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SOLITONS ON A FINITE-GAP BACKGROUND IN BULLOUGH-DODD-JIBER-SHABAT MODEL

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ABSTRACT. The determinant formula for N-soliton solutions of the Bullough-Dodd-Jiber-Shabat equation on a finite-gap background is obtained. Nonsingularity conditions for them and their asymptotics are investigated.

1. Introduction

Found in the early works on an inverse scattering method [1, 2] the Bullough-Dodd-Jiber-Shabat equation[1]

$$u_{xt} = e^{u} - e^{-2u} \tag{1.1}$$

makes a good alternative to the well-known Sine-Gordon equation as one more model of the two-dimensional integrable relativistic field theory with the self-action. In [3] the authors had suggested the construction of the finite-gap solutions for this equation. These are periodic and/or almost periodic function in x, t associated with special classes of double sheet ramified coverings of Riemann surfaces and represented by explicit formulae in terms of Prym theta-functions of such coverings. In this paper we construct Hirota type (see [4]) determinant formula for N-soliton solutions of (1.1) and use it for the investigation of the space-time asymptotics of them in space-time variables where (1.1) has a form

$$u_{tt} - u_{xx} = e^{u} - e^{-2u}. \tag{1.2}$$

It is known that Eqs. (1.1) and (1.2) have no fast-decreasing soliton solutions being the same time real-valued and nonsingular. The presence of finite-gap background, as shown below, makes it possible for nonsingular solitons to decrease rapidly to that background. This has a transparent explanation in connection with the topology of appropriate Riemann surface.

2. Main Construction

First we touch briefly on the construction of finite-gap solutions of (1.1) being the background in the following considerations (for more details, see [3]). Equation

[1] S. P. Tsarev discovered that this equation was first found by Tzitzeica in [12]. Now it is called Tzitzeica equation. See also [13] for more details.

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(1.1) has "zero curvature" representation $[\partial_x - L,\, \partial_t - A] = 0$ found in [5] with the matrix Lax operators of the form

$$L = \begin{Vmatrix} -u_x & 0 & \lambda \\ 1 & u_x & 0 \\ 0 & 1 & 0 \end{Vmatrix}, \qquad A = \begin{Vmatrix} 0 & e^{-2u} & 0 \\ 0 & 0 & e^u \\ \lambda^{-1}\, e^u & 0 & 0 \end{Vmatrix}. \tag{2.1}$$

According to the general scheme of finite-gap integration (see review [6]) one should find the Baker-Achiezer vector function $\psi = e(x,t,P)$ solving the spectral equations

$$\psi_x = L\psi, \qquad \psi_t = A\psi \tag{2.2}$$

and depending on a point P of some compact Riemann surface Γ. As soon as matrices A and L are not of general form, the so-called "reduction problem" arises. In [3] we managed to solve it in our particular case in a way similar to that of [7] and [8] where the reduction of the general Schrodinger operator in magnetic field to the purely potential Schrodinger operator was found. Let Γ be the compact Riemann surface of the even genus g with the meromorphic function $\lambda(P)$ having the only pole P_∞ of the third order and the only zero P_0 of the third order. In addition, let us suppose that Γ admits a holomorphic involution σ with the property $\lambda(\sigma P) = -\lambda(P)$ and the antiholomorphic involution τ with property $\lambda(\tau P) = \overline{\lambda(P)}$. For this case, we can choose the local parameters $k^{-1}(P)$ and $q^{-1}(P)$ in the neighborhoods of marked points P_∞ and P_0 defined by the following conditions $k^3 = \lambda$, $q^{-3} = \lambda$, $k(\tau P) = \overline{k(P)}$, $q(\tau P) = \overline{q(P)}$. Baker-Achiezer functions satisfying (2.1) then are defined by fixing their essential singularities at P_∞ and P_0

$$\begin{Vmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{Vmatrix} \sim \begin{Vmatrix} k^{-1} \\ k^{-2} \\ k^{-3} \end{Vmatrix} \cdot e^{kx}, \qquad \begin{Vmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{Vmatrix} \sim \begin{Vmatrix} q\, e^{-u} \\ q^2\, e^u \\ q^3 \end{Vmatrix} \cdot e^{qt} \tag{2.3}$$

and by fixing the divisor D of their poles. Divisor D of degree g should satisfy the following limitations

$$D + \sigma D - P_0 - P_\infty \sim C, \qquad \tau D = D, \tag{2.4}$$

where C is a divisor of canonical class on Γ. The above limitations imposed on the choice of Γ and D make sure that the matrices L and A are of the form (2.1) and enable us to find the explicit formula for appropriate finite-gap solution $u = v(x,t)$ of Eq. (1.1) in terms of the Prym theta-function $\mu(z)$ of the covering r $\Gamma \to \Gamma/\sigma$:

$$e^{v(x,t)} = c(\Gamma) - 2\,\partial_{xt} \ln \mu(Ux + Vt + z_D). \tag{2.5}$$

Vector z_D is determined by the divisor D, constant $c(\Gamma)$ depends only on Riemann surface Γ. Vectors U and V are determined by the normalized Abelian differentials of the second kind Ω_∞ and Ω_0, the main parts of them at P_∞ and P_0 are of the form $\Omega_\infty = dk + \ldots$ and $\Omega_0 = dq + \ldots$ respectively.

Let Γ be the M-curve with respect to anti-involution τ (see [9]). In this case condition (2.4) is compatible with the choice of D such that each invariant cycle of τ contains only one point of D except the cycle containing P_∞ and P_0 where there is no point of divisor D. It is the very choice which provides the real and nonsingular finite-gap solutions of Eq. (1.1) in (2.5).

In order to construct the N-soliton solutions of (1.1) let us fix an extra set of parameters consisting of

(a) a set of numbers $\lambda_1, \dots, \lambda_N$ such that $\lambda_i^2 \neq \lambda_j^2$ for $i \neq j$;
(b) a set of nonzero constants $C_1, \dots, C_N$.

For general λ_i, each equation $\lambda(P) = \lambda_i$, has exactly three solutions. Let us denote two of them as Λ_i, and $\sigma\Lambda_i^*$, and then define the new N-soliton Baker-Achiezer function $\Psi(x,t,P)$ by the following requirements:

(A) $\Psi(x,t,P)$ is analytic everywhere on Γ except at P_∞, P_0 and at points of the divisor $D + \Lambda_1 + \ldots + \Lambda_n + \Lambda_1^* + \ldots + \Lambda_N^*$;
(B) it has essential singularities at P_∞ and P_0 of the same form (2.3) as a purely finite-gap Baker-Achiezer function;
(C) divisor $\mathcal{D} = D + \Lambda_1 + \ldots + \Lambda_n + \Lambda_1^* + \ldots + \Lambda_N^*$ is its divisor of poles and

$$\begin{aligned}
&\operatorname*{Res}_{P=\Lambda_j} \big(3\,\Psi_i(P)\,\lambda^2(P)\,\omega(P)\big) = C_j \cdot \Psi_i(\sigma\Lambda_j^*),\\
&\operatorname*{Res}_{P=\Lambda_j^*} \big(3\,\Psi_i(P)\,\lambda^2(P)\,\omega(P)\big) = -C_j \cdot \Psi_i(\sigma\Lambda_j).
\end{aligned} \tag{2.6}$$

Here $\omega(P) = \omega_{0,\infty}(P)$ is an Abelian differential of the third kind with the divisor of zeros and poles $D + \sigma D - P_0 - P_\infty$ and with the residue $+1$ at P_0 and the opposite residue at P_∞. The above conditions (A)-(C) fix up the function Baker-Achiezer $\Psi(x,t,P)$ uniquely. From (2.6) we have

$$\sum_{j=1}^{N} \operatorname*{Res}_{\substack{\Lambda_j,\Lambda_j^*,\\ \sigma\Lambda_j,\sigma\Lambda_j^*}} \big(\Psi_1(P)\,\Psi_2(\sigma P)\,\lambda(P)\,\omega(P)\big) = 0,$$

$$\sum_{j=1}^{N} \operatorname*{Res}_{\substack{\Lambda_j,\Lambda_j^*,\\ \sigma\Lambda_j,\sigma\Lambda_j^*}} \big(\Psi_3(P)\,\Psi_3(\sigma P)\,\lambda^2(P)\,\omega(P)\big) = 0.$$

This is the duality condition for $\Psi(P)$ and $\Psi(\sigma P)$ which provides the equations (2.1) for N-soliton Baker-Achiezer functions.

In order to find the explicit formula for N-soliton solution of Eq. (1 1) we shall use the technique developed in [10] and [11]. For each pair of vector functions ψ and ϕ we define the pairing $\langle \psi \,|\, \phi \rangle$ as follows:

$$\langle \psi \,|\, \phi \rangle = -\psi_1\,\phi_2 + \psi_2\,\phi_1 + \lambda(P)\,\psi_3\,\phi_3. \tag{2.7}$$

The following properties of the pairing (2.7) are checked up immediately:

$$\begin{aligned}
\partial_x \,\langle e(P) \,|\, e(\sigma Q) \rangle &= \big(\lambda(P) - \lambda(Q)\big)\, e_2(P)\, e_3(P),\\
\partial_t \,\langle e(P) \,|\, e(\sigma Q) \rangle &= e^u \left(1 - \frac{\lambda(P)}{\lambda(Q)}\right) e_3(P)\, e_1(P).
\end{aligned} \tag{2.8}$$

Because of (2.8) the value of $\langle e(P)\,|\,e(\sigma P)\rangle$ does not depend on x and t. One can calculate this value explicitly:

$$\langle e(P)\,|\,e(\sigma P)\rangle = \frac{d\lambda}{3\,\lambda^2\,\omega}. \tag{2.9}$$

Consider the following function depending on x, t and on the pair of points P and Q on Riemann surface Γ:

$$\Omega(x,t,P,Q) = \frac{\langle e(x,t,P)\,|\,e(x,t\sigma Q)\rangle}{\lambda(P)-\lambda(Q)}. \tag{2.10}$$

When Q is fixed function Ω has the pole divisor $D+Q$ and essential singularities at P_0 and P_∞ of the form

$$\begin{aligned} &\Omega(P,Q) \sim -\frac{q^2\,e^{qt}\,e_1(\sigma Q)}{\lambda(Q)} \quad \text{as} \quad q\to\infty,\\ &\Omega(P,Q) \sim -k^{-4}\,e^{kx}\,e_2(\sigma Q) \quad \text{as} \quad k\to\infty. \end{aligned} \tag{2.11}$$

Moreover from (2.10) we derive

$$\operatorname*{Res}_{P=Q}\bigl(3\,\Omega(x,t,P,Q)\,\lambda^2(P)\,\omega(P)\bigr) = 1. \tag{2.12}$$

Properties of $\Omega(x,t,P,Q)$ just mentioned give us the opportunity to use it for finding one of the components of N-soliton Baker-Achiezer function $\Psi_3(x,t,P)$. We $\Psi_3(x,t,P)$ construct via the following ansatz

$$\begin{aligned} \Psi_3(x,t,P) = e_3(x,t,P) &+ \sum_{j=1}^{N}\Omega(x,t,P,\Lambda_j)\cdot\alpha_j(x,t)\,+\\ &+\sum_{j=1}^{N}\Omega(x,t,P,\Lambda_j^*)\cdot\alpha_j^*(x,t) \end{aligned} \tag{2.13}$$

with the parameters α_j, α_j^* yet undefined. We shall define them with the use of (2.6) by applying technique from [11]. N-soliton solution $u(x,t)$ of Eq. (1.1) then is found on a base of equivalences

$$\partial_t e_3(P) \sim e^v\,k^{-4}\,e^{kx}, \qquad\qquad \partial_t\Psi_3(P) \sim e^u\,k^{-4}\,e^{kx}$$

as $P\to P_\infty$. Taking (2.11) into account, one gets

$$e^u = e^v - \partial\left(\sum_{j=1}^{N} e_2(\sigma\Lambda_j)\cdot\alpha_j + \sum_{j=1}^{N} e_2(\sigma\Lambda_j^*)\cdot\alpha_j^*\right). \tag{2.14}$$

In general, to find a_j and a_j^* from (2.6) we should solve some system of linear differential equations, the special form of sums on the right part of (2.14) however makes it possible to eliminate this step and leads us to the formula

$$e^u = e^v - \partial_{xt}\ln\det(1-\Omega\,C). \tag{2.15}$$

Here $C = diag(C_1, \dots, C_N, -C_1, \dots, -C_N)$ is a diagonal matrix matrix built of constants C_j from (b) (see above). Matrix Ω is defined by $\Omega(x, t, P, \Lambda_j)$ and $\Omega(x, t, P, \Lambda_j^*)$ with $P = \sigma\Omega_j$ and $P = \sigma\Lambda_j^*$. Therefore its components could be evaluated explicitly via Riemann theta functions. This matrix is composed of blocks

$$\Omega = \begin{array}{|c|c|} \hline A_{ij} = \Omega(x, t, \sigma\Lambda_i, \Lambda_j) & B_{ij} = \Omega(x, t, \sigma\Lambda_i, \Lambda_j^*) \\ \hline C_{ij} = \Omega(x, t, \sigma\Lambda_i^*, \Lambda_j) & D_{ij} = \Omega(x, t, \sigma\Lambda_i^*, \Lambda_j^* \\ \hline \end{array} \tag{2.16}$$

Formula (2.15) with the matrix (2.16) is a direct analog of Hirota's determinant formula from [4].

3. Problem of Nonsingularity

In order to get real and nonsingular N-soliton solutions in (2.15) one should set more restrictions when choosing parameters in (a) and (b). Let the Riemann surface

Fig. 1. The quarter of the Riemann surface Γ unfolded onto the plane, τ acts as a reflection downward, σ acts as a rotation around the point P_∞.

Γ and divisor D be chosen so that they correspond to the real and nonsingular finite-gap solution $v(x, t)$ of Eq. (1.1). In this case surface Γ can be cut and then unwrapped onto a plane as shown on Fig. 1. Points where $\lambda(P)$ is purely real are shown with solid lines while purely imaginary values of $\lambda(P)$ are shown with dashed lines. A set of $\lambda(P)$ values on canonic a-cycles (i. e. invariant cycles of τ except cycle a_0 containing P_0 and P_∞) consists of g intervals. When λ_j are in that interval the canonic choice Λ_j, Λ_j^* pairs preserving $\lambda(\Lambda_j) = \lambda(\sigma\Lambda_j^*) > 0$ is possible. Each pair Λ_j, Λ_j^* together with the point P_i, of divisor D define the orientation i_j on a cycle a_i, in the direction from Λ_j to $\sigma\Lambda_j$ through P_i. Thus we can prescribe the sign to Abelian differential $\omega(P)$ with respect to orientation i_j just introduced. Let us choose constants C_j, from (b) being real with the signs defined by

$$\operatorname{sign}(C_j) = \operatorname{sign}(\omega(P)|i_j)\Big|_{P=\Lambda_j}. \tag{3.1}$$

It is easy to check that this rule gives an opposite sign on an opposite cycle σa_i being in accordance with (2.6).

Lemma 3.1. *When sign rule* (3.1) *holds, the number of zeros of the N-soliton Baker-Achiezer function $\Psi(P)$ on each cycle a_i is equal to the number of its poles on that cycle.*

Since $\Psi(P)$ is real on a_i, (2.6) and (3.1) and simple geometrical considerations yield us the statement of lemma. This means that all zeros of $\Psi(P)$ are on a-cycles, hence they are separated from P_0 and P_∞, i. e. they never coincide nor come close to those points. The last fact in turn leads us to the following theorem.

Theorem 3.1. *The above canonical choice of Λ_j, Λ_j^* and the sign rule* (3.1) *for real constants C_j are enough to provide nonsingular N-soliton solution of* (1.1) *in* (2.15).

Such facts are well-known in the theory of finite-gap integration (see [9]). In the case of fast-decreasing solitons Γ has a topology of sphere with the only cycle a_0 containing P_0 and P_∞, and as a result we can find no place on Γ to set Λ_j and Λ_j^* as required by the theorem above.

Formula (2.15) together with the following estimates for purely finite-gap Baker-Achiezer function $e(x,t,P)$

$$|e(x,t,P)| < \mathrm{const}\cdot\exp\bigl(\kappa_\infty(P)\,x + \kappa_0(P)\,t\bigr) \tag{3.2}$$

via the Abelian integrals

$$\kappa_\infty(P) = \operatorname{Re}\int^P \Omega_\infty, \qquad\qquad \kappa_0(P) = \operatorname{Re}\int^P \Omega_0$$

give us the opportunity to find asymptotics for N-soliton solutions of Eq. (1.1) as $x, t \to \pm\infty$. Because of (3.2) we may divide entire rows and columns of matrix Ω in (2.15) by exponential factors of growth

$$\exp\bigl(2(\kappa_\infty(\sigma\Lambda_j^*) - \kappa_\infty(\sigma\Lambda_j))\,x + 2(\kappa_0(\sigma\Lambda_j^*) - \kappa_0(\sigma\Lambda_j))\,t\bigr).$$

Vanishing of $(\kappa_\infty(\sigma\Lambda_j^*) - \kappa_\infty(\sigma\Lambda_j))\,x + 2(\kappa_0(\sigma\Lambda_j^*) - \kappa_0(\sigma\Lambda_j))\,t$ determines the free soliton trajectories. Their velocities are

$$v_j = -\frac{\kappa_\infty(\sigma\Lambda_j^*) - \kappa_\infty(\sigma\Lambda_j)}{\kappa_0(\sigma\Lambda_j^*) - \kappa_0(\sigma\Lambda_j)} \tag{3.3}$$

For Eq. (1.2) in space-time coordinates soliton velocities V_j, are bound with v_j as

$$V_j = \frac{e^{\varepsilon}\, v_j + e^{-\varepsilon}}{e^{-\varepsilon} - e^{\varepsilon}\, v_j} \tag{3.4}$$

Here ε is a scalar parameter depending on the choice of particular system of Lorentzian coordinates. The matter that V_j are less than light velocity $c = 1$ here is a consequence of (3.3) and (3.4) and the following lemma.

Lemma 3.2. *When points Λ_j, $\sigma\Lambda_j^*$ are placed as shown on Fig. 1, the inequalities $\kappa_\infty(\sigma\Lambda_j^*) - \kappa_\infty(\sigma\Lambda_j), 0$ and $\kappa_0(\sigma\Lambda_j^*) - \kappa_0(\sigma\Lambda_j) < 0$ hold.*

The observers being at rest in almost all Lorentzian systems of coordinates will see the similar pictures of soliton interaction: before and after the interaction solitons are free. They are separated by finite-gap background potentials $v(x, t, z_j)$ with the common spectrum Γ but different phases z_j, peculiar to each interval between solitons. The background phase shift obtained while crossing the path of j-th soliton is $\triangle z_j = U(\Lambda_j + \Lambda_j^* - \sigma\Lambda_j + \sigma\Lambda_j^*)$. Here $\triangle z_j$ is a vector of Prym variety of the covering $\Gamma \to \Gamma/\sigma$ and U is a map from subset of $\mathrm{Div}(\Gamma)$ into $\mathrm{Prym}(\Gamma)$ built in paper [3].

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