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ON THE SEPARATE ALGEBRAICITY ALONG THE FAMILIES OF ALGEBRAIC CURVES.

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Abstract. A new generalization of the classical separate algebraicity theorem is suggested and proved.

1. Introduction.

The theorem on separate holomorphy (Hartogs theorem) is one of the basic theorems in the theory of functions of several complex variables (see, for example, in [1]). It's formulated as follows.

Theorem 1. *Let $f(z) = f(z_1, \dots, z_n)$ be a function in the domain $D \subset \mathbb{C}^n$, which is holomorphic in each variable z_i for any fixed values of other variables. Then it is holomorphic function in D.*

Theorem 1 has certain modifications for polynomial, rational and algebraic functions (see [2]). In the last case we have the following statement.

Theorem 2. *Let $f(z) = f(z_1, \dots, z_n)$ be a holomorphic function in the domain $D \subset \mathbb{C}^n$, which is algebraic in each variable z_i for any fixed values of other variables. Then it is holomorphic branch in D for some algebraic function given by a polynomial equation $P(f, z_1, \dots, z_n) = 0$.*

Let's study the assumptions of these two theorems. In both cases we may consider the following families of complex lines, one per each variable:

$$\begin{cases} z_i = c_i^{(m)} = \text{const} & \text{for } i \neq m, \\ z_m = t \in \mathbb{C} & \text{for } i = m. \end{cases} \tag{1.1}$$

These are the coordinate lines represented in a parametric form, $t \in \mathbb{C}$ is a parameter. Restricting $f(z)$ to such lines, we obtain the following functions:

$$f_m(t) = f(c_1^{(m)}, \dots, c_{m-1}^{(m)}, t, c_{m+1}^{(m)}, \dots, c_n^{(m)}), \tag{1.2}$$

which are holomorphic in t under the assumptions of theorem 1 and algebraic in t under the assumptions of theorem 2.

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In terms of coordinate lines (1.1) the above classical theorem 2 can be formulated as follows: every holomorphic function f(z), which is algebraic along each coordinate line in D, is algebraic as a function of several complex variables. In the present paper we consider some generalized versions of this statement, when coordinate lines are replaced by certain classes of complex curves. The necessity of such generalization was fist recognized by A.B. Sukhov [3, 4] in connection with the problem of Poincaré–Alexander (see [5–8]) and its generalizations (see [9–13]). As a first step in [3, 4] coordinate lines were replaced by quadrics. Second step was done in [14]. Here instead of coordinate lines arbitrary algebraic curves are considered. They are taken in a parametric form with parameter $t \in \mathbb{C}$:

$$\begin{cases} z_1 = R_1^{(m)}(t, c_1^{(m)}, \ldots, c_{n-1}^{(m)}), \\ \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \\ z_n = R_n^{(m)}(t, c_1^{(m)}, \ldots, c_{n-1}^{(m)}). \end{cases} \tag{1.3}$$

Algebraic curves (1.3) form n families $R^{(1)}, \ldots, R^{(n)}$, index m numerates these families. Coordinate functions $R_i^{(m)}$ in (1.3) are algebraic in t. Other complex parameters $c_1^{(m)}, \ldots, c_{n-1}^{(m)}$ are introduced in order to determine particular curve within m–th family.

Note that in general the whole set of curves (1.3) can't be mapped to the coordinate lines by an algebraic map. This is why we should develop some special tools for this case.

Given a function $f(z)$ in D, we may consider its restriction to the curves (1.3). This gives us n functions analogous to (1.2):

$$f_m(t) = f(R_1^{(m)}(t, c_1^{(m)}, \ldots, c_{n-1}^{(m)}), \ldots, R_n^{(m)}(t, c_1^{(m)}, \ldots, c_{n-1}^{(m)})) \tag{1.4}$$

We say that the function $f(z)$ is *algebraic along the curves* (1.3), if each function $f_m(t)$ in (1.4) depends algebraically on t. In terms of curves (1.3) the separate algebraicity theorem can be stated as follows.

Theorem 3. *Let $f(z) = f(z_1, \ldots, z_n)$ be a holomorphic function in a domain $D \subset \mathbb{C}^n$, which is algebraic along algebraic curves forming n regular*[1)] *families being in general position*[2)]*. Then it is holomorphic branch in D for some algebraic function given by a polynomial equation $P(f, z_1, \ldots, z_n) = 0$.*

This theorem 3 was proved in [14], but in a more restricted form. Coordinate functions $R_i^{(m)}$ in (1.3) were assumed to be algebraic not only in t, but in whole set of their arguments $t, c_1^{(m)}, \ldots, c_{n-1}^{(m)}$. In the present paper we exclude this extra hypothesis concerning $c_1^{(m)}, \ldots, c_{n-1}^{(m)}$, but make more restrictive assumptions concerning the main variable t in (1.3) and (1.4). The functions $z_1(t), \ldots, z_n(t)$ and $f_m(t)$ are supposed to be polynomials in t. Below is the main theorem that we prove in this paper.

Theorem 4. *Let $f(z) = f(z_1, \ldots, z_n)$ be a holomorphic function in a domain $D \subset \mathbb{C}^n$, which is polynomial along polynomial curves forming n regular families*

[1,2)] Precise definitions see in section 2.

being in general position. Then it is holomorphic branch in D for some algebraic function given by a polynomial equation $P(f, z_1, \ldots, z_n) = 0$.

As for the most general theorem 3 it still remains an unproved conjecture. The proof of this theorem is the subject for the next publication.

Our paper is organized as follows. In section 2 we give precise definitions and preliminary discussions. Sections 3 contains the sketch of the proof of the main theorem 4. Here we omit all details and technicalities. Other sections are devoted to the strict proofs for all auxiliary results we need to prove the main theorem.

2. Precise definitions and some preliminaries.

Parametric equations (1.3) define our basic families of polynomial curves. To fit the theorem 4 they should be regular in the sense of the following definition.

Definition 1. We say that m–th family of polynomial curves (1.3) is regular in a domain $D \subset \mathbb{C}^n$, if corresponding map

$$R^{(m)} : (t, c_1^{(m)}, \ldots, c_{n-1}^{(m)}) \to (z_1, \ldots, z_n) \in D \tag{2.1}$$

in (1.3) is the biholomorphic diffeomorphism of some domain $U_m \subset \mathbb{C}^n$ and D.

If m–th family is regular, we may treat $t, c_1^{(m)}, \ldots, c_{n-1}^{(m)}$ as new curvilinear complex coordinates in D. Then the following map

$$\varphi_\tau^{(m)} : (t, c_1^{(m)}, \ldots, c_{n-1}^{(m)}) \to (t + \tau, c_1^{(m)}, \ldots, c_{n-1}^{(m)}) \tag{2.2}$$

is a translation along the curves of m–th family and τ is a magnitude of translation. By means of diffeomorphism (2.1) we may rewrite (2.2) in cartesian coordinates. Here this map $\varphi_\tau^{(m)} : z \to \tilde{z}$ is given by holomorphic functions

$$\begin{cases} \tilde{z}_1 = \varphi_1^{(m)}(\tau, z_1, \ldots, z_n), \\ \;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;. \\ \tilde{z}_n = \varphi_n^{(m)}(\tau, z_1, \ldots, z_n). \end{cases} \tag{2.3}$$

From (2.2) we find that $\varphi_\tau^{(m)}(\varphi_\theta^{(m)}(z)) = \varphi_{\tau+\theta}^{(m)}(z)$. Therefore $\varphi_\tau^{(m)}$ define the holomorphic one–parametric local group of diffeomorphisms in D. This local group generates the holomorphic vector field $\mathbf{X}_m$ in D, tangent to the curves of m–th family. Under the assumptions of theorem 4 we have n holomorphic vector–fields:

$$\mathbf{X}_1, \ldots, \mathbf{X}_n. \tag{2.4}$$

In terms of curvilinear coordinates $t, c_1^{(m)}, \ldots, c_{n-1}^{(m)}$ in $\tilde{D}$ the m-th vector–field $\mathbf{X}_m$ from (2.4) is represented by the following differential operator:

$$\mathbf{X}_m = \frac{\partial}{\partial t}. \tag{2.5}$$

Definition 2. Given n regular families of polynomial curves in D we shall say that they are in general position, if corresponding vector–fields (2.4) are linearly independent at each point $z \in D$.

Now let's consider the polynomial curves (1.3), which are supposed to form n regular families in general position. Denote by $d_{1m}, \ldots, d_{nm}$ and p_m the degrees of polynomials $z_1(t), \ldots, z_n(t)$ and $f_m(t)$ in (1.3) and (1.4). They depend on $c_1^{(m)}, \ldots, c_{n-1}^{(m)}$. Because of regularity of m–th family of curves in D one can treat them as functions of a point $z \in D$:

$$\begin{aligned} d_{im}(z) &= \deg R_i^{(m)}(t),\ i = 1, \ldots, n,\ m = 1, \ldots, n, \\ p_m(z) &= \deg f_m(t),\ m = 1, \ldots, n. \end{aligned} \tag{2.6}$$

Functions (2.6) are integer–valued functions in D, which remain constant along the curves of appropriate family. In section 4 we shall prove the following theorem.

Theorem 5. *Under the assumptions of theorem 4 one can find a smaller subdomain $\tilde{D} \subset D$, such that all functions* (2.6) *are constant in $\tilde{D}$.*

Now we are ready to explain the idea for the proof of the main theorem 4.

3. Proof of the main theorem 4.

Note that in theorems 5 we contract the domain D to $\tilde{D} \subset D$. This doesn't affect the ultimate result, since holomorphic function $f(z)$ in D, which is algebraic in some smaller subdomain of D, is algebraic in D. Therefore, we can reduce our domain as many times as we need.

According to the theorem 6 we have the subdomain $\tilde{D} \subset D$, where the degrees of polynomials in (1.3) and (1.4) are constant:

$$\deg R_i^{(m)}(t) = d_{im}, \qquad\qquad \deg f_m(t) = p_m.$$

Choose some other non–negative integer numbers $k_1, \ldots, k_n$ and q and consider the following monomial in $\tilde{D}$:

$$M(k_1, \ldots, k_n, q) = f^q \cdot (z_1)^{k_1} \cdot \ldots \cdot (z_n)^{k_n}. \tag{3.1}$$

where $f = f(z)$. Considered as a function in $\tilde{D}$ monomial (3.1) is holomorphic. We restrict it to the some arbitrary curve of m–th family in $\tilde{D}$. Then $M(k_1, \ldots, k_n, q)$ becomes a polynomial in t, its degree is given by the formula:

$$q_m = \deg M(k_1, \ldots, k_n, q) = q\, p_m + \sum_{i=1}^{n} k_i\, d_{im}. \tag{3.2}$$

If $N > q_m$ is some integer number, then, applying N-th power of the differential operator (2.5) to the monomial (3.1), we get

$$(\mathbf{X}_m)^N M(k_1, \ldots, k_n, q) = 0. \tag{3.3}$$

Respective to $M(k_1, \dots, k_n, q)$ the equality (3.3) is a linear differential equation of N–th order with holomorphic coefficients. Now we unite all these equations (3.3) into the system

$$(\mathbf{X}_m)^N \varphi(z) = 0,\ m = 1, \dots, n, \tag{3.4}$$

and denote by $V(N)$ the set of their holomorphic solutions $\varphi(z)$ in $\tilde{D}$:

$$V(N) = \{\varphi \in \mathcal{H}(\tilde{D}) : (\mathbf{X}_m)^N \varphi = 0 \text{ for all } m = 1, \dots, n\}. \tag{3.5}$$

Since the equations (3.4) are linear and homogeneous, the set of their solutions $V(N)$ forms a linear space over the field of complex numbers.

Theorem 6. *For any integer $N > 0$ the complex linear space* (3.5) *has finite dimension and* $\dim V(N) \le N^n$.

In section 4 we shall prove this theorem giving the estimate for $\dim V(N)$. Now denote by $M(N)$ the number of monomials (3.1), for which the degrees (3.2) of their restrictions to the curves are less than N, i.e. $q_m < N$ for all $m = 1, \dots, n$. If for some value of N we find that $M(N) > N^n$, then we obtain that certain number of monomials (3.1) are linearly dependent over $\mathbb{C}$. This will give a polynomial equation $P(f, z^1, \dots, z_n) = 0$ for the function $f(z)$ and will terminate the proof of theorem 4.

According to the above conclusion the last step in the proof of theorem 4 is to be the proper estimate for $M(N)$ at least for some certain value of N. Suppose $p = \max\{p_1, \dots, p_n\}$ and $d_i = \max\{d_{i1}, \dots, d_{in}\}$ (see formula (3.2)). Choose some arbitrary integer number $K > 1$ and let

$$N = N(K) = 1 + K\,p + \sum_{i=1}^{n} K\,d_i. \tag{3.6}$$

Then for $q = 1, \dots, K$ and for $k_i = 1, \dots, K$ the degrees of corresponding monomials (3.1) are less than N. For the number of such monomials we have

$$M(N) \ge K^{n+1}.$$

For the dimension of V(N) from (3.6) we derive another estimate

$$\dim V(N) \le N^n \le \text{const} \cdot K^n, \text{ as } K \to \infty.$$

Comparing these to estimates we conclude that $M(N) > \dim V(N)$ for some large enough value of K.

So the main theorem 4 is proved provided the theorems 5 and 6 hold. The rest part of paper is devoted to proof of these two theorems.

4. Proof of the theorems 5 and 6.

Let's study the integer-valued functions $d_{im}(z)$ and $p_m(z)$ in (2.6). The whole set of these functions can be treated as a map

$$\nu : D \to \mathbb{Z}^r, \tag{4.1}$$

where $r = n(n+1)$. Here in (4.1) $S = \mathbb{Z}^r$ is a countable set. Therefore we may apply the following theorem to the map (4.1).

Theorem 7. *For any map $\nu : D \to S$ from some domain $V \subset \mathbb{C}^{n-1}$ to the countable set S one can find at least one value $s \in S$, such that its inverse image $\nu^{-1}(s) = \{v \in V :\ \nu(v) = s\}$ is a dense set in some subdomain $\tilde{D} \subset D$.*

The domain $\tilde{D}$ given by the theorem 7 is exactly the domain we need to prove the theorem 5. So we reduced theorem 5 to the topological theorem 7. We shall not prove this topological theorem, since it is direct consequence of the theorem of Baire (see [15] chapter 3 section 5 theorem 8 and [16] chapter 2 section 3). Similar theorems were used in [2] and [14].

The next is the proof of the theorem 6. Let's consider the vector fields (2.4) and their local one–parametric groups (2.3). Fix some point $z_0 \in D$ and apply the first map (2.2) to this point. As a result we obtain some one–parametric set of points $z(t_1) = \varphi^{(1)}_{t_1}(z_0)$, i.e. the holomorphic complex curve in D. Then we apply another map $\varphi^{(2)}_{t_2}$ to the points of this line and obtain a complex surface

$$z(t_1, t_2) = \varphi^{(2)}_{t_2}(z(t_1)).$$

Continuing this process on the n–th step we obtain the map

$$(t_1, \dots , t_n) \to z = z(t_1, \dots , t_n) \in D \tag{4.2}$$

Since curves (1.3) are in general position, this map (4.2) can be treated as holomorphic change of coordinates in some neighborhood $\tilde{D}$ of the point $z_0 = z(0, \dots , 0)$.

In general case vector fields (2.4) do not commutate. Therefore their local groups of diffeomorphisms (2.2) also don't commute. Nevertheless we can use the map (4.2) to simplify the differential equations (3.4). Because of (2.5) the last n–th equation (3.5) is written as:

$$\left(\frac{\partial}{\partial t_n}\right)^N \varphi(t_1, \dots , t_{n-1}, t_n) = 0.$$

The previous $(n-1)$–th equation (3.5) can be simplified only for $t_n = 0$:

$$\left(\frac{\partial}{\partial t_{n-1}}\right)^N \varphi(t_1, \dots , t_{n-1}, 0) = 0.$$

Continuing this process backward from n–th equation to the first equation (3.5), we receive the following system of differential equations for the function $\varphi(z)$:

$$\begin{cases} \left(\dfrac{\partial}{\partial t_n}\right)^N \varphi(t_1, t_2, \dots , t_{n-1}, t_n) = 0, \\ \dots\dots\dots\dots\dots\dots \\ \left(\dfrac{\partial}{\partial t_1}\right)^N \varphi(t_1, 0, \dots , 0, 0) = 0. \end{cases} \tag{4.3}$$

One can easily derive the formula for the general solution of the system (4.3). It is the following polynomial with arbitrary constant coefficients:

$$\varphi = \sum_{i_1=0}^{N-1} \dots \sum_{i_n=0}^{N-1} C_{i_1,\dots,i_n} (t_1)^{i_1} \cdot \ldots \cdot (t_n)^{i_n} \tag{4.4}$$

From (4.4) one can easily count the number of linearly independent solutions for (4.3). It's exactly N^n. But the number of linearly independent solutions of (3.5) can be less than N^n. The equations (4.3) follow from (3.5), but they are not completely equivalent to (3.5), except for the case when vector–fields (2.3) are commuting. The estimate $\dim V(N) \leq N^n$ is proved. This completes the proof of the theorem 6.

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