BURGERS SPACE VERSUS REAL SPACE
in the nonlinear theory of dislocations.

Ruslan Sharipov

Abstract. Some double space tensorial quantities in the nonlinear theory of dislocations are considered. Their real space counterparts are introduced.

1. Introduction.

The Burgers vector is a quantitative characteristic of each dislocation line in a crystal. It is similar to the charge or the mass of elementary particles, but it is a vectorial quantity. The Burgers space $\mathbb{B}$ is introduced as a container for all of the Burgers vectors (see [1]). The Burgers space is an imaginary space, it is different from the real space $\mathbb{E}$ where the actual evolution of a crystal occurs. The Burgers space can be understood as a copy of the real space filled with the infinite ideal (non-distorted) crystalline grid that does not move and does not evolve in any other way. Like the real space, it is equipped with the metric given by the metric tensor and the dual metric tensor. Their components $\overset{\star}{g}_{ij}$ and $\overset{\star}{g}{}^{ij}$ are related to some Cartesian coordinates x^1, x^2, x^3 in $\mathbb{B}$. The components of the metric tensor and the dual metric tensor in the real space $\mathbb{E}$ are denoted by g_{pq} and g^{pq}, i. e. without the star. For the sake of simplicity, below we assume g_{pq} and g^{pq} to be related to some Cartesian coordinates y^1, y^2, y^3 in $\mathbb{E}$ (though we could choose curvilinear coordinates either). Due to the choice of Cartesian coordinates the components of metric tensors are constants (see [2]).

The basic differential equations describing the kinematics of a dislocated crystal are written in terms of the tensor fields $\hat{\mathbf{T}}$, $\mathbf{T}$, $\boldsymbol{\rho}$, $\mathbf{j}$, and $\mathbf{w}$ (see [1]):

$$\frac{\partial \hat{T}^i_k}{\partial t} + j^{\,i}_{\,k} = -\nabla_k w^i, \tag{1.1}$$

$$\frac{\partial T^i_k}{\partial t} = -\sum_{p=1}^{3} \nabla_k(v^p\ T^i_p), \tag{1.2}$$

$$w^i = \sum_{p=1}^{3} v^p\ \hat{T}^i_p, \tag{1.3}$$

$$\sum_{q=1}^{3}\sum_{p=1}^{3}\sum_{r=1}^{3}\sum_{m=1}^{3} \omega_{kqp}\ g^{qr}\ g^{pm}\,\nabla_r \hat{T}^i_m = \rho^i_k. \tag{1.4}$$

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The third equation (1.3) is based on the conjecture 4.1 suggested in [3]. If this conjecture appears to be not valid, then (1.3) will be replaced by some other equation. However, this will not affect the methods we use in the present paper. With this explicit reservation, we continue assuming below that the conjecture from [3] is valid and that (1.3) is a true expression for w^i.

The tensor fields $\hat{\mathbf{T}}$, $\mathbf{T}$, $\boldsymbol{\rho}$, $\mathbf{j}$, and $\mathbf{w}$ are double space tensor fields. Their upper index i in (1.1), (1.2), (1.3), and (1.4) is associated with the Burgers space $\mathbb{B}$, while their components are functions of the coordinates y^1, y^2, y^3 in the real space $\mathbb{E}$:

$$\begin{aligned} \hat{T}^i_k &= \hat{T}^i_k(t, y^1, y^2, y^3),\\ T^i_k &= T^i_k(t, y^1, y^2, y^3),\\ j^{\,i}_{\,k} &= j^{\,i}_{\,k}(t, y^1, y^2, y^3),\\ \rho^i_k &= \rho^i_k(t, y^1, y^2, y^3),\\ w^i &= w^i(t, y^1, y^2, y^3). \end{aligned} \tag{1.5}$$

Note that v^1, v^2, v^3 are the components of the velocity vector $\mathbf{v}$ describing the motion of a medium. Unlike (1.5), they are components of a real space tensor:

$$v^p = v^p(t, y^1, y^2, y^3). \tag{1.6}$$

The index p in (1.2), (1.3), and in (1.6) is associated with the real space. Apart from (1.5) and (1.6), in (1.4) we see the components of the so-called *volume tensor*, they are denoted ω_{kqp}. Due to the choice of the Cartesian coordinates y^1, y^2, y^3 in $\mathbb{E}$ they are constants: $\omega_{kqp} = \text{const}$.

The concept of the Burgers space and the double space tensor fields associated with it are convenient tools in understanding the microscopic structure of dislocations (in terms of a crystalline grid and interatomic bonds). They are also convenient in deriving the basic differential equations (1.1), (1.2), and (1.4). However, macroscopically, e. g. in explaining real stress-strain curves or in computer simulation of the behavior of real crystals, it would be better to deal with purely real space tensor fields. The main goal of the present paper is to find some purely real space substitutes for the tensor fields (1.5) and derive the differential equations for them in place of (1.1), (1.2), and (1.4). In part, this work is already done in [1]. Below we complete this work and thus lay a foundation for the further development of our approach to the theory of dislocations.

2. The elastic and plastic deformation tensors.

In [1] the *elastic* and the *plastic* deformation tensors $\hat{\mathbf{G}}$ and $\check{\mathbf{G}}$ for a dislocated crystalline matter were suggested. They are given by the formulas

$$\hat{G}_{pq} = \sum_{i=1}^{3} \sum_{j=1}^{3} \overset{\star}{g}_{ij}\ \hat{T}^i_p\ \hat{T}^j_q, \tag{2.1}$$

$$\check{G}^p_q = \sum_{i=1}^{3} \hat{S}^p_i\ T^i_q, \tag{2.2}$$

where $\hat{S}^p_i$ are the components of the inverse matrix $\hat{\mathbf{S}} = \hat{\mathbf{T}}^{-1}$. The *total* deformation tensor $\mathbf{G}$ is defined by the similar formula

$$G_{pq} = \sum_{i=1}^{3}\sum_{j=1}^{3} \overset{\star}{g}_{ij}\ T^i_p\ T^j_q. \tag{2.3}$$

From (2.1), (2.2), and (2.3) one easily derives the equality

$$G_{pq} = \sum_{r=1}^{3}\sum_{s=1}^{3} \check{G}^{\,r}_p\ \hat{G}_{rs}\ \check{G}^{\,s}_q. \tag{2.4}$$

The equality (2.4) is known as the *multiplicative decomposition* of the total deformation tensor into the elastic and plastic parts. This decomposition was first suggested in [4] for the theory of plastic glassy media.

Both deformation tensors $\hat{\mathbf{G}}$ and $\check{\mathbf{G}}$ are purely real space tensors, and so is the total deformation tensor $\mathbf{G}$. The tensor $\hat{\mathbf{G}}$ is considered as a real space substitute for $\hat{\mathbf{T}}$, and $\mathbf{G}$ is such a substitute for $\mathbf{T}$. Passing from $\hat{\mathbf{T}}$ to $\hat{\mathbf{G}}$, we loose a part of information contained in $\hat{\mathbf{T}}$. However, this is that very part of information which is inessential. Indeed, if $\mathbf{O}$ is a rotation matrix in the Burgers space, i. e. if

$$\sum_{i=1}^{3}\sum_{j=1}^{3} \overset{\star}{g}_{ij}\ O^i_p\ O^j_q = \overset{\star}{g}_{pq},$$

then the distorsion tensor $\hat{\mathbf{T}}' = \mathbf{O}\cdot\hat{\mathbf{T}}$ represent the same extent of deformation in interatomic bonds as the the tensor $\hat{\mathbf{T}}$. Similarly, a crystalline body with the distorsion field $\mathbf{T}' = \mathbf{O}\cdot\mathbf{T}$ has the same shape as if it had the distorsion field $\mathbf{T}$ (the matrix $\mathbf{O}$ is assumed to be a constant rotation matrix).

The deformation tensors $\mathbf{G}$ and $\hat{\mathbf{G}}$ satisfy the following partial differential equations describing their time evolution:

$$\frac{\partial G_{pq}}{\partial t} + \sum_{r=1}^{3} v^r\,\nabla_r G_{pq} = -\sum_{r=1}^{3} G_{rq}\,\nabla_p v^r - \sum_{r=1}^{3} G_{pr}\,\nabla_q v^r, \tag{2.5}$$

$$\begin{aligned}\frac{\partial \hat{G}_{pq}}{\partial t} + \sum_{r=1}^{3} v^r\,\nabla_r \hat{G}_{pq} = -\sum_{r=1}^{3} \nabla_p v^r\,\hat{G}_{rq} - \sum_{r=1}^{3} \hat{G}_{pr}\,\nabla_q v^r +\\ +\sum_{r=1}^{3} \theta^{\,r}_p\,\hat{G}_{rq} + \sum_{r=1}^{3} \hat{G}_{pr}\,\theta^{\,r}_q.\end{aligned} \tag{2.6}$$

Here $\theta^{\,r}_p$ and $\theta^{\,r}_q$ are the components of another one purely real space tensor field $\boldsymbol{\theta}$. They are given by the following formula:

$$\theta^{\,r}_q = \nabla_q v^r - \sum_{i=1}^{3} \hat{S}^r_i\,j^{\,i}_q - \sum_{i=1}^{3} \hat{S}^r_i\,\nabla_q w^i + \sum_{i=1}^{3}\sum_{p=1}^{3} v^p\,\hat{S}^r_i\,\nabla_p \hat{T}^i_q. \tag{2.7}$$

The differential equation (2.5) is derived from (1.2), while (2.6) is derived from (1.1) (see [1] for details). The components of the plastic deformation tensor (2.2) satisfy the following differential equation similar to (2.6):

$$\frac{\partial \check{G}^p_q}{\partial t}+\sum_{r=1}^{3} v^r\,\nabla_r\check{G}^p_q=\sum_{r=1}^{3}\bigl(\check{G}^r_q\,\nabla_r v^p-\nabla_q v^r\,\check{G}^p_r\bigr)-\sum_{r=1}^{3}\theta^p_r\,\check{G}^r_q. \tag{2.8}$$

In deriving (2.8) both equations (1.1) and (1.2) are used (see [1]).

Note that the equations (2.6) and (2.8) do coincide with the corresponding equations in the theory of amorphous materials (see [4] where these equations were written for the first time). The tensorial parameter $\boldsymbol{\theta}$ is proposed as a new parameter of media like the density, the specific heat capacity, the dielectric permittivity, the elastic moduli, etc (see [5] where some experimental schemes for measuring this parameter $\boldsymbol{\theta}$ are discussed in brief). According to [5], the parameter $\boldsymbol{\theta}$ is interpreted as a tensor that determines the *stress relaxation rate*.

Looking at (2.2) and (2.7), we see that the components of the inverse matrix $\hat{\mathbf{S}}=\hat{\mathbf{T}}^{-1}$ are used for to transform the index i, which is associated with the Burgers space, into the real space indices p and r respectively. By analogy, we can use $\hat{S}^p_i$ in order to define the pair of purely real space tensor fields $\mathbf{R}$ and $\mathbf{J}$:

$$R^p_q=\sum_{i=1}^{3}\hat{S}^p_i\;\rho^{\,i}_q, \qquad\qquad J^p_q=\sum_{i=1}^{3}\hat{S}^p_i\;j^{\,i}_q. \tag{2.9}$$

Like $\boldsymbol{\rho}$ and $\mathbf{j}$, the tensor fields (2.9) characterize the density and the flow of dislocations respectively. Now, under the assumption that the conjecture 4.1 from [3] is valid, if we substitute (1.3) into (2.7), we derive

$$\theta^{\,r}_q=-J^{\,r}_q+\sum_{i=1}^{3}\sum_{p=1}^{3}v^p\,\hat{S}^r_i\,(\nabla_p\hat{T}^i_q-\nabla_q\hat{T}^i_p). \tag{2.10}$$

In order to exclude the double space tensors $\hat{\mathbf{S}}$ and $\hat{\mathbf{T}}$ from (2.10) we introduce the tensor field $\hat{\mathbf{R}}$ with the following components:

$$\hat{R}^{\,r}_{pq}=\sum_{i=1}^{3}\hat{S}^r_i\,(\nabla_p\hat{T}^i_q-\nabla_q\hat{T}^i_p). \tag{2.11}$$

The tensor field $\hat{\mathbf{R}}$ is closely related to the tensor field $\mathbf{R}$ with the components (2.9). Indeed, taking into account the well-known identity

$$\sum_{s=1}^{3}\omega_{slc}\,\omega^{spq}=\delta^p_l\,\delta^q_c-\delta^p_c\,\delta^q_l,$$

from the equalities (2.11) and (1.4) we derive

$$\hat{R}^{\,r}_{pq}=\sum_{i=1}^{3}\hat{S}^r_i\,(\nabla_p\hat{T}^i_q-\nabla_q\hat{T}^i_p)=\sum_{i=1}^{3}\sum_{m=1}^{3}\sum_{n=1}^{3}\hat{S}^r_i\,(\delta^m_p\,\delta^n_q-\delta^m_q\,\delta^n_p)\,\nabla_m T^i_n=$$

$$= \sum_{i=1}^{3}\sum_{s=1}^{3}\sum_{m=1}^{3}\sum_{n=1}^{3} \hat{S}^r_i\,\omega^{smn}\,\omega_{spq}\,\nabla_m T^i_n = \sum_{i=1}^{3}\sum_{s=1}^{3}\sum_{m=1}^{3}\sum_{n=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3}\sum_{c=1}^{3} \hat{S}^r_i\,\omega_{klc}\,g^{sk}\,\times$$
$$\times\, g^{lm}\,g^{cn}\,\omega_{spq}\,\nabla_m T^i_n = \sum_{i=1}^{3}\sum_{s=1}^{3}\sum_{k=1}^{3} \hat{S}^r_i\,g^{sk}\,\omega_{spq}\,\rho^i_k,$$

Now, if we remember (2.9), the result of the above calculations can be written as

$$\hat{R}^{\,r}_{pq} = \sum_{s=1}^{3}\sum_{k=1}^{3}\omega_{spq}\;g^{sk}\,R^r_k. \tag{2.12}$$

By means of the other well-known identity

$$\sum_{p=1}^{3}\sum_{q=1}^{3}\omega_{spq}\;\omega^{kpq} = 2\,\delta^k_s$$

one can easily derive the converse equality to (2.12) that expresses $\mathbf{R}$ through $\hat{\mathbf{R}}$:

$$R^r_k = \frac{1}{2}\sum_{s=1}^{3}\sum_{p=1}^{3}\sum_{q=1}^{3} g_{sk}\;\omega^{spq}\;\hat{R}^{\,r}_{pq}.$$

Applying (2.11) to (2.10) and taking into account (2.12), we get

$$\theta^{\,r}_q = -J^{\,r}_q + \sum_{s=1}^{3}\sum_{p=1}^{3}\sum_{k=1}^{3}\omega_{spq}\;v^p\;g^{sk}\,R^r_k. \tag{2.13}$$

Now let's remember the theorem 3.2 from [3]. In terms of the tensor fields (2.9) this theorem is formulated as follows.

Theorem 2.1. *In the case of frozen dislocations the tensor fields* $\mathbf{R}$ *and* $\mathbf{J}$ *are related to each other by the formula*

$$J^{\,r}_q = \sum_{s=1}^{3}\sum_{p=1}^{3}\sum_{k=1}^{3}\omega_{spq}\;v^p\;g^{sk}\,R^r_k. \tag{2.14}$$

Comparing the equalities (2.14) and (2.13), we derive another theorem.

Theorem 2.2. *The case of frozen dislocations is that very case when* $\boldsymbol{\theta} = 0$*, i. e. when no stress relaxation occurs.*

Note that the equality (2.13) depend on the conjecture 4.1 from [3]. If this conjecture is not valid, we could have some extra terms in its right hand side. However, due to the theorem 3.4 from [3], the above theorem 2.2 holds irrespective to the conjecture 4.1 from [3].

3. Spatial derivatives of the deformation tensors.

In the previous section 2 we have rewritten the three basic equations (1.1), (1.2), and (1.3) in terms of the purely real space tensor fields $\mathbf{T}$, $\hat{\mathbf{T}}$, $\mathbf{R}$, and $\mathbf{J}$. The equation (1.1) has been transformed into (2.5), the equation (1.2) — into (2.6), and the equation (1.3) — into (2.13). In deriving (2.13) from (1.3) we used (1.4). However, the equation (1.4) itself is not yet transformed to the form free of double space tensors. This problem is considered below in the present section.

The equation (1.4) contains only the spatial derivatives of the distorsion tensor $\hat{\mathbf{T}}$. Therefore, it is convenient to denote

$$\hat{Z}^{\,r}_{pq}=\sum_{i=1}^{3}\hat{S}^r_i\ \nabla_p\hat{T}^i_q,\qquad\qquad Z^{\,r}_{pq}=\sum_{i=1}^{3}S^r_i\ \nabla_pT^i_q.\tag{3.1}$$

Here $\hat{S}^r_i$ are the components of the inverse matrix $\hat{\mathbf{S}}=\hat{\mathbf{T}}^{-1}$, and S^r_i are the components of the other inverse matrix $\mathbf{S}=\mathbf{T}^{-1}$. The quantities $\hat{Z}^{\,i}_{pq}$ and $Z^{\,i}_{pq}$ introduced in (3.1) are the components of two purely real space tensor fields $\hat{\mathbf{Z}}$ and $\mathbf{Z}$. Despite to the similarity of the formulas (3.1), the tensor fields $\hat{\mathbf{Z}}$ and $\mathbf{Z}$ are somewhat different. Since $\mathbf{Z}$ is produced by the compatible distorsion tensor, it is symmetric:

$$Z^{\,r}_{pq}=Z^{\,r}_{qp}.\tag{3.2}$$

Indeed, by definition, the components of the compatible distorsion tensor $\mathbf{T}$ are given by partial derivatives (see [1] and [3]):

$$T^i_q=\frac{\partial x^i}{\partial y^q}.\tag{3.3}$$

From (3.3) in Cartesian coordinates y^1, y^2, y^3 we immediately get

$$\nabla_pT^i_q=\frac{\partial^2x^i}{\partial y^p\,\partial y^q}=\frac{\partial^2x^i}{\partial y^q\,\partial y^p}=\nabla_qT^i_p.\tag{3.4}$$

Applying (3.4) to the second equality (3.1), we find that the above symmetry condition (3.2) is a valid equality. As for the tensor $\hat{\mathbf{Z}}$ produced from the incompatible distorsion tensor $\hat{\mathbf{T}}$, its components are usually not symmetric.

In the next step we apply the operator $\nabla_p=\partial/\partial y^p$ to the components G_{qk} of the deformation tensor $\mathbf{G}$. From (2.3), since $\overset{\star}{g}_{ij}=\text{const}$, we derive

$$\nabla_pG_{qk}=\sum_{r=1}^{3}Z^{\,r}_{pq}\,G_{rk}+\sum_{r=1}^{3}Z^{\,r}_{pk}\,G_{rq}.\tag{3.5}$$

By means of cyclic transposition of indices $p\to q\to k\to p$ in (3.5) we get

$$\nabla_qG_{kp}=\sum_{r=1}^{3}Z^{\,r}_{qk}\,G_{rp}+\sum_{r=1}^{3}Z^{\,r}_{qp}\,G_{rk},\tag{3.6}$$

$$\nabla_kG_{pq}=\sum_{r=1}^{3}Z^{\,r}_{kp}\,G_{rq}+\sum_{r=1}^{3}Z^{\,r}_{kq}\,G_{rp}.\tag{3.7}$$

By adding (3.5) and (3.6), then subtracting (3.7) and taking into account (3.2) we derive the following equality for the quantities $Z^{\,r}_{pq}$:

$$\nabla_p G_{qk} + \nabla_q G_{kp} - \nabla_k G_{pq} = 2\sum_{r=1}^{3} Z^{\,r}_{pq}\, G_{rk}$$

If we denote by $[\,G^{-1}]^{kr}$ the components of the inverse matrix $\mathbf{G}^{-1}$, we can write

$$Z^{\,r}_{pq} = \sum_{k=1}^{3} \frac{\nabla_p G_{qk} + \nabla_q G_{kp} - \nabla_k G_{pq}}{2}\;[\,G^{-1}]^{kr}. \tag{3.8}$$

By applying $\nabla_p = \partial/\partial y^p$ to the components $\hat{G}_{qk}$ of the elastic deformation tensor $\hat{\mathbf{G}}$ one can derive the equality similar to (3.5):

$$\nabla_p \hat{G}_{qk} = \sum_{r=1}^{3} \hat{Z}^{\,r}_{pq}\, \hat{G}_{rk} + \sum_{r=1}^{3} \hat{Z}^{\,r}_{pk}\, \hat{G}_{rq}. \tag{3.9}$$

However, $\hat{Z}^{\,r}_{pq}$ are not symmetric in p and q. Therefore, (3.9) is insufficient for expressing the tensor $\hat{\mathbf{Z}}$ through $\hat{\mathbf{G}}$ and $\nabla\hat{\mathbf{G}}$. The number of components $\hat{Z}^{\,r}_{pq}$ is equal to 27, while (3.9) is symmetric in k and q. This means that in (3.9) we have only 18 linear algebraic equations for $\hat{Z}^{\,r}_{pq}$ different from each other. Nine equations are obviously lacking.

Let's expand the tensor field $\hat{\mathbf{Z}}$ into symmetric and skew-symmetric parts, the skew-symmetric part being given by the tensor $\hat{\mathbf{R}}$ with the components (2.11):

$$\hat{\mathbf{Z}} = \hat{\mathbf{H}} + \frac{1}{2}\,\hat{\mathbf{R}}. \tag{3.10}$$

Due to (3.10) we can rewrite the equation (3.9) as follows:

$$\begin{gathered}\sum_{r=1}^{3} \hat{H}^{\,r}_{pq}\, \hat{G}_{rk} + \sum_{r=1}^{3} \hat{H}^{\,r}_{pk}\, \hat{G}_{rq} = \\ = \nabla_p \hat{G}_{qk} - \frac{1}{2}\sum_{r=1}^{3} \hat{R}^{\,r}_{pq}\, \hat{G}_{rk} - \frac{1}{2}\sum_{r=1}^{3} \hat{R}^{\,r}_{pk}\, \hat{G}_{rq}.\end{gathered} \tag{3.11}$$

Because of the symmetry of $\hat{\mathbf{H}}$ we can apply to (3.11) the same method as used in deriving (3.8) from (3.5). As a result we obtain

$$\begin{gathered}\hat{H}^{r}_{pq} = \sum_{k=1}^{3} \frac{\nabla_p \hat{G}_{qk} + \nabla_q \hat{G}_{kp} - \nabla_k \hat{G}_{pq}}{2}\;[\,\hat{G}^{-1}]^{kr} - \\ -\frac{1}{2}\sum_{k=1}^{3}\sum_{s=1}^{3} \hat{R}^{s}_{pk}\, \hat{G}_{sq}\,[\,\hat{G}^{-1}]^{kr} - \frac{1}{2}\sum_{k=1}^{3}\sum_{s=1}^{3} \hat{R}^{s}_{qk}\, \hat{G}_{sp}\,[\,\hat{G}^{-1}]^{kr}.\end{gathered} \tag{3.12}$$

The formula (3.12) shows that we cannot express both $\hat{\mathbf{H}}$ and $\hat{\mathbf{R}}$ through $\hat{\mathbf{G}}$ and $\nabla\hat{\mathbf{G}}$. However, if we take the components of $\hat{\mathbf{R}}$ for independent variables, then we can express $\mathbf{H}$ through $\hat{\mathbf{G}}$, $\nabla\hat{\mathbf{G}}$, and $\hat{\mathbf{R}}$. From (3.12) and (3.10) we derive

$$\begin{aligned}\hat{Z}^{\,r}_{pq}=\sum^3_{k=1}\frac{\nabla_p\hat{G}_{qk}+\nabla_q\hat{G}_{kp}-\nabla_k\hat{G}_{pq}}{2}\,\,[\hat{G}^{-1}]^{kr}+\frac{1}{2}\,\hat{R}^r_{pq}\,-\\ -\frac{1}{2}\sum^3_{k=1}\sum^3_{s=1}\hat{R}^s_{pk}\,\hat{G}_{sq}\,[\hat{G}^{-1}]^{kr}-\frac{1}{2}\sum^3_{k=1}\sum^3_{s=1}\hat{R}^s_{qk}\,\hat{G}_{sp}\,[\hat{G}^{-1}]^{kr}.\end{aligned}\tag{3.13}$$

Now let's substitute (2.12) into (3.13). As a result we obtain

$$\begin{aligned}\hat{Z}^{\,r}_{pq}=\sum^3_{k=1}\frac{\nabla_p\hat{G}_{qk}+\nabla_q\hat{G}_{kp}-\nabla_k\hat{G}_{pq}}{2}\,\,[\hat{G}^{-1}]^{kr}\,+\\ +\,\frac{1}{2}\,\sum^3_{s=1}\sum^3_{k=1}\omega_{spq}\,\,g^{sk}\,R^r_k\,-\\ -\,\frac{1}{2}\sum^3_{k=1}\sum^3_{s=1}\sum^3_{m=1}\sum^3_{n=1}\omega_{mpk}\,\,g^{mn}\,R^s_n\,\hat{G}_{sq}\,[\hat{G}^{-1}]^{kr}\,-\\ -\,\frac{1}{2}\sum^3_{k=1}\sum^3_{s=1}\sum^3_{m=1}\sum^3_{n=1}\omega_{mqk}\,\,g^{mn}\,R^s_n\,\hat{G}_{sp}\,[\hat{G}^{-1}]^{kr}.\end{aligned}\tag{3.14}$$

As we already mentioned above, passing from $\hat{\mathbf{T}}$ to the deformation tensor $\hat{\mathbf{G}}$, we loose a part of information contained in $\hat{\mathbf{T}}$. Nine components of the tensor $\mathbf{R}$ in (3.14) is an exact quantitative measure for that loss. We cannot express $\mathbf{R}$ through $\hat{\mathbf{G}}$ and $\nabla\hat{\mathbf{G}}$, hence, we cannot write the equation (1.4) in terms of the tensor fields $\hat{\mathbf{G}}$ and $\mathbf{R}$. However, we can do it for some equations derived from (1.4) and (1.1).

4. Accessory differential relationships.

Let's consider the equation (1.4) again. One can write it in a formal, but more easily understandable way as follows (see [1]):

$$\operatorname{rot}\hat{\mathbf{T}}=\boldsymbol{\rho}.\tag{4.1}$$

Similarly, the equation (1.1) is written as

$$\frac{\partial\hat{\mathbf{T}}}{\partial t}+\mathbf{j}=-\operatorname{grad}\mathbf{w}.\tag{4.2}$$

From (4.1) and (4.2) one easily derives the following two relationships:

$$\operatorname{div}\boldsymbol{\rho}=0,\tag{4.3}$$

$$\frac{\partial\boldsymbol{\rho}}{\partial t}+\operatorname{rot}\mathbf{j}=0.\tag{4.4}$$

Our present goal in this section is to rewrite (4.3) and (4.4) in terms of the real space tensor fields $\hat{\mathbf{G}}$, $\mathbf{R}$, and $\mathbf{J}$.

The tensor $\mathbf{R}$ is defined by the first formula (2.9). Its components are expressed through the components of the tensor $\boldsymbol{\rho}$. By means of the inverse matrix $\hat{\mathbf{T}} = \hat{\mathbf{S}}^{-1}$ one can express ρ^i_q back through R^m_q:

$$\rho^i_q = \sum_{m=1}^{3} \hat{T}^i_m\, R^m_q. \tag{4.5}$$

The equation (4.3) in coordinate form is written as follows:

$$\sum_{p=1}^{3}\sum_{q=1}^{3} g^{pq}\,\nabla_p \rho^{\,i}_q = 0. \tag{4.6}$$

Substituting (4.5) into (4.6), we immediately derive the equality

$$\sum_{m=1}^{3}\sum_{p=1}^{3}\sum_{q=1}^{3} g^{pq}\,\hat{T}^i_m\,\nabla_p R^m_q + \sum_{m=1}^{3}\sum_{p=1}^{3}\sum_{q=1}^{3} g^{pq}\,R^m_q\,\nabla_p \hat{T}^i_m = 0. \tag{4.7}$$

Multiplying (4.7) by $\hat{S}^r_i$ and summing up on the index i, we get

$$\sum_{p=1}^{3}\sum_{q=1}^{3} g^{pq}\,\nabla_p R^r_q + \sum_{m=1}^{3}\sum_{p=1}^{3}\sum_{q=1}^{3} g^{pq}\,R^m_q\,\hat{Z}^r_{pm} = 0. \tag{4.8}$$

If we remember that $\hat{Z}^r_{pm}$ can be obtained from the formula (3.14), we see that (4.8) is a differential equation written in terms of the tensor fields $\mathbf{R}$ and $\hat{\mathbf{G}}$. This equation is a purely real space substitute for (4.3).

In order to perform the same transformations with the equation (4.4), let's apply the time derivative $\partial/\partial t$ to (4.5). As a result we get

$$\sum_{k=1}^{3}\frac{\partial \hat{T}^i_k}{\partial t}\,R^k_q + \sum_{k=1}^{3}\hat{T}^i_k\,\frac{\partial R^k_q}{\partial t} + \sum_{m=1}^{3}\sum_{r=1}^{3}\sum_{s=1}^{3} g_{qm}\,\omega^{mrs}\,\nabla_r j^{\,i}_s = 0. \tag{4.9}$$

For $j^{\,i}_s$ in (4.9) there is a formula similar to (4.5). It is derived from (2.9):

$$j^{\,i}_s = \sum_{m=1}^{3}\hat{T}^i_m\,J^m_s. \tag{4.10}$$

Substituting (4.10) into (4.9) and applying (1.1) and (1.3), we obtain

$$\begin{gathered}\frac{\partial R^k_q}{\partial t} - \sum_{m=1}^{3} J^{\,k}_m\,R^m_q - \sum_{m=1}^{3}\nabla_m v^k\,R^m_q - \sum_{m=1}^{3}\sum_{p=1}^{3} v^p\,\hat{Z}^k_{mp}\,R^m_q\,+\\ +\sum_{m=1}^{3}\sum_{r=1}^{3}\sum_{s=1}^{3} g_{qm}\,\omega^{mrs}\,\nabla_r J^{\,k}_s + \sum_{m=1}^{3}\sum_{r=1}^{3}\sum_{s=1}^{3}\sum_{p=1}^{3} g_{qm}\,\omega^{mrs}\,\hat{Z}^k_{rp}\,J^{\,p}_s = 0.\end{gathered} \tag{4.11}$$

Like (4.8), the equation (4.11) is a purely real space substitute for the equation (4.4). Though (4.8) and (4.11) are derived from the accessory equations (4.3) and (4.4), now they play more substantial role because of the absence of a real space substitute for the equation (4.1).

The equations (4.3) and (4.4) are compatible. This means that one cannot produce new equations of the same or lower order by differentiating them and combining the resulting equalities. Otherwise, if that were not the case, the newly produced equations would be called the *compatibility conditions*. Though the equations (4.8) and (4.11) are derived from (4.3) and (4.4), their compatibility is not so evident. The study of all possible compatibility conditions in the real space version of the theory of dislocations is the subject for a separate paper. It should be done with the use of differential geometric methods and terminology.

5. Acknowledgments.

I am grateful to Gregg Allen whose remark during the rehearsal of my report at The University of Akron in January 2004 stimulated my interest to strain-stress curves in the material science. The ultimate goal of the series of papers initiated by [1] is to calculate numerically and thus predict the strain-stress curves of crystalline materials. The same activity for amorphous materials was initiated in [4].

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Rabochaya street 5, 450003 Ufa, Russia
E-mail address: R_Sharipov@ic.bashedu.ru
r-sharipov@mail.ru
ra_sharipov@lycos.com
URL: http://www.geocities.com/r-sharipov