\,r}$ are given by the formula

$$U^k_{s\,p\,r} = \nabla_p \hat{Z}^k_{r\,s} + \sum_{m=1}^{3} \hat{Z}^m_{r\,s}\ \hat{Z}^k_{p\,m}. \tag{5.4}$$

Due to the symmetry condition (5.3) the following sum is obviously equal to zero:

$$\sum_{p=1}^{3}\sum_{r=1}^{3}\sum_{s=1}^{3} \omega^{p\,r\,s}\, U^k_{s\,p\,r} = 0. \tag{5.5}$$

Indeed, $U^k_{s\,p\,r}$ is symmetric, while $\omega^{p\,r\,s}$ is skew-symmetric with respect to p and r. The next transformation of the identity (5.5) is also quite obvious:

$$\sum_{q=1}^{3}\sum_{n=1}^{3}\sum_{p=1}^{3}\sum_{r=1}^{3}\sum_{s=1}^{3} g^{p\,q}\, g_{qn}\, \omega^{n\,r\,s}\, U^k_{s\,p\,r} = 0. \tag{5.6}$$

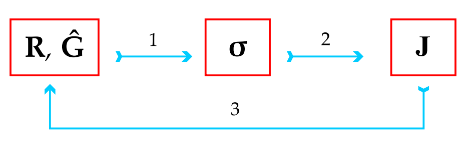
Now it is sufficient to substitute (5.4) into (5.6). As a result we obtain

$$\begin{gathered}
\sum_{p=1}^{3}\sum_{q=1}^{3} g^{p\,q} \sum_{n=1}^{3}\sum_{r=1}^{3}\sum_{s=1}^{3} g_{qn}\, \omega^{n\,r\,s}\, \nabla_p \hat{Z}^k_{r\,s} + \\
+\sum_{p=1}^{3}\sum_{q=1}^{3} g^{p\,q} \sum_{m=1}^{3} \left(\sum_{n=1}^{3}\sum_{r=1}^{3}\sum_{s=1}^{3} g_{qn}\, \omega^{n\,r\,s}\, \hat{Z}^m_{r\,s}\right) \hat{Z}^k_{p\,m} = 0.
\end{gathered} \tag{5.7}$$

Applying (5.2) to (5.7), we find that the equality (5.7) is equivalent to the zero-divergency condition (5.1). Thus, the theorem 5.1 is proved.

6. Conclusions.

As a main result we can formulate the following statement: the elastic deformation tensor $\hat{\mathbf{G}}$ and the tensor of Burgers vector density $\mathbf{R}$ are two basic tensor fields describing completely the deformation state of a crystal. The time evolution of $\hat{\mathbf{G}}$ and $\mathbf{R}$ is determined by the differential equations (4.4) and (4.7)[1]. However, (4.4) and (4.7) do not form the closed system of differential equations — they are only the kinematic equations. They should be completed with the dynamic equations relating $\mathbf{J}$ (the density of the Burgers vector flow) to $\hat{\mathbf{G}}$ and $\mathbf{R}$. Qualitatively, the process of a crystal deformation is expressed by the following diagram:



The elastic deformation $\hat{\mathbf{G}}$ produces the stress $\boldsymbol{\sigma}$, this phenomenon is expressed by the arrow 1 (Hooke's law or its nonlinear generalization). The stress $\boldsymbol{\sigma}$ causes the dislocations to move producing their flow $\mathbf{J}$, see the arrow 2 on the above diagram.

[1] Or by the equations (4.1) and (1.1), which are equivalent to (4.4) and (4.7).

And finally, the moving dislocations rearrange the interatomic bonds causing $\hat{\mathbf{G}}$ and the stress $\boldsymbol{\sigma}$ to relax. This phenomenon is expressed by the arrow 3 and described by the equations (4.4) and (4.7). The plastic deformation tensor $\check{\mathbf{G}}$ is not presented on the diagram. However, it can be implicitly present in the arrows 1 and 2. The detailed quantitative description of the phenomena associated with these arrows is the subject of the separate paper.

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