

## GAUGE OR NOT GAUGE?

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ABSTRACT. The analogy of the nonlinear dislocation theory in crystals and the electromagnetism theory is studied. The nature of some quantities is discussed.

## 1. BURGERS VECTORS AND THE BURGERS SPACE.

Dislocations are one-dimensional defects of a crystalline grid used to explain the plasticity in crystals. Each dislocation line is characterized by its Burgers vector  $\mathbf{b}$ , while the dislocated medium in whole is characterized by the so-called *incompatible distortion tensor*  $\hat{\mathbf{T}}$  (see [1]). The components of the Burgers vector

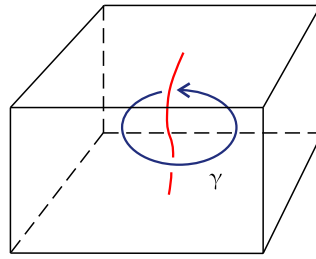


Fig. 1.1

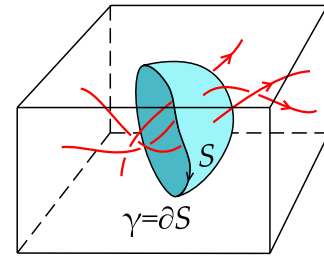


Fig. 1.2

for a dislocation line are determined by the following path integral along some closed contour encircling this dislocation line (see Fig. 1.1):

$$b^i = \oint_{\gamma} \sum_{j=1}^3 \hat{T}_j^i dy^j. \quad (1.1)$$

The Burgers vector  $\mathbf{b}$  with components (1.1) is treated as a vector of a special space, it is called the *Burgers space*. The Burgers space is an imaginary space, it is assumed to be filled with the infinite ideal (non-distorted) crystalline grid. The tensor  $\hat{\mathbf{T}}$  in (1.1) is a *double space tensor*: its upper index  $i$  is associated with some Cartesian coordinate system in the Burgers space, its lower index  $j$  is a traditional tensorial index associated with some coordinates  $y^1, y^2, y^3$  (no matter Cartesian or curvilinear) in the real space where the crystalline medium moves.

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In the continual limit, when the number of dislocation lines is macroscopically essential, separate dislocation lines are replaced by their distribution  $\boldsymbol{\rho}$  (see [1] for more details). Then (1.1) is replaced by the following integral equality:

$$\oint_{\partial S} \sum_{j=1}^3 \hat{T}_j^i dy^j = \int_S \sum_{j=1}^3 \rho_j^i n^j dS. \quad (1.2)$$

Here  $S$  is some imaginary surface within the medium,  $n^j$  are the components of the unit normal vector to  $S$ , and  $\gamma = \partial S$  is the boundary of  $S$  (see Fig. 1.2). According to (1.2), the double space tensorial quantity  $\boldsymbol{\rho}$  is interpreted as the Burgers vector per unit area. Applying the Stokes formula to (1.2), we get the differential equality

$$\boldsymbol{\rho} = \text{rot } \hat{\mathbf{T}}. \quad (1.3)$$

Apart from (1.3), we have the following equality (see [1]):

$$\mathbf{j} = -\text{grad } \mathbf{w} - \frac{\partial \hat{\mathbf{T}}}{\partial t}. \quad (1.4)$$

The double space tensorial quantity  $\mathbf{j}$  is interpreted as the Burgers vector crossing the unit length of a contour per unit time due to the moving dislocations (see [1]). However, the interpretation of the quantity  $\mathbf{w}$  was not clarified in [1]. This is the goal of the present paper. For this purpose below we study two special cases.

## 2. PLASTIC RELAXATION.

Let's consider a two-dimensional model of a crystalline medium with square cells (see Fig. 2.1). On the preliminary stage the crystal was distorted as shown on

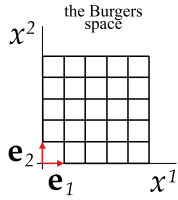


Fig. 2.1

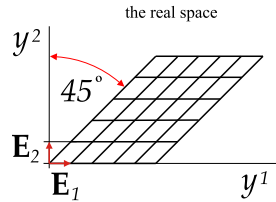


Fig. 2.2

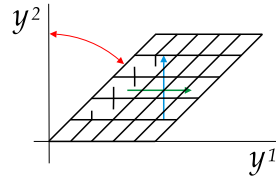


Fig. 2.3

Fig. 2.2. This distortion is described by the following deformation map:

$$\begin{cases} x^1 = x^1(y^1, y^2) = y^1 - y^2, \\ x^2 = x^2(y^1, y^2) = y^2. \end{cases} \quad (2.1)$$

Here we assume that  $x^1, x^2$  are Cartesian coordinates in the Burgers space and  $y^1, y^2$  are Cartesian coordinates in the real space, both are associated with the

orthonormal bases  $\mathbf{e}_1, \mathbf{e}_2$  and  $\mathbf{E}_1, \mathbf{E}_2$  respectively. By differentiating (2.1) we find the components of the *compatible distortion tensor*  $\mathbf{T}$  (see [1]):

$$T_k^i = \frac{\partial x^i}{\partial y^k} = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix}. \quad (2.2)$$

They are constants since the further evolution of our crystal goes without the displacement of atoms (see Fig. 2.3, Fig. 2.4, Fig. 2.5, Fig. 2.6). Hence, we have

$$\frac{\partial \mathbf{T}}{\partial t} = 0. \quad (2.3)$$

Initially, our crystal has no dislocations at all. Therefore, the compatible and incompatible distortion tensors are initially equal to each other:

$$\hat{\mathbf{T}} \Big|_{t=t_0} = \mathbf{T} \Big|_{t=t_0}. \quad (2.4)$$

On Fig. 2.3 three pairs of the edge dislocations arise. Their Burgers vectors are  $\mathbf{b} = \mathbf{e}_1$  and  $\mathbf{b} = -\mathbf{e}_1$  in each pair. Therefore, the total Burgers vector of this group

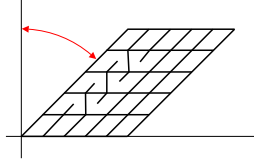


Fig. 2.4

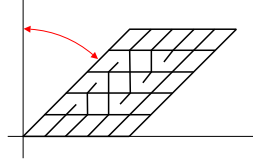


Fig. 2.5

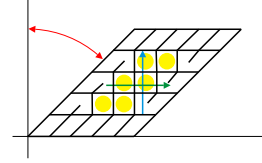


Fig. 2.6

of dislocations is equal to zero. On Fig. 2.4, Fig. 2.5, Fig. 2.6 the dislocations with negative Burgers vectors  $\mathbf{b} = -\mathbf{e}_1$  move to the right. During the evolution time the blue arrow of the length 3 (see Fig. 2.3 and Fig. 2.6) is crossed by three dislocations with total Burgers vector  $\mathbf{b} = -3\mathbf{e}_1$ . The green arrow of the same length is not crossed by the moving dislocations at all. Therefore, we have the equality

$$\int_{t_0}^{t_1} j_k^i dt = \begin{vmatrix} 0 & -1 \\ 0 & 0 \end{vmatrix} \quad (2.5)$$

at the center of our crystal. Behind the moving dislocations we find the undistorted cells, they are marked by yellow spots on Fig. 2.6. This means that

$$\hat{T}_k^i \Big|_{t=t_1} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \quad (2.6)$$

at the center of our crystal. Combining (2.2), (2.3), (2.4), (2.5), and (2.6), we get

$$\hat{\mathbf{T}} \Big|_{t_0}^{t_1} + \int_{t_0}^{t_1} \mathbf{j} dt = 0. \quad (2.7)$$

In the continuous limit, when the moving dislocations form the homogeneous and constant flow the above equality (2.7) can be transformed to the following one:

$$\frac{\partial \hat{\mathbf{T}}}{\partial t} + \mathbf{j} = 0. \quad (2.8)$$

Comparing (2.8) with (1.4), we conclude that  $\mathbf{v} = 0$  and  $\mathbf{T} = \text{const}$  implies  $\mathbf{w} = 0$  in our first example.

### 3. FROZEN DISLOCATIONS.

As the second example, we consider a three-dimensional crystal where the dislocation lines move together with the medium like water-plants frozen into the ice.

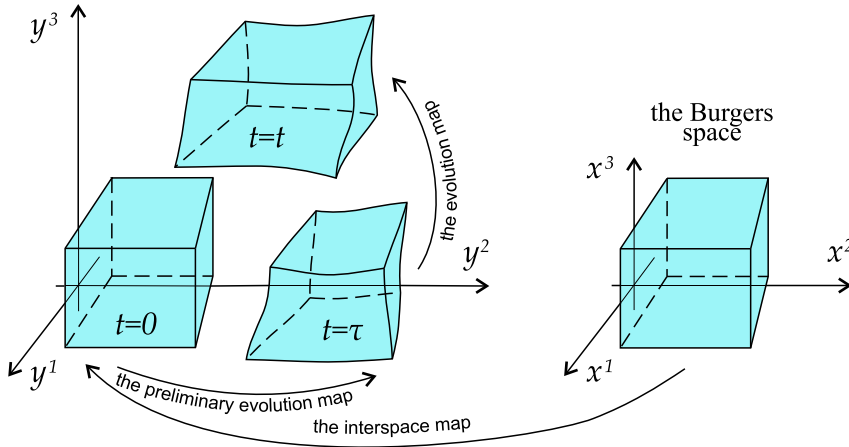


Fig. 3.1

The choice of Cartesian coordinates  $x^1, x^2, x^3$  in the Burgers space is obligatory (see [1]). In the real space we could choose either Cartesian or curvilinear coordinates. Below we choose Cartesian coordinates  $y^1, y^2, y^3$  for the sake of simplicity. Then the interspace map and its inverse map are given by the following formulas:

$$\begin{cases} \tilde{y}^1 = x^1, \\ \tilde{y}^2 = x^2, \\ \tilde{y}^3 = x^3, \end{cases} \quad \begin{cases} x^1 = \tilde{y}^1, \\ x^2 = \tilde{y}^2, \\ x^3 = \tilde{y}^3. \end{cases} \quad (3.1)$$

The evolution of the crystal is subdivided into two stages (see Fig. 3.1). In the first stage, which is a preliminary one, the dislocations are produced:

$$\begin{cases} \bar{y}^1 = \bar{y}^1(\tau, \tilde{y}^1, \tilde{y}^2, \tilde{y}^3), \\ \bar{y}^2 = \bar{y}^2(\tau, \tilde{y}^1, \tilde{y}^2, \tilde{y}^3), \\ \bar{y}^3 = \bar{y}^3(\tau, \tilde{y}^1, \tilde{y}^2, \tilde{y}^3), \end{cases} \quad \begin{cases} \tilde{y}^1 = \tilde{y}^1(\tau, \bar{y}^1, \bar{y}^2, \bar{y}^3), \\ \tilde{y}^2 = \tilde{y}^2(\tau, \bar{y}^1, \bar{y}^2, \bar{y}^3), \\ \tilde{y}^3 = \tilde{y}^3(\tau, \bar{y}^1, \bar{y}^2, \bar{y}^3). \end{cases} \quad (3.2)$$

In the second stage the dislocations are frozen and move together with the medium:

$$\begin{cases} y^1 = y^1(t, \tau, \bar{y}^1, \bar{y}^2, \bar{y}^3), \\ y^2 = y^2(t, \tau, \bar{y}^1, \bar{y}^2, \bar{y}^3), \\ y^3 = y^3(t, \tau, \bar{y}^1, \bar{y}^2, \bar{y}^3), \end{cases} \quad \begin{cases} \bar{y}^1 = \bar{y}^1(t, \tau, y^1, y^2, y^3), \\ \bar{y}^2 = \bar{y}^2(t, \tau, y^1, y^2, y^3), \\ \bar{y}^3 = \bar{y}^3(t, \tau, y^1, y^2, y^3). \end{cases} \quad (3.3)$$

The compatible distortion tensor  $\mathbf{T}$  at the time instant  $t$  is determined by the composite map including all of the three above maps (3.1), (3.2), and (3.3):

$$T_k^i(t, y^1, y^2, y^3) = \frac{\partial x^i}{\partial y^k} \quad (3.4)$$

(see (2.2) for comparison). Similarly, at the time instant  $t = \tau$  we have

$$T_q^i(\tau, \bar{y}^1, \bar{y}^2, \bar{y}^3) = \frac{\partial x^i}{\partial \bar{y}^q}. \quad (3.5)$$

From (3.4) and (3.5), applying the chain rule to (3.3), we derive the relationship

$$T_k^i(t, y^1, y^2, y^3) = \sum_{q=1}^3 T_q^i(\tau, \bar{y}^1, \bar{y}^2, \bar{y}^3) \frac{\partial \bar{y}^q}{\partial y^k}. \quad (3.6)$$

The relationship (3.6) expresses the evolution rule for the compatible distortion tensor  $\mathbf{T}$ . If the dislocations are frozen into the material, then the incompatible distortion tensor  $\hat{\mathbf{T}}$  should obey the same evolution rule:

$$\hat{T}_k^i(t, y^1, y^2, y^3) = \sum_{q=1}^3 \hat{T}_q^i(\tau, \bar{y}^1, \bar{y}^2, \bar{y}^3) \frac{\partial \bar{y}^q}{\partial y^k}. \quad (3.7)$$

For the sake of simplicity, in the further calculations we denote

$$\bar{T}_k^q(t, \tau, y^1, y^2, y^3) = \frac{\partial \bar{y}^q}{\partial y^k}. \quad (3.8)$$

The quantities (3.8) form a non-degenerate square matrix  $\bar{T}$ . Let  $\bar{S} = \bar{T}^{-1}$  be the inverse matrix for  $\bar{T}$  and let  $S_k^q = S_k^q(t, \tau, y^1, y^2, y^3)$  be the components of this inverse matrix. Note that  $\bar{T}_k^q$  are not the components of a tensor field. They are not the components of a double space tensor in the sense of [1] as well. We shall not discuss the tensorial properties of the quantities  $\bar{T}_k^q$ , we shall use them only as the notations for the partial derivatives (3.8).

The following functional identity with two parameters  $t$  and  $\tau$  is quite obvious for the pair of mutually inverse maps given by the functions (3.3):

$$y^k(t, \tau, \bar{y}^1(t, \tau, y^1, \dots, y^3), \dots, \bar{y}^3(t, \tau, y^1, \dots, y^3)) = y^k. \quad (3.9)$$

By differentiating (3.9) with respect to the time variable  $t$  we easily derive

$$\frac{\partial \bar{y}^q}{\partial t} = - \sum_{r=1}^3 v^r \bar{T}_r^q. \quad (3.10)$$

Here  $v^1, v^2, v^3$  are the components of the velocity vector  $\mathbf{v} = \mathbf{v}(t, y^1, y^2, y^3)$  of a point of the medium. Applying the partial derivative  $\partial/\partial y^k$  to (3.10) and taking into account (3.8), we derive another useful equality:

$$\frac{\partial \bar{T}_k^q}{\partial t} = - \sum_{r=1}^3 \frac{\partial(v^r \bar{T}_r^q)}{\partial y^k}. \quad (3.11)$$

As for the equalities (3.6) and (3.7), now they are written as follows:

$$\begin{aligned} T_k^i &= \sum_{q=1}^3 T_q^i(\tau, \bar{y}^1, \bar{y}^2, \bar{y}^3) \bar{T}_k^q, \\ \hat{T}_k^i &= \sum_{q=1}^3 \hat{T}_q^i(\tau, \bar{y}^1, \bar{y}^2, \bar{y}^3) \bar{T}_k^q. \end{aligned} \quad (3.12)$$

**Theorem 3.1.** *In the case of frozen dislocations the density of the Burgers vector  $\rho$  obeys the following evolution rule:*

$$\rho_k^i = \det \bar{T} \sum_{p=1}^3 \sum_{q=1}^3 \sum_{r=1}^3 \rho_q^i(\tau, \bar{y}^1, \bar{y}^2, \bar{y}^3) g^{qr} \bar{S}_r^p g_{pk}. \quad (3.13)$$

Here  $g_{pk}$  and  $g^{qr}$  are the components of the metric tensor and the dual metric tensor respectively (see [2] for details).

*Proof.* The proof is pure calculations. The density of the Burgers vector is defined by the formula (1.3). In coordinate form this formula is written as

$$\rho_k^i = \sum_{r=1}^3 \sum_{p=1}^3 \sum_{q=1}^3 g_{kr} \omega^{rpq} \frac{\partial \hat{T}_q^i}{\partial y^p}, \quad (3.14)$$

where  $\omega^{rpq}$  are the components of the so-called *volume tensor* (see [2]). The quantities  $\omega^{rpq}$  in (3.14) and the quantities  $g^{qr}$  and  $g_{pk}$  in (3.13) are constants because we chose the Cartesian coordinates  $y^1, y^2, y^3$  (see Fig. 3.1). From (3.12) we derive

$$\frac{\partial \hat{T}_q^i}{\partial y^p} = \sum_{m=1}^3 \frac{\partial \hat{T}_m^i(\tau, \bar{y}^1, \bar{y}^2, \bar{y}^3)}{\partial \bar{y}^p} \bar{T}_q^m + \sum_{m=1}^3 \hat{T}_m^i(\tau, \bar{y}^1, \bar{y}^2, \bar{y}^3) \frac{\partial \bar{T}_q^m}{\partial y^p}. \quad (3.15)$$

Then we apply the chain rule to the first term and the formula (3.8) to the second term in the right hand side of the equality (3.15). As a result we get

$$\frac{\partial \hat{T}_q^i}{\partial y^p} = \sum_{m=1}^3 \sum_{n=1}^3 \frac{\partial \hat{T}_m^i(\tau, \bar{y}^1, \bar{y}^2, \bar{y}^3)}{\partial \bar{y}^n} \frac{\partial \bar{y}^n}{\partial y^p} \bar{T}_q^m + \sum_{m=1}^3 \hat{T}_m^i(\tau, \bar{y}^1, \bar{y}^2, \bar{y}^3) \frac{\partial^2 \bar{y}^m}{\partial y^p \partial y^q}.$$

Applying the formula (3.8) again, we derive

$$\frac{\partial \hat{T}_q^i}{\partial y^p} = \sum_{m=1}^3 \sum_{n=1}^3 \frac{\partial \hat{T}_m^i(\tau, \bar{y}^1, \bar{y}^2, \bar{y}^3)}{\partial \bar{y}^n} \bar{T}_p^n \bar{T}_q^m + \sum_{m=1}^3 \hat{T}_m^i(\tau, \bar{y}^1, \bar{y}^2, \bar{y}^3) \frac{\partial^2 \bar{y}^m}{\partial y^p \partial y^q}.$$

Now we substitute the above equality into (3.14). The second term in its right hand side is symmetric in  $p$  and  $q$ . It vanishes when substituted into (3.14) because of the skew symmetry of  $\omega^{rpq}$  (see [2]). For  $\rho_k^i$  now we get

$$\rho_k^i = \sum_{r=1}^3 \sum_{p=1}^3 \sum_{q=1}^3 \sum_{m=1}^3 \sum_{n=1}^3 g_{kr} \omega^{rpq} \frac{\partial \hat{T}_m^i(\tau, \bar{y}^1, \bar{y}^2, \bar{y}^3)}{\partial \bar{y}^n} \bar{T}_p^n \bar{T}_q^m. \quad (3.16)$$

In the next step we use the following well-known identity:

$$\det \bar{T} \cdot \omega^{lnm} = \sum_{r=1}^3 \sum_{p=1}^3 \sum_{q=1}^3 \omega^{rpq} \bar{T}_r^l \bar{T}_p^n \bar{T}_q^m.$$

Since  $\bar{S} = \bar{T}^{-1}$ , from this identity we immediately derive

$$\det \bar{T} \sum_{l=1}^3 \bar{S}_l^r \omega^{lnm} = \sum_{p=1}^3 \sum_{q=1}^3 \omega^{rpq} \bar{T}_p^n \bar{T}_q^m. \quad (3.17)$$

Comparing (3.16) and (3.17) we can write the following formula for  $\rho_k^i$ :

$$\rho_k^i = \det \bar{T} \sum_{r=1}^3 \sum_{l=1}^3 \sum_{m=1}^3 \sum_{n=1}^3 g_{kr} \bar{S}_l^r \omega^{lnm} \frac{\partial \hat{T}_m^i(\tau, \bar{y}^1, \bar{y}^2, \bar{y}^3)}{\partial \bar{y}^n}. \quad (3.18)$$

Now let's rewrite the formula (3.14) for the time instant  $t = \tau$ :

$$\rho_q^i(\tau, \bar{y}^1, \bar{y}^2, \bar{y}^3) = \sum_{l=1}^3 \sum_{n=1}^3 \sum_{m=1}^3 g_{ql} \omega^{lnm} \frac{\partial \hat{T}_m^i(\tau, \bar{y}^1, \bar{y}^2, \bar{y}^3)}{\partial \bar{y}^n}. \quad (3.19)$$

Comparing (3.18) and (3.19) we can rewrite (3.18) as follows:

$$\rho_k^i = \det \bar{T} \sum_{r=1}^3 \sum_{l=1}^3 \sum_{q=1}^3 g_{kr} \bar{S}_l^r g^{lq} \rho_q^i(\tau, \bar{y}^1, \bar{y}^2, \bar{y}^3). \quad (3.20)$$

Now it is easy to see that, in essential, (3.20) coincides with the equality (3.13) which we needed to prove. So the proof is over.  $\square$

In the case of frozen dislocations the motion of the dislocation lines is completely determined by the motion of the medium. Therefore,  $\mathbf{j}$  should be expressed through  $\mathbf{v}$  and  $\boldsymbol{\rho}$ . In order to find this expression let's remember that  $\mathbf{j}$  by definition

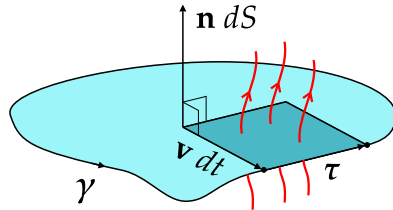


Fig. 3.2

is the total Burgers vector of the moving dislocations that cross the unit length of a contour  $\gamma$  per unit time. It is clear that all of the dislocations passing through the dark parallelogram on Fig. 3.2 will cross the segment  $\tau$  during the next time interval  $dt$ . The total Burgers vector of such dislocations is determined by formula

$$db^i = \sum_{k=1}^3 \rho_k^i n^k dS, \quad (3.21)$$

where  $\mathbf{n} dS = [\mathbf{v}, \boldsymbol{\tau}] dt$  and  $[\mathbf{v}, \boldsymbol{\tau}]$  is the cross product (vector product) of  $\mathbf{v}$  and  $\boldsymbol{\tau}$ . On the other hand, the same Burgers vector is given by another formula

$$db^i = \sum_{c=1}^3 j_c^i \tau^c dt. \quad (3.22)$$

If we write the equality  $\mathbf{n} dS = [\mathbf{v}, \boldsymbol{\tau}] dt$  in coordinate form

$$n^k dS = \sum_{s=1}^3 \sum_{l=1}^3 \sum_{c=1}^3 g^{ks} \omega_{slc} v^l \tau^c dt,$$

then from (3.21) and (3.22) we derive the following formula for  $\mathbf{j}$ :

$$j_c^i = \sum_{s=1}^3 \sum_{l=1}^3 \sum_{k=1}^3 \omega_{slc} v^l g^{sk} \rho_k^i. \quad (3.23)$$

**Theorem 3.2.** *In the case of frozen dislocation  $\boldsymbol{\rho}$  and  $\mathbf{j}$  are related to each other by the formula (3.23).*

In order to continue the above calculations, now we substitute (3.16) into (3.23). As a result for the components of  $\mathbf{j}$  we derive the following expression:

$$j_c^i = \sum_{s=1}^3 \sum_{l=1}^3 \sum_{p=1}^3 \sum_{q=1}^3 \sum_{m=1}^3 \sum_{n=1}^3 \omega_{slc} v^l \omega^{spq} \frac{\partial \hat{T}_m^i(\tau, \bar{y}^1, \bar{y}^2, \bar{y}^3)}{\partial \bar{y}^n} \bar{T}_p^n \bar{T}_q^m.$$

Applying the well-known identity<sup>1</sup>

$$\sum_{s=1}^3 \omega_{slc} \omega^{spq} = \delta_l^p \delta_c^q - \delta_c^p \delta_l^q$$

to this expression we can decrease the multiplicity of summation symbols in it:

$$\begin{aligned} j_c^i &= \sum_{p=1}^3 \sum_{m=1}^3 \sum_{n=1}^3 \bar{T}_c^m v^p \bar{T}_p^n \frac{\partial \hat{T}_m^i(\tau, \bar{y}^1, \bar{y}^2, \bar{y}^3)}{\partial \bar{y}^n} - \\ &- \sum_{q=1}^3 \sum_{m=1}^3 \sum_{n=1}^3 \bar{T}_c^m v^q \bar{T}_q^n \frac{\partial \hat{T}_m^i(\tau, \bar{y}^1, \bar{y}^2, \bar{y}^3)}{\partial \bar{y}^n}. \end{aligned} \quad (3.24)$$

**Theorem 3.3.** *In the case of frozen dislocation  $\mathbf{j}$  and  $\hat{\mathbf{T}}$  are related to each other by the formula (3.24).*

Now we are going to calculate the time derivative of  $\hat{\mathbf{T}}$  using (3.12). Upon doing this, we will be able to verify the equality (1.4) and calculate  $\mathbf{w}$ :

$$\frac{\partial \hat{T}_c^i}{\partial t} = \sum_{q=1}^3 \frac{\partial \hat{T}_q^i(\tau, \bar{y}^1, \bar{y}^2, \bar{y}^3)}{\partial t} \bar{T}_c^q + \sum_{q=1}^3 \hat{T}_q^i(\tau, \bar{y}^1, \bar{y}^2, \bar{y}^3) \frac{\partial \bar{T}_c^q}{\partial t} =$$

<sup>1</sup> Here  $\delta_l^p$ ,  $\delta_c^q$ ,  $\delta_c^p$ , and  $\delta_l^q$  are the Kronecker symbols.



$$\begin{aligned}
&= \sum_{q=1}^3 \sum_{r=1}^3 \frac{\partial \hat{T}_q^i(\tau, \bar{y}^1, \bar{y}^2, \bar{y}^3)}{\partial \bar{y}^r} \frac{\partial \bar{y}^r}{\partial t} \bar{T}_c^q - \sum_{q=1}^3 \sum_{p=1}^3 \hat{T}_q^i(\tau, \bar{y}^1, \bar{y}^2, \bar{y}^3) \frac{\partial (v^p \bar{T}_p^q)}{\partial y^c} = \\
&= - \sum_{q=1}^3 \sum_{r=1}^3 \sum_{p=1}^3 \frac{\partial \hat{T}_q^i(\tau, \bar{y}^1, \bar{y}^2, \bar{y}^3)}{\partial \bar{y}^r} v^p \bar{T}_p^r \bar{T}_c^q - \sum_{q=1}^3 \sum_{p=1}^3 \hat{T}_q^i(\tau, \bar{y}^1, \bar{y}^2, \bar{y}^3) \frac{\partial (v^p \bar{T}_p^q)}{\partial y^c}.
\end{aligned}$$

In the above calculations we used the identities (3.10) and (3.11). In order to make the resulting expression similar to (3.24) we change some summation indices:

$$\begin{aligned}
\frac{\partial \hat{T}_c^i}{\partial t} &= - \sum_{p=1}^3 \sum_{m=1}^3 \sum_{n=1}^3 \bar{T}_c^m v^p \bar{T}_p^n \frac{\partial \hat{T}_m^i(\tau, \bar{y}^1, \bar{y}^2, \bar{y}^3)}{\partial \bar{y}^n} - \\
&- \sum_{m=1}^3 \sum_{q=1}^3 \hat{T}_m^i(\tau, \bar{y}^1, \bar{y}^2, \bar{y}^3) \frac{\partial v^q}{\partial y^c} \bar{T}_q^m - \sum_{m=1}^3 \sum_{q=1}^3 \hat{T}_m^i(\tau, \bar{y}^1, \bar{y}^2, \bar{y}^3) v^q \frac{\partial \bar{T}_q^m}{\partial y^c}.
\end{aligned} \tag{3.25}$$

Now let's return to the equation (1.4). In coordinate form it looks like

$$\frac{\partial \hat{T}_c^i}{\partial t} + j_c^i = - \frac{\partial w^i}{\partial y^c}. \tag{3.26}$$

Substituting (3.24) and (3.25) into (3.26), we derive

$$\begin{aligned}
\frac{\partial w^i}{\partial y^c} &= \sum_{q=1}^3 \sum_{m=1}^3 \sum_{n=1}^3 \bar{T}_c^n v^q \bar{T}_q^m \frac{\partial \hat{T}_m^i(\tau, \bar{y}^1, \bar{y}^2, \bar{y}^3)}{\partial \bar{y}^n} + \\
&+ \sum_{m=1}^3 \sum_{q=1}^3 \hat{T}_m^i(\tau, \bar{y}^1, \bar{y}^2, \bar{y}^3) \frac{\partial v^q}{\partial y^c} \bar{T}_q^m + \sum_{m=1}^3 \sum_{q=1}^3 \hat{T}_m^i(\tau, \bar{y}^1, \bar{y}^2, \bar{y}^3) v^q \frac{\partial \bar{T}_q^m}{\partial y^c}.
\end{aligned} \tag{3.27}$$

Let's remember the formula (3.8) and apply it to  $\bar{T}_c^n$  in (3.27):

$$\begin{aligned}
\frac{\partial w^i}{\partial y^c} &= \sum_{m=1}^3 \sum_{q=1}^3 \left( \sum_{n=1}^3 \frac{\partial \hat{T}_m^i(\tau, \bar{y}^1, \bar{y}^2, \bar{y}^3)}{\partial \bar{y}^n} \frac{\partial \bar{y}^n}{\partial y^c} \right) v^q \bar{T}_q^m + \\
&+ \sum_{m=1}^3 \sum_{q=1}^3 \hat{T}_m^i(\tau, \bar{y}^1, \bar{y}^2, \bar{y}^3) \frac{\partial v^q}{\partial y^c} \bar{T}_q^m + \sum_{m=1}^3 \sum_{q=1}^3 \hat{T}_m^i(\tau, \bar{y}^1, \bar{y}^2, \bar{y}^3) v^q \frac{\partial \bar{T}_q^m}{\partial y^c}.
\end{aligned} \tag{3.28}$$

Looking at the right hand side of (3.28), one easily recognizes the partial derivative of the product of three terms. Therefore, (3.28) is written as

$$\frac{\partial w^i}{\partial y^c} = \frac{\partial}{\partial y^c} \left( \sum_{m=1}^3 \sum_{q=1}^3 \hat{T}_m^i(\tau, \bar{y}^1, \bar{y}^2, \bar{y}^3) v^q \bar{T}_q^m \right).$$

Now, if we remember the formula (3.12), we obtain an even more simple equality:

$$\frac{\partial w^i}{\partial y^c} = \frac{\partial}{\partial y^c} \left( \sum_{q=1}^3 v^q \hat{T}_q^i \right). \quad (3.29)$$

Integrating the equality (3.29), we define the components of the vector  $\mathbf{w}$ :

$$w^i = \sum_{q=1}^3 v^q \hat{T}_q^i. \quad (3.30)$$

This definition is unique up to some inessential terms depending only on the time variable:  $w^i \rightarrow w^i + \omega^i(t)$ . These terms can be omitted since  $\mathbf{w}$  is used only in the form of its gradient (see the equation (1.4)).

**Theorem 3.4.** *In the case of frozen dislocation the vectorial parameter  $\mathbf{w}$  is determined by the formula (3.30).*

**Theorem 3.5.** *In the case of frozen dislocation the time evolution of  $\hat{\mathbf{T}}$ ,  $\boldsymbol{\rho}$ ,  $\mathbf{j}$ , and  $\mathbf{w}$  is determined by the formulas (3.12), (3.13), (3.24), and (3.30) respectively. Due to these formulas both differential equations (1.3) and (1.4) are fulfilled identically.*

#### 4. THE ANALOGY TO ELECTROMAGNETISM AND THE GAUGE TRANSFORMATIONS.

The basic differential equations (1.3) and (1.4) describing the kinematics of dislocations in crystals are similar to the equations expressing the electric and magnetic fields  $\mathbf{E}$  and  $\mathbf{H}$  through the scalar potential  $\varphi$  and the vector potential  $\mathbf{A}$ :

$$\mathbf{H} = \text{rot } \mathbf{A}, \quad \mathbf{E} = -\text{grad } \varphi - \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t}. \quad (4.1)$$

These differential equations (4.1) admit the following gauge transformations

$$\mathbf{A} \rightarrow \mathbf{A} + \text{grad } \psi, \quad \varphi \rightarrow \varphi - \frac{1}{c} \frac{\partial \psi}{\partial t} \quad (4.2)$$

that change potentials, but do not change the actual physical fields  $\mathbf{E}$  and  $\mathbf{H}$  (see [3] for the reference). By analogy we can write the gauge transformations

$$\hat{\mathbf{T}} \rightarrow \hat{\mathbf{T}} + \text{grad } \psi, \quad \mathbf{w} \rightarrow \mathbf{w} - \frac{\partial \psi}{\partial t} \quad (4.3)$$

for  $\hat{\mathbf{T}}$  and  $\mathbf{w}$  in the differential equations (1.3) and (1.4). Despite to the striking similarity of (4.2) and (4.3) their roles are absolutely different. The matter is that  $\mathbf{A}$  and  $\varphi$  in the electromagnetism are not actual physical fields, they are derived mathematically from the Maxwell equations (see [3]). Unlike  $\mathbf{A}$ , the incompatible distorsion field  $\hat{\mathbf{T}}$  is an actual physical field describing the state of a medium. Applying the gauge transformation (4.3) to it, we get another field describing some other state of the medium. For this reason, the nonlinear theory of dislocations is not a gauge theory or, at least, it is a gauge theory with the different gauge group, other than (4.3).

The physical nature of the parameter  $\mathbf{w}$  is still misty. The following conjecture opens a way to clarify it. It is based on the theorem 3.4.

**Conjecture 4.1.** *The vectorial parameter  $\mathbf{w}$  in the equation (1.4) is always determined by the formula (3.30).*

If we admit the conjecture 4.1, then the differential equation (1.4) describing the time evolution of the incompatible distortion tensor  $\hat{\mathbf{T}}$  can be written as

$$\frac{\partial \hat{T}_k^i}{\partial t} + \sum_{p=1}^3 \nabla_p \hat{T}_k^{pi} = -j_k^i, \quad (4.4)$$

where  $\nabla_p = \partial/\partial y^p$  since we use the Cartesian coordinates  $y^1, y^2, y^3$ , and

$$\hat{T}_k^{pi} = \sum_{q=1}^3 v^q \hat{T}_q^i \delta_k^p. \quad (4.5)$$

In this form (4.4) the differential equation (1.4) is quite similar to the balance equations which are traditional in the mechanics of continuous media (see the mass balance equation, the momentum balance equation, and the energy balance equation in [4] for comparison). The tensor (4.5) is interpreted as the density of the incompatible distortion flow.

A conjecture is not a theorem yet. Therefore, we need to discuss all of the possible options for the parameter  $\mathbf{w}$  in the equation (1.4):

- (1) the parameter  $\mathbf{w}$  is determined by the formula (3.30);
- (2) the parameter  $\mathbf{w}$  is determined by the formula

$$w^i = \sum_{q=1}^3 v^q T_q^i = -\frac{\partial T_q^i}{\partial t}; \quad (4.6)$$

- (3) the parameter  $\mathbf{w}$  is determined by some formula other than (3.30) and (4.6);
- (4) the parameter  $\mathbf{w}$  is an independent parameter of a dislocated medium.

The option (1) is my favorite option. It is supported by the above two examples considered in the sections 2 and 3.

The option (2) is often chosen in many papers. Substituting (4.6) into the equation (1.4), we obtain the following equality:

$$\frac{\partial \mathbf{T}}{\partial t} = \frac{\partial \hat{\mathbf{T}}}{\partial t} + \mathbf{j}. \quad (4.7)$$

The equality (4.7) is referred to as the *additive decomposition* of the total distortion into the *elastic* and *plastic* parts. It is consistent in the linear theory, where both  $\mathbf{T}$  and  $\hat{\mathbf{T}}$  are approximately equal to the unit matrix. In the nonlinear case the equality (4.6) is not a valid option since it contradicts the theorem 3.4. The *multiplicative decomposition* of the total deformation tensor suggested in [4] is more preferable.

The choice of the option (3) or the option (4) produces more questions than the answers. One should carry out a special research in order to exclude both these options or to prove one of them.

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