

DYNAMICAL SYSTEMS ACCEPTING THE NORMAL SHIFT.

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March 9, 1993.

ABSTRACT. Classical Bianchi-Lie, Bäcklund and Darboux transformations are considered. Their generalizations for the dynamical systems are discussed. For the transformation being the generalization of the normal shift the special class of dynamical systems is defined. The effective criterion for separating such systems in form of partial differential equations is found.

1. INTRODUCTION

Bäcklund transformations which are well known to the specialists in integrable nonlinear equations originally first arose in the works of classics of differential geometry in last century. Let's remind some of their results. Consider 2-dimensional surface S in \mathbb{R}^3 with the standard scalar product. Let us map each point M of the surface S onto the point M' of the other surface S' so that the following conditions are fulfilled

- (1) The distance $|MM'|$ is constant and it is equal to R ;
- (2) The tangent planes τ and τ' of the surfaces S and S' in the points M and M' are orthogonal: $\angle(\tau, \tau') = 90^\circ$;
- (3) The segment MM' lies in both tangents planes τ and τ' of the surfaces S and S' respectively.

Bianchi [1] showed that if S is the surface of constant negative Gaussian curvature $K = -1/R^2$ then in assumptions (1)–(3) the second surface S' also is the surface of constant negative Gaussian curvature $K' = -1/R^2$. Lie [2] strengthened Bianchi's result having shown that construction (1)–(3) can be realized only on the surface S of constant negative Gaussian curvature $K = -1/R^2$.

Bäcklund [3] generalized the construction (1)–(3) having substituted condition (2) by more weak condition of constancy of the angle between the tangent planes τ and τ' . Such kind of transformation also is defined only on the surfaces of constant negative curvature and it leads to the surfaces S' of the same curvature. Darboux [4] offered further generalization of this construction having substituted condition (3) by condition of constancy of the angles between line MM' and both tangent planes τ and τ' . However thereby some differences there appeared: Darboux transformation is realized on the surfaces where some linear combination of Gaussian and mean curvatures is constant. The result of transformation is the surface where some other linear combination of Gaussian and mean curvatures is constant.

Differential equations which are obtained in considering the transformations (1)–(3) as well as their generalizations became the objects of the numerous investigations. For the modern state of such investigations one can enquire the monographs [5] and [6]. The generalization of the above transformations for the multidimensional spaces and submanifolds in them is made in papers [7], [8] and [9]. The generalizations for the submanifolds in the arbitrary Riemannian manifold of constant sectional curvature is considered in [10] and [11].

The normal shift (or the Bonnet transformation) is some particular case in the general Darboux construction. In this case tangent planes τ and τ' are parallel while the segment MM' is orthogonal to them both. This is degenerate particular case since it can be realized on any surface imposing no limitations for its curvatures.

In this paper we discuss the generalization of the Bianchi-Lie, Bäcklund and Darboux transformation for the spacially anisotropic case substituting the straight line segment or geodesic segment MM' by the segment of trajectory for some dynamical system. Let's consider the following dynamical system in \mathbb{R}^n

$$(1.1) \quad \ddot{\mathbf{r}} = \mathbf{F}(\mathbf{r}, \dot{\mathbf{r}})$$

This is the natural second order dynamical system defined by the force \mathbf{F} . For each point M on some submanifold $S \subset \mathbb{R}^n$ let us consider the particle starting from it with some initial velocity \mathbf{v} . In the end of some time interval (same for all particles) these particles form some other submanifold $S' \subset \mathbb{R}^n$. In most general form the problem may be stated as follows: what kind of limitations for the dynamical system (1.1) itself and for the choice of submanifold S and initial velocities of particles on it arise if one require the angles defining the mutual arrangement of tangent spaces τ, τ' and particle trajectories to be constant. Such problem has a lot of possible specializations one of which leading to the meaningful results is considered below.

2. THE NORMAL SHIFT ALONG THE DYNAMICAL SYSTEM.

Let's consider the dynamical system (1.1) in Euclidean space \mathbb{R}^n with the standard scalar product. Let S be the submanifold of codimension 1 in \mathbb{R}^n and let \mathbf{n} be the vector field of unit normal vectors on S . We direct the initial velocity of particles along \mathbf{n} defining the modulus of velocity as some smooth function $v = v(M)$ on the submanifold S . Then the dynamical system (1.1) produces the family of submanifolds S_t

$$(2.1) \quad f_t : S \longrightarrow S_t$$

together with the diffeomorphisms f_t binding S with S_t .

Definition 1. Each transformation $f = f_t$ of the family (2.1) is called the normal shift along the dynamical system (1.1) if each trajectory of (1.1) crosses each submanifold S_t along its normal vector \mathbf{n} .

Definition 2. Dynamical system (1.1) is called the dynamical system accepting the normal shift of submanifolds of codimension 1 if for any submanifold S of codimension 1 there is the function $v = v(M)$ on S such that the transformation (2.1) defined by the system (1.1) and the initial velocity function $|\mathbf{v}(M)| = v(M)$ is the transformation of normal shift.

Note that the transformation (2.1) may implement the normal shift for some particular submanifolds even when they do not satisfy the definition 2. Dynamical systems accepting the normal shift for arbitrary submanifold form the special class of dynamical systems narrow enough to be described in much details. In the following two sections we consider such dynamical systems in \mathbb{R}^2 and derive the partial differential equation for the force function \mathbf{F} of them.

3. DYNAMICAL SYSTEMS IN \mathbb{R}^2 ACCEPTING THE NORMAL SHIFT.

Let's consider the second order dynamical system (1.1) in Euclidean space \mathbb{R}^2 with the standard scalar product. Phase space of the system (1.1) is four-dimensional in this case. The locus of points where $\mathbf{v} = \dot{\mathbf{r}} = 0$ is the two-dimensional plane in it. Everywhere out of this locus we define the unit vector

$$(3.1) \quad \mathbf{N} = \mathbf{N}(\mathbf{v}) = \frac{\mathbf{v}}{|\mathbf{v}|}$$

and the unit vector $\mathbf{M}(\mathbf{v})$ perpendicular to $\mathbf{N}(\mathbf{v})$. Using (3.1) the right hand side of (1.1) can be rewritten as follows

$$(3.2) \quad \mathbf{F}(\mathbf{r}, \mathbf{v}) = A(\mathbf{r}, \mathbf{v})\mathbf{N}(\mathbf{v}) + B(\mathbf{r}, \mathbf{v})\mathbf{M}(\mathbf{v})$$

Cartesian components of $\mathbf{N}(\mathbf{v})$ and $\mathbf{M}(\mathbf{v})$ satisfy the following differential equations

$$(3.3) \quad \frac{\partial N^k}{\partial v^i} = \frac{M_i M^k}{|\mathbf{v}|} \quad \frac{\partial M^k}{\partial v^i} = -\frac{M_i N^k}{|\mathbf{v}|}$$

Gradients of the functions $A(\mathbf{r}, \mathbf{v})$ and $B(\mathbf{r}, \mathbf{v})$ in (3.2) also can be expressed in terms of components of $\mathbf{N}(\mathbf{v})$ and $\mathbf{M}(\mathbf{v})$

$$(3.4) \quad \frac{\partial A}{\partial r^i} = \alpha_1 N_i + \alpha_2 M_i \quad \frac{\partial A}{\partial v^i} = \alpha_3 N_i + \alpha_4 M_i$$

$$(3.5) \quad \frac{\partial B}{\partial r^i} = \beta_1 N_i + \beta_2 M_i \quad \frac{\partial B}{\partial v^i} = \beta_3 N_i + \beta_4 M_i$$

For the dynamical system (1.1) with the right hand side (3.2) we consider the Cauchy problem with the initial data depending on some scalar parameter s

$$(3.6) \quad \mathbf{r}(t, s)|_{t=0} = \mathbf{r}(s) \quad \partial \mathbf{r}(t, s)|_{t=0} = \mathbf{v}(s)$$

Let $\boldsymbol{\tau}(t, s)$ be the derivative of $\mathbf{r}(t, s)$ by the parameter s . Differentiating (1.1) we derive the following equation for the components of the vector $\boldsymbol{\tau}(t, s)$

$$(3.7) \quad \ddot{\tau}^k = \frac{\partial F^k}{\partial r^i} \tau^i + \frac{\partial F^k}{\partial v^i} \dot{\tau}^i$$

Using the expression (3.2), the differential equations (3.3) and the formulae (3.4) and (3.5) we may write the partial derivatives in (3.7) in the following form

$$(3.8) \quad \frac{\partial F^k}{\partial r^i} = (\alpha_1 N_i + \alpha_2 M_i) N^k + (\beta_1 N_i + \beta_2 M_i) M^k$$

$$(3.9) \quad \frac{\partial F^k}{\partial v^i} = \alpha_3 N_i N^k + \left(\alpha_4 - \frac{B}{|\mathbf{v}|} \right) M_i N^k + \beta_3 N_i M^k + \left(\beta_4 + \frac{A}{|\mathbf{v}|} \right) M_i M^k$$

Time derivative for the vector \mathbf{v} is defined by the equation (1.1). For the length of this vector then we have

$$(3.10) \quad \partial_t |\mathbf{v}| = A \quad \partial_t (|\mathbf{v}|^{-1}) = -\frac{A}{|\mathbf{v}|^2}$$

For the derivatives of A and B in (3.4) and (3.5) we obtain

$$(3.11) \quad \begin{aligned} \partial_t A &= \alpha_1 |\mathbf{v}| + \alpha_3 A + \alpha_4 B \\ \partial_t B &= \beta_1 |\mathbf{v}| + \beta_3 A + \beta_4 B \end{aligned}$$

Taking into account (3.4) and (3.5) from (3.3) and (3.10) we derive the formulae for the time derivatives of \mathbf{N} and \mathbf{M}

$$(3.12) \quad \partial_t \mathbf{N} = \frac{B\mathbf{N}}{|\mathbf{v}|} \quad \partial_t \mathbf{M} = -\frac{B\mathbf{N}}{|\mathbf{v}|}$$

Time dynamics of the vector $\boldsymbol{\tau}$ due to (3.7) and the relationships (3.12) determine the dynamics of the scalar products $\langle \boldsymbol{\tau}, \mathbf{N} \rangle$ and $\langle \boldsymbol{\tau}, \mathbf{M} \rangle$. Let's denote them as follows

$$(3.13) \quad \langle \boldsymbol{\tau}, \mathbf{N} \rangle = \varphi \quad \langle \boldsymbol{\tau}, \mathbf{M} \rangle = \psi$$

Differentiating (3.13) and taking into account (3.12) we obtain

$$(3.14) \quad \begin{aligned} \langle \partial_t \boldsymbol{\tau}, \mathbf{N} \rangle &= \partial_t \varphi - \frac{B}{|\mathbf{v}|} \psi \\ \langle \partial_t \boldsymbol{\tau}, \mathbf{M} \rangle &= \partial_t \psi + \frac{B}{|\mathbf{v}|} \varphi \end{aligned}$$

Differentiating the left hand sides in (3.14) we get

$$(3.15) \quad \begin{aligned} \partial_t \langle \partial_t \boldsymbol{\tau}, \mathbf{N} \rangle &= \langle \partial_{tt} \boldsymbol{\tau}, \mathbf{N} \rangle + \langle \partial_t \boldsymbol{\tau}, \partial_t \mathbf{N} \rangle = \\ &= \langle \partial_{tt} \boldsymbol{\tau}, \mathbf{N} \rangle + \frac{B}{|\mathbf{v}|} \cdot \partial_t \psi + \frac{B^2}{|\mathbf{v}|^2} \varphi \end{aligned}$$

$$(3.16) \quad \begin{aligned} \partial_t \langle \partial_t \boldsymbol{\tau}, \mathbf{M} \rangle &= \langle \partial_{tt} \boldsymbol{\tau}, \mathbf{M} \rangle + \langle \partial_t \boldsymbol{\tau}, \partial_t \mathbf{M} \rangle = \\ &= \langle \partial_{tt} \boldsymbol{\tau}, \mathbf{M} \rangle - \frac{B}{|\mathbf{v}|} \partial_t \varphi + \frac{B^2}{|\mathbf{v}|^2} \psi \end{aligned}$$

Differentiating the right hand sides in the same equations (3.14) and taking into account (3.10), (3.11) and (3.12) we have

$$(3.17) \quad \begin{aligned} \partial_t \langle \partial_t \boldsymbol{\tau}, \mathbf{N} \rangle &= \partial_{tt} \varphi - \frac{B}{|\mathbf{v}|} \partial_t \psi + \frac{BA}{|\mathbf{v}|^2} \psi - \\ &\quad - \beta_1 \psi - \beta_3 \frac{A}{|\mathbf{v}|} \psi - \beta_4 \frac{B}{|\mathbf{v}|} \psi \end{aligned}$$

$$(3.18) \quad \begin{aligned} \partial_t \langle \partial_t \boldsymbol{\tau}, \mathbf{M} \rangle &= \partial_{tt} \psi + \frac{B}{|\mathbf{v}|} \partial_t \varphi - \frac{BA}{|\mathbf{v}|^2} \varphi + \\ &\quad + \beta_1 \varphi + \beta_3 \frac{A}{|\mathbf{v}|} \varphi + \beta_4 \frac{B}{|\mathbf{v}|} \varphi \end{aligned}$$

The second derivative of the vector $\boldsymbol{\tau}$ in (3.15) and (3.16) can be calculated on the base of (3.7), (3.8) and (3.9). Then we get

$$(3.19) \quad \begin{aligned} \langle \partial_{tt} \boldsymbol{\tau}, \mathbf{N} \rangle &= \alpha_1 \varphi + \alpha_2 \psi + \alpha_3 \left(\partial_t \varphi - \frac{B}{|\mathbf{v}|} \varphi \right) + \\ &\quad + \left(\alpha_4 - \frac{B}{|\mathbf{v}|} \right) \cdot \left(\partial_t \psi + \frac{B}{|\mathbf{v}|} \varphi \right) \end{aligned}$$

$$(3.20) \quad \begin{aligned} \langle \partial_{tt} \boldsymbol{\tau}, \mathbf{M} \rangle &= \beta_1 \varphi + \beta_2 \psi + \beta_3 \left(\partial_t \varphi - \frac{B}{|\mathbf{v}|} \psi \right) + \\ &\quad + \left(\beta_4 + \frac{A}{|\mathbf{v}|} \right) \cdot \left(\partial_t \psi + \frac{B}{|\mathbf{v}|} \varphi \right) \end{aligned}$$

As a result of equating (3.15) with (3.17) and (3.16) with (3.18) after substituting (3.19) and (3.20) for φ and ψ we obtain

$$(3.21) \quad \begin{aligned} \partial_{tt} \varphi - \frac{B}{|\mathbf{v}|} \partial_t \psi + \frac{BA}{|\mathbf{v}|^2} \psi - \beta_1 \psi - \beta_3 \frac{A\psi}{|\mathbf{v}|} - \beta_4 \frac{B\psi}{|\mathbf{v}|} = \\ = \alpha_1 \varphi + \alpha_2 \psi + \alpha_3 \left(\partial_t \varphi - \frac{B}{|\mathbf{v}|} \psi \right) + \alpha_4 \left(\partial_t \psi + \frac{B}{|\mathbf{v}|} \varphi \right) \end{aligned}$$

$$(3.22) \quad \begin{aligned} \partial_{tt} \psi + \frac{B}{|\mathbf{v}|} \partial_t \varphi - \frac{BA}{|\mathbf{v}|^2} \psi + \beta_1 \varphi + \beta_3 \frac{A\varphi}{|\mathbf{v}|} + \beta_4 \frac{B\varphi}{|\mathbf{v}|} = \\ = \beta_1 \varphi + \beta_2 \psi + \left(\beta_3 - \frac{B}{|\mathbf{v}|} \right) \cdot \left(\partial_t \varphi - \frac{B}{|\mathbf{v}|} \psi \right) + \\ + \alpha_4 \left(\beta_4 + \frac{A}{|\mathbf{v}|} \right) \cdot \left(\partial_t \psi + \frac{B}{|\mathbf{v}|} \varphi \right) \end{aligned}$$

Let the dynamical system (1.1) with the force field (3.2) be accepting the normal shift in \mathbb{R}^2 . Submanifolds of codimension 1 in \mathbb{R}^2 are the plane curves. It is convenient to use the natural parameter on them measuring the arc length referenced to some fixed point on the curve. Let the parameter s in the Cauchy problem initial data (3.6) do coincide with the natural parameter on the curve. Then for the transformations of the normal shift these initial data may be rewritten as follows

$$(3.23) \quad \mathbf{r}(t, s)|_{t=0} = \mathbf{r}(s) \quad \partial_t \mathbf{r}(t, s)|_{t=0} = v(s) \mathbf{n}(s)$$

The derivative $\partial_s \mathbf{r}(s) = \boldsymbol{\tau}(s)$ is the unit tangent vector for the curve while $\mathbf{n}(s)$ is the unit normal vector for it.

From (3.23), (3.14) and from the orthogonality of the vectors $\boldsymbol{\tau}(s)$ and $\mathbf{n}(s)$ we derive the following initial data for the function φ introduced in (3.13)

$$(3.24) \quad \varphi|_{t=0} = 0 \quad \partial_t \varphi|_{t=0} = \partial_s v(s) + \frac{B}{v(s)}$$

By the proper choice of the function $v(s)$ in (3.23) and (3.24) (see also the definition 2) one can vanish the right hand side of the second initial data statement in (3.24). Such proper choice is defined by the following differential equation

$$(3.25) \quad v'_s = -\frac{B(\mathbf{r}(s), v(s)\mathbf{n}(s))}{v(s)}$$

The right hand side of (3.25) as well as the function $v(s)$ itself then depend on the form of the curve. Once the choice (3.25) for $v(s)$ is made the initial data (3.14) take the form

$$(3.26) \quad \varphi|_{t=0} = 0 \quad \partial_t \varphi|_{t=0} = 0$$

Now let's come back to the differential equations (3.21) and (3.22). They are linear with respect to φ and ψ and altogether they form the complete system of the differential equations. As for the initial data (3.26) they are not enough to determine the Cauchy problem for this system. However one can make the special choice of the functions A and B in (3.2) such that the coefficients of ψ and $\partial_t \psi$ in (3.21) vanish making (3.21) the separate equation of the second order with respect to the function φ . Because of the linearity the Cauchy problem (3.26) for this equation has the unique solution being identically zero. In this case vectors $\boldsymbol{\tau}$ and \mathbf{N} are perpendicular for all s and t which is just the condition of normal shift. The proper choice of A and B leading to this case is described by the following theorem.

Theorem 1. *Dynamical system (1.1) with the right hand side of the form (3.2) is the system accepting the normal shift in \mathbb{R}^2 if and only if the following relationships*

$$(3.27) \quad B = -|\mathbf{v}|\alpha_4$$

$$(3.28) \quad \frac{BA}{|\mathbf{v}|^2} - \beta_1 - \beta_3 \frac{A}{|\mathbf{v}|} - \beta_4 \frac{B}{|\mathbf{v}|} = \alpha_2 - \alpha_3 \frac{B}{|\mathbf{v}|}$$

are fulfilled. Here $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and $\beta_1, \beta_2, \beta_3, \beta_4$ are the coefficients in (3.4) and (3.5).

4. SOME EXAMPLES IN \mathbb{R}^2 .

Because of the derivatives in (3.4) and (3.5) the equations (3.27) and (3.28) are the system of the partial differential equations with respect to A and B defining in turn the force field \mathbf{F} according to (3.2). In order to make these equations more explicit one should use the explicit form of the vectors \mathbf{N} and \mathbf{M}

$$\mathbf{N} = \frac{1}{|\mathbf{v}|} \begin{vmatrix} v^1 \\ v^2 \end{vmatrix} \quad \mathbf{M} = \frac{1}{|\mathbf{v}|} \begin{vmatrix} -v^2 \\ v^1 \end{vmatrix}$$

For the function B then from (3.27) we may obtain its expression via the function A

$$(4.1) \quad B = v^2 \frac{\partial A}{\partial v^1} - v^1 \frac{\partial A}{\partial v^2}$$

Further substitution of (4.1) into (3.5) and (3.28) let us bring the system of equation (3.27) and (3.28) to the form of one equation with respect to A . We do not write this equation here because it is very huge. But in place of it we consider some examples when this equation can be simplified to rather observable size.

Case 1. Spatially homogeneous force field directed along the vector \mathbf{v} of velocity. Function A in this case does not depend on \mathbf{r} and B is equal to zero identically. From (4.1) then we obtain that A depends only on the modulus of velocity $A = A(|\mathbf{v}|)$. Therefore $\alpha_2 = 0$ and the equation (3.28) become the identity. Note that this case is trivial from geometrical point of view since trajectories of particles are straight lines and associated transformation in (2.1) coincides with the classical normal shift.

Case 2. Spatially homogeneous but anisotropic force field. Both functions A and B do not depend on \mathbf{r} . Let's denote $v = |\mathbf{v}|$ the modulus of velocity and denote via θ the angle between \mathbf{v} and some fixed direction in space. Then $B = -\partial_\theta A = -A_\theta$ and the equation (3.28) is written as follows

$$(4.2) \quad AA_\theta - vAA_{\theta v} + A_\theta A_{\theta\theta} = -vA_\theta A_v$$

Equation (4.2) has the particular solution with the separated variables $A = A(v) \cos(\theta)$. Force field then is of the form

$$(4.3) \quad \mathbf{F} = A(v)\mathbf{N} \cos(\theta) + A(v)\mathbf{M} \sin(\theta)$$

The modulus of force here $|\mathbf{F}| = A(v)$ is some arbitrary function of v . The direction of force \mathbf{F} form with the fixed direction in space the angle 2θ twice as greater than the angle between \mathbf{v} and that direction (see fig. 1).

Place for fig.1.

Place for fig.2.

Case 3. Spacially unhomogeneous force field with the central point. Here it is convenient to introduce new variables ρ , γ , v and θ according to the fig. 2

$$(4.4) \quad \begin{aligned} r^1 &= \rho \cos(\gamma) & v^1 &= v \cos(\gamma + \theta) \\ r^2 &= \rho \sin(\gamma) & v^2 &= v \sin(\gamma + \theta) \end{aligned}$$

From (4.4) it is not difficult to derive the following formulae for different functions in the equations (3.27) and (3.28)

$$\begin{aligned} \alpha_2 &= -\sin(\theta)A_\rho + \frac{\cos(\theta)}{\rho}(A_\gamma - A_\theta) & \alpha_4 &= \frac{1}{v}A_\theta & \alpha_3 &= A_v \\ \beta_1 &= \cos(\theta)B_\rho + \frac{\sin(\theta)}{\rho}(B_\gamma - B_\theta) & \beta_4 &= \frac{1}{v}B_\theta & \beta_3 &= B_v \end{aligned}$$

Let's substitute these expressions into (3.27) and (3.28) and impose one additional condition $A_\gamma = 0$ (the condition of isotropy for all rays coming out from the central point). As a result we obtain the following equations

$$(4.5) \quad \begin{aligned} B &= -A_\theta \\ -\frac{AA_\theta}{v^2} + \cos(\theta)A_{\theta\rho} - \frac{\sin(\theta)}{\rho}A_{\theta\theta} + \frac{AA_{\theta v}}{v} - \frac{A_\theta A_{\theta\theta}}{v^2} &= \\ &= -\sin(\theta)A_\rho - \frac{\cos(\theta)}{\rho}A_\theta + \frac{A_\theta A_v}{v} \end{aligned}$$

The solution of the equations (4.5) with separated variables here has the following form

$$A = \frac{A(v)}{\rho} \cos(\theta) \quad B = \frac{A(v)}{\rho} \sin(\theta)$$

where $A(v)$ is some arbitrary function of modulus of velocity \mathbf{v} . Corresponding dynamical system has the following force field

$$(4.6) \quad \mathbf{F} = \frac{A(v)\mathbf{N} \cos(\theta) + A(v)\mathbf{M} \sin(\theta)}{\rho}$$

(see fig. 2). Note, that both dynamical systems with force fields (4.3) and (4.6) are integrable via quadratures. Moreover for some special choice of $A(v)$ they are explicitly integrable. For instance when $A(v) = \text{const}$ the trajectories of the system (4.3) are the cycloids.

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