ON CR MAPPINGS BETWEEN ALGEBRAIC CAUCHY–RIEMANN MANIFOLDS AND SEPARATE ALGEBRAICITY FOR HOLOMORPHIC FUNCTIONS.

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Abstract. We prove the algebraicity of smooth CR-mappings between algebraic Cauchy–Riemann manifolds. A generalization of separate algebraicity principle is established.

1. Introduction

The well-known Webster theorem [W1] asserts that a local CR diffeomorphism of a real algebraic Levi-nondegenerate hypersurface in $\mathbb{C}^n$, for $n > 1$, extends to an algebraic mapping of all $\mathbb{C}^n$. In this paper we generalize this result for real algebraic Cauchy–Riemann manifolds of higher codimensions (Theorem 1). Our technique is based on the Webster reflection principle modified in the spirit of [Su] and a generalization of the classical separate algebraicity theorem, where we replace the parallel lines by families of algebraic curves (Theorem 2). We hope that this second result is of self-interest.

The paper is organized as follows. In section 2 we give the precise definitions and statements of our results. In sections 3 and 4 we prove Theorem 1 provided Theorem 2 holds. Section 5 is devoted to the proof of Theorem 2.

2. The results.

Let Ω be a domain in $\mathbb{C}^n$. A closed subset M of Ω is called a generic real algebraic manifold of codimension $d \geq 1$ if

$$M = \{z \in \Omega : \rho_j(z, \bar{z}) = 0, j = 1, \dots, d\} \tag{2.1}$$

where ρ_j are real polynomials and $\bar{\partial}\rho_1 \wedge \dots \wedge \bar{\partial}\rho_d \neq 0$ in Ω. They are called the defining functions of M. We denote by T_pM and T_p^cM the real and complex tangent spaces of M at the point $p \in M$ (recall that $T_p^cM = T_pM \cap J(T_pM)$, where J is the complex structure operator in $\mathbb{C}^n$). This is well known that for a generic manifold the complex dimension of T_p^cM does not depend on $p \in M$ and is equal to $n - d$; it is called the CR-dimension of M.

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We denote by $H_p(\rho_j, u, v)$ the value of the Levi form (complex hessian) of the function ρ_j on vectors u, v at the point $p \in M$, i.e.

$$H_p(\rho_j, u, v) = \sum_{\nu,\mu=1}^{n} \frac{\partial^2 \rho_j}{\partial z_\nu \, \partial \bar{z}_\mu}(p)\, u_\nu \bar{v}_\mu \tag{2.2}$$

The Levi cone (at $p \in M$) of the manifold M of the form (2.1) is said to be the convex hull of the set

$$\{\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{R}^d : \alpha_j = H_p(\rho_j, u, u), u \in T^c_p M\} \tag{2.3}$$

If the Levi cone of M has a non-empty interior in $\mathbb{R}^d$, then we say that M possess a non-degenerate Levi cone at p. Evidently this condition does not depend on the choice of defining functions and is invariant with respect to changes of coordinates.

The vector valued hermitian form

$$L_p(u, v) = (H_p(\rho_1, u, v), \dots, H_p(\rho_d, u, v))$$

is called the Levi form of M at p.

Along with M we consider a domain Ω' in $\mathbb{C}^{n'}$ and a generic real algebraic manifold M' of codimension $d' \geq 1$ of the form

$$M' = \{z' \in \Omega' : \rho'_j(z', \bar{z}') = 0, j = 1, \dots, d'\} \tag{2.4}$$

where ρ'_j are real polynomials and $\bar\partial \rho'_1 \wedge \ldots \wedge \bar\partial \rho'_{d'} \neq 0$ in Ω'. Fix a hermitian scalar product in $\mathbb{C}^{n'}$ that is denoted by $bigl < \cdot, \cdot\big\rangle'$. Fix a point $p' \in M'$. With every defining function ρ'_j we associate the Levi operator $L^j_{p'} : T^c_{p'} M' \longrightarrow T^c_{p'} M'$ determined by $H_{p'}(\rho'_j, u, v) = \big\langle L^j_{p'}(u),\ v\big\rangle'$ for all $u, v \in T^c_{p'} M'$. Of course, this definition of the Levi operator depends on the choice of hermitian scalar product. But as we shall see further this dependence is unessential In what follows we use the similar notation for the Levi operators of M (without primes).

Let U be an open connected subset of the manifold M. The mapping $F : U \longrightarrow M'$ of the smoothness class C^1 is called the CR-mapping if for any point $p \in U$ the tangent map dF_p is $\mathbb{C}$-linear after restriction to the complex tangent space $T^c_p M$ (in this case $dF_p(T^c_p M) \subset T^c_{p'} M'$). We say that F extends to an algebraic mapping of all $\mathbb{C}^n$ if the graph of F is a part of n-dimensional complex algebraic manifold in $\mathbb{C}^{n+n'}$.

Now we can formulate the first main result of our paper. It is given by the following theorem.

Theorem 1. *Let $\Omega \in \mathbb{C}^n$ and $\Omega' \in \mathbb{C}^{n'}$ be domains, $M \subset \Omega$ and $M' \subset \Omega'$ be generic real algebraic manifolds of the form* (2.1) *and* (2.4) *respectively, M having the non-degenerate Levi cone at some point $p \in M$. Suppose $U \subset M$ is an open connected subset of M containing p and let $F : U \longrightarrow M'$ be a CR-mapping of class C^1 satisfying the following condition*

$$\sum_{j=1}^{d'} L^j_{p'}(dF_p(T^c_p M)) = T^c_{p'} M', \ \textit{where}\ p' = F(p) \tag{2.5}$$

Then F extends to an algebraic mapping of the whole space $\mathbb{C}^n$.

For the first note that (2.5) does not depend on the choice of the hermitian scalar product in $\mathbb{C}^{n'}$ by means of which we defined the Levi operators $L^j_{p'}$. If $\tilde{L}^j_{p'}$ is defined by another scalar product, then it is connected with $L^j_{p'}$ by the equality $\tilde{L}^j_{p'} = A\,L^j_{p'}$, where A is a non-degenerate $\mathbb{C}$-linear operator in $T^c_{p'}M'$. Therefore (2.5) holds for operators $\tilde{L}^j_{p'}$ as well.

Recall that the Levi form of M is called non-degenerate if $L_p(u, v) = 0$ for any $v \in T^c_p M$ implies $u = 0$ [W2].

Corollary. *Let $F : M \longrightarrow M'$ be a CR diffeomorphism of class C^1 between two real algebraic manifolds in $\mathbb{C}^n$ with non-degenerate Levi forms and non-degenerate Levi cones. Then F extends to an algebraic mapping on all $\mathbb{C}^n$.*

This assertion follows by theorem 1 quite similar to [Su] We emphasize that theorem 1 treats a considerably more general situation, since M and M' are allowed to have different CR dimensions. Our proof of theorem 1 is based on the modification of the Webster reflection principle [W1]. The crucial technical tool here is a "curved" version of the following classical separate algebraicity principle.

Claim. *Let $f(\mathbf{z}) = f(z_1, \ldots, z_n)$ be a function in some domain $D \subset C^n$. If $f(\mathbf{z})$ is algebraic in each separate variable z_i for any fixed values of other variables, then $f(\mathbf{z})$ is an algebraic function in D.*

Proof of this classical theorem can be found in [BM]. Function $f(\mathbf{z})$ in this assertion is algebraic along the straight lines, parallel to the coordinate axis. The domain D foliates into n families of such lines. In order to generalize this result for our purposes we introduce n families of algebraic curves in D

$$
\begin{aligned}
z_1 &= R_1^{(m)}(t_m, c_1^{(m)}, \ldots, c_{n-1}^{(m)}) \\
&\cdots\cdots\cdots\cdots\cdots \\
z_n &= R_n^{(m)}(t_m, c_1^{(m)}, \ldots, c_{n-1}^{(m)})
\end{aligned}
\tag{2.6}
$$

Here $m = 1, \ldots, n$ is the number of the family, t_m is a parameter on each particular curve of m-th family ($R_i^{(m)}$ depends on t_m algebraically). Parameters $c_1^{(m)}, \ldots, c_{n-1}^{(m)}$ identify curves of m-th family.

Definition 1. The family of algebraic curves (2.6) is called *algebraically depending on parameters* if each of the defining functions $R_i^{(m)}$, $i = 1, \ldots, n$ in (2.6) is algebraic in $c_1^{(m)}, \ldots, c_{n-1}^{(m)}$.

In this case because of above classical separate algebraicity principle functions $R_i^{(m)}$ in (2.6) are algebraic in whole set of their arguments $t_m, c_1^{(m)}, \ldots, c_{n-1}^{(m)}$.

Definition 2. The family of curves (2.6) is called nonsingular in D if curves of this family fills the whole domain D and the mapping $R^{(m)} : \ (t_m, c_1^{(m)}, \ldots, c_{n-1}^{(m)}) \longrightarrow \mathbf{z}$ is the local diffeomorphism.

Definition 3. Let (2.6) define n nonsingular families of algebraic curves in D. Then we have n tangent vectors to the curves at each point of D. We shall say that n families of curves (2.6) are in general position if these vectors are linearly independent at any point $\mathbf{z} \in D$.

Now we can state the "curved" separate algebraicity principle which partially generalizes the classical one and form the second main result of our paper.

Theorem 2. *Let $D \subset \mathbb{C}^n$ be a domain equipped with n families (2.6) of nonsingular algebraic curves algebraically depending on parameters and being in general position. Then each holomorphic function $f(\mathbf{z})$ in D, which is algebraic in t_m after restriction to any particular curve from any one of these families, extends to an algebraic function on $\mathbb{C}^n$.*

3. Tangent CR fields and the main equations.

First we shall recall briefly some facts of the theory of CR-structures (reader can find more details in [Ch]). Let $(T^c_pM)_{\mathbb{R}}$ be the complex tangent space T^c_pM considered as a vector space over real numbers $\mathbb{R}$. Then $(T^c_pM)_{\mathbb{R}} \overset{\mathbb{R}}{\otimes} \mathbb{C}$ is a complexification for $(T^c_pM)_{\mathbb{R}}$. Operator J of canonical complex structure in this complexification is $\mathbb{C}$-linear with respect to the own complex structure of T^c_pM. Therefore it has two eigenvalues $+i$ and $-i$. Then we have the decomposition $(T^c_pM)_{\mathbb{R}} \otimes \mathbb{C} = T^c_pM^{1,0} \oplus T^c_pM^{0,1}$, where

$$T^c_pM^{1,0} = \{(v, -iv) : v \in T^c_pM\} \text{ and } T^c_pM^{0,1} = \{(v, +iv) : v \in T^c_pM\}$$

are the eigenspaces for J corresponding to the eigenvalues $+i$ and $-i$ respectively. The following two maps $v \longmapsto (v, -iv)$ and $v \longmapsto (v, +iv)$ realize the canonical isomorphisms $T^c_pM \equiv T^c_pM^{1,0}$ and $\overline{T^c_pM} \equiv T^c_pM^{0,1}$. Thus we have $T^c_pM_{\mathbb{R}} \otimes \mathbb{C} = T^c_pM \oplus \overline{T^c_pM}$. Because of the last decomposition the sections of the vector-bundles T^cM and $\overline{T^cM}$ are called the CR vector fields of the types $(1,0)$ and $(0,1)$ respectively. It is easy to check that the field $V = \sum_{j=1}^n v_j(z)\partial/\partial z_j$ in $\mathbb{C}^n$ is the vector field of the type $(1,0)$ on M if and only if the vector $(v_1(a), \ldots, v_n(a))$ is in T^c_aM for any $a \in M$.

Let us come back to theorem 1. Note that without loss of generality we can take $p = 0$ and $F(p) = 0$. Taking $z = (x, y)$ and $z' = (x', y')$ and denoting $\mathbb{C}^n = \mathbb{C}^k_x \times \mathbb{C}^d_y$ and $\mathbb{C}^{n'} = \mathbb{C}^{k'}_{x'} \times \mathbb{C}^{d'}_{y'}$ one can bring the defining polynomials for M and M' to the form

$$\begin{aligned} p_j &= y_j + \bar{y}_j + o(|z|), \quad j = 1, \ldots, d \\ p'_j &= y'_j + \bar{y}'_j + o(|z|), \quad j = 1, \ldots, d' \end{aligned} \tag{3.1}$$

For this choice of coordinates we have $T^c_0M = \mathbb{C}^k_x = \{(x, y) : y = 0\}$ and $T^c_0M' = \mathbb{C}^k_{x'} = \{(x', y') : y' = 0\}$ Now let us consider vector fields T_q, $q = 1, \ldots, k$ of the following form

$$T_q = \Delta(z, \bar{z})\frac{\partial}{\partial x_q} - \sum_{j=1}^{d} a_{jq}(z, \bar{z})\frac{\partial}{\partial y_j} \tag{3.2}$$

where Δ is the determinant of the following matrix

$$\Delta_{sj} = \left(\frac{\partial \rho_s}{\partial y_j}\right)^{j=1,\ldots,d}_{s=1,\ldots,d} \tag{3.3}$$

Everywhere in this paper we shall obey the following rule for denoting the matrix elements: lower index outside the right bracket is the row number and upper index is the column number. The coefficients a_{jq} in (3.4) we define as follows

$$a_{jq} = \sum_{s=1}^{d} \Delta\, b_{js}\, \frac{\partial \rho_s}{\partial x_q} \tag{3.4}$$

where b_{js} is the matrix inverse to the matrix (3.3). According to the elementary facts from linear algebra the matrix with the elements $\Delta\, b_{js}$ is an conjugate matrix for (3.3) i.e. its elements are the algebraic cofactors for the elements of transpose of the matrix (3.3). Therefore the coefficients of the vector fields in (3.2) are polynomials in x_i and y_i.

Clear that the restrictions of the fields T_q, $q = 1, \ldots, k$ form a base of the bundle $T^c M$ over a neighborhood of the origin in M. Moreover, it is obvious that

$$\Delta(0) = 1, a_{jq}(0) = 0, j = 1, \ldots, d, q = 1, \ldots, k \tag{3.5}$$

Note that because of $dF_0(T_0^c M) \subset T_0^c M'$, we get by (3.1)

$$\partial F_j / \partial x_q(0) = 0, j = k' + 1, \ldots, n', q = 1, \ldots, k \tag{3.6}$$

Together with the fields (3.2) let us consider the conjugate fields $\overline{T_q}$. Recall that a C^1-function h defined on an open connected subset $U \subset M$ is called a CR-function if for any $z = (x, y) \in U$ one has

$$\overline{T_q} h = \Delta\, \frac{\partial h}{\partial \bar{x}_q} - \sum_{j=1}^{d} \bar{a}_{jq}\, \frac{\partial h}{\partial \bar{y}_j} = 0, q = 1, \ldots, k \tag{3.7}$$

These are the tangent Cauchy-Riemann equations. This is well known that a C^1-mapping $F : M \longrightarrow M'$ is a CR-mapping (in the sense of the section 2) if and only if each its component is a CR-function on M. Indeed, (3.7) means that $\bar{\partial} h(V^q) = 0, q = 1, \ldots, k$,(where $\{V^q\}_{q=1}^{k}$ is a base of $T_p^c M$) for any point $p \in M$. Therefore $\bar{\partial} h | T_p^c M = 0$ and (3.7) means that the restriction $dh | T_p^c M$ is a $\mathbb{C}$-linear function for any $p \in M$.

According to the Boggess-Polking theorem [BP] from the non-degeneracy of the Levi cone of M at the origin $0 \in M$ we derive that the mapping F extends holomorphically to a wedge with the edge M. Using the results of [Su] we obtain that F extends holomorphically to a neighborhood of the origin in $\mathbb{C}^n$. Therefore everywhere below we take F being holomorphic in a neighborhood $\Omega \ni 0$ in $\mathbb{C}^n$ and suppose $U = M \cap \Omega$.

The condition $F(U) \subset M'$ means that $\rho'_j(F, \bar{F}) = 0$ for $z \in U$ and $j = 1, \ldots, d'$. Applying the tangent operators (3.2) to both sides of these equalities we obtain

$$T_q \rho'_j(F, \bar{F}) = 0, q = 1, \ldots, k, j = 1, \ldots, d' \text{ for } z \in U \tag{3.8}$$

Now we introduce the vector-function $D(F)$ holomorphic in Ω

$$D(F) = \left(\frac{\partial F_1}{\partial z_1}, \ldots, \frac{\partial F_{n'}}{\partial z_1}, \ldots, \frac{\partial F_1}{\partial z_n}, \ldots, \frac{\partial F_{n'}}{\partial z_n} \right)$$

Its values are in $\mathbb{C}^{nn'}$.

Left hand sides of (3.8) can be considered as polynomials in z, $\bar z$, F, $\bar F$ and $\partial F_j/\partial z_s$, $\partial \bar F_j/\partial z_s$. But since F is holomorphic, we have $\partial \bar F/\partial z_s = 0$. Thus, the following expressions

$$\Phi_{qj}(z,\bar z,F,\bar F,D(F)) = T_q\rho'_j(F,\bar F) \tag{3.9}$$

are polynomials in z, $\bar z$, F, $\bar F$, $D(F)$. By $\tilde 0$ we denote the point $(0,0,0,0,D(F)(0))$ in $\mathbb{C}^n\times\mathbb{C}^n\times\mathbb{C}^{n'}\times\mathbb{C}^{n'}\times\mathbb{C}^{nn'}$.

Lemma 3.1. *In the set of functions* (3.9) *with* $q=1,\dots,k, j=1,\dots,d'$ *one can choose a subset* $\Phi_1=\Phi_{q(1)j(1)},\dots,\Phi_{k'}=\Phi_{q(k')j(k')}$ *such that the following Jacobian matrix is of rank* k'

$$\left(\frac{\partial\Phi_j}{\partial\bar F_s}(\tilde 0)\right)^{s=1,\dots,k'}_{j=1,\dots,k'}$$

Proof. : Let us fix j, consider the set of functions Φ_{qj}, $q=1,\dots,k$ and form the following Jacobi matrix

$$\left(\frac{\partial\Phi_{qj}}{\partial\bar F_s}(\tilde 0)\right)^{s=1,\dots,k}_{j=1,\dots,k'}$$

It follows by (3.1), (3.2), (3.5), (3.6), (3.8), (3.9) that this matrix can be rewritten in the form

$$\left(\sum_{r=1}^{k'}\frac{\partial^2\rho'_j}{\partial z'_r\,\partial\bar z'_s}(0)\frac{\partial F_r}{\partial z_q}(0)\right)^{s=1,\dots,k}_{j=1,\dots,k'} \tag{3.11}$$

The condition (1.5) does not depend on the choice of scalar product in $\mathbb{C}^{n'}$. Therefore one can take the hermitian scalar product, which defines Levi operators L^j_0, being canonical in the coordinates (3.1). Then each operator L^j_p in the standard basis e'_r, $r=1,\dots,k'$ of $\mathbb{C}^{k'}=T^c_0M'$ has the matrix of the following form

$$\left(\frac{\partial^2\rho'_j}{\partial z'_j\,\partial\bar z'_s}(0)\right)^{s=1,\dots,k'}_{j=1,\dots,k'}$$

Therefore q-th the row of the matrix (3.11) consists of the coordinates of the vector $L^j_0(dF_0(e_q))$, where e_q, $q=1,\dots,k$ is the standard basis of $\mathbb{C}^{k'}=T^c_0M'$, i.e.

$${}^t(\partial\Phi_{qj}/\partial\bar F_1(\tilde 0),\dots,\partial\Phi_{qj}/\partial\bar F_k(\tilde 0)) = L^j_0(dF_0(e_q))$$

where ${}^t(\ )$ denotes the transpose matrix. According to (2.5) the rank of the set of vectors $L^j_0(dF_0(e_q))$, $j=1,\dots,d'$, $q=1,\dots,k$, is equal to k. Thus we get the proof of the lemma. □

Let Φ_j, $j=1,\dots,k'$ be the functions chosen according to lemma 3.1. For $z\in U$ we have

$$\begin{aligned}&\Phi_j(z,\bar z,F,\bar F,D(F))=0, j=1,\dots,k'\\&\rho'_s(F,\bar F)=0, s=1,\dots,d'\end{aligned} \tag{3.12}$$

Lemma 3.2. *The rank of the Jacobian matrix for* (3.12) *with respect to* $\bar{F}$ *at the point* $\tilde{0}$ *is equal to* n'.

Proof. The Jacobi matrix, which is mentioned in lemma, has the following form

$$\left(\begin{array}{c|c} \dfrac{\partial \Phi_j(\tilde{0})}{\partial \bar{F}_s} & * \\ \hline \tilde{0}^{k'}_{d'} & I_{d'} \end{array}\right) \tag{3.13}$$

Upper left block of the matrix (3.13) is formed by the matrix (3.10), **0** is zero rectangular matrix $d' \times k'$ and I is a square unit matrix $d' \times k'$. Therefore because of lemma 3.1 the rank of matrix (3.13) is equal to $k' + d' = n'$. $\square$

Applying the complex conjugation to the first k' equations (3.13), we get

$$\begin{aligned} &P_j(z, \bar{z}, F, \bar{F}, \overline{D(F)}) = 0,\ j = 1, \ldots, k' \\ &\rho'_s(F, \bar{F}) = 0,\ s = 1, \ldots, d' \end{aligned} \tag{3.14}$$

This system of the equations is of crucial importance in what follows.

4. Geometry of Segre surfaces.

For a real algebraic manifold M of the form (2.1) the Segre surface associated to a fixed point $z \in \mathbb{C}^n$ is a complex algebraic set in $\mathbb{C}^n$ of the form

$$Q(z) = \{w \in \mathbb{C}^n :\ \rho_j(w, \bar{z}) = 0,\ j = 1, \ldots, d\} \tag{4.1}$$

We denote by A the graph of the mapping F over a neighborhood $\Omega \ni 0$ in $\mathbb{C}^n$. Let also

$$A_\zeta = \{(z, z') \in \mathbb{C}^{n+n'} :\ z' = F(z), \rho_j(z, \bar{\zeta}) = 0,\ j = 1, \ldots, d\} \tag{4.2}$$

denote the graph of the restriction of F to the Segre surface $Q(\zeta)$. Evidently, every A_ζ is a k-dimensional complex manifold ($k = n - d$) in $\Omega \times \mathbb{C}^{n'}$.

Lemma 4.1. *For any point* $\zeta \in \Omega$ *the complex manifold* A_ζ *is a piece of a complex* p*-dimensional algebraic variety* $\tilde{A}_\zeta$ *in* $\mathbb{C}^{n+n'}$.

Proof. It follows by lemma 3.2 that one can apply the implicit function theorem to the system (3.14). We get $F(z) = R(z, \bar{z})$ for $z \in M \cap \Omega$ (where R is a real analytic function in Ω algebraic in z). By (3.1) and the implicit function theorem we get $M \cap \Omega = \{z = (x, y) \in \Omega :\ y = \phi(x, \bar{z})\}$. Therefore,

$$F(x, \phi(x, \bar{z})) = R(x, \phi(x, \bar{z}), \bar{z}), \tag{4.3}$$

for $z = (x, y) \in M \cap \Omega$. Consider antiholomorphic functions $F^\star(\theta, \xi) = F(\bar{\theta}, \phi(\bar{\theta}, \bar{\xi}))$ and $R^\star(\theta, \xi) = R(\bar{\theta}, \phi(\bar{\theta}, \bar{\xi}), \bar{\xi})$, where $\theta \in \mathbb{C}^k$, $\xi \in \mathbb{C}^n$. Then (4.3) means that these functions coincide on the manifold

$$\hat{M} = \{(\theta, \xi) :\ \bar{\theta} = (\xi_1, \ldots, \xi_k),\ \xi \in M\},$$

that obviously is generic in a neighborhood of the origin in $\mathbb{C}^{n+k}$. Now it follows by the uniqueness theorem [P] that $F(\bar\theta, \phi(\bar\theta, \bar\xi)) \equiv R(\bar\theta, \phi(\bar\theta, \bar\xi), \bar\xi)$ in a neighborhood of the origin in $\mathbb{C}^{n+k}$. Hence,

$$F(x, \phi(x, \bar\zeta)) = R(x, \phi(x, \bar\zeta), \bar\zeta)$$

for any fixed ζ from a neighborhood of the origin in $\mathbb{C}^n$. But the following set

$$\{(x, \phi(x, \bar\zeta)) : \ x \text{ runs over a neighborhood of the origin}\}$$

coincide with the Segre surface $Q(\zeta) = \{z : \rho_j(z, \bar\zeta) = 0, \ j = 1, \dots, d\}$. Thus we get $F(z) = R(z, \bar\zeta)$ for $z \in Q(\zeta)$. Since R was obtained by (3.14), the set

$$A_\zeta = \{(z, z') : \ z \in Q(\zeta), \ z' = F(z)\}$$

is contained in $(n-d)$-dimensional complex algebraic manifold of the form

$$\begin{aligned} &P_j(z, \bar\zeta, z', \overline{F}(\zeta), \overline{DF(\zeta)}) = 0, \ j = 1, \dots, k', \\ &\rho'_s(z', \overline{F}(\zeta)) = 0, \ s = 1, \dots, d', \\ &\rho_l(z, \bar\zeta) = 0, \ l = 1, \dots, d, \end{aligned} \tag{4.5}$$

that proves the desired assertion. □

Fix $\theta \in \mathbb{C}^k$ and consider the d-parametric family of the Segre surfaces $Q(\theta, \tau)$, $\tau \in \mathbb{C}^d$.

Lemma 4.2. *There exists a neighborhood $U \ni 0$ in $\mathbb{C}^n$ of the form $U = U_x \times U_y$, $U_x \subset \mathbb{C}^k$, $U_y \subset \mathbb{C}^d$ such that for any fixed $\theta \in U_x$, the family of Segre surfaces $Q(\theta, \tau)$, $\tau \in U_y$ has the following properties:*

1. *for any $\tau', \tau'' \in U_y$ the intersection $Q(\theta, \tau') \cap Q(\theta, \tau'')$ is empty;*
2. *for any $z = (x, y) \in U$ there exists the unique $\tau \in U_y$ such that $(x, y) \in Q(\theta, \tau)$.*

Proof. One can represent the Segre surface as $Q(\theta, \tau) = \{z \in U : \ \bar\tau = S(z, \bar\theta)\}$, where S is an analytic function and U is a neighborhood of the origin. Now, if $z = (x, y)$ is in $Q(\theta, \tau') \cap Q(\theta, \tau'')$, then $\bar\tau' = S(z, \bar\theta) = \bar\tau''$; this implies (1). For $z = (x, y)$ set $\tau = \bar S(z, \bar\theta)$. Then $z \in Q(\theta, \tau)$ and we get (2). □

By the implicit function theorem

$$Q(\theta, \tau) \cap U = \{(x, y) \in U : \ y = R(x, \bar\theta, \bar\tau)\},$$

where R is an algebraic function, i.e. locally $Q(\theta, \tau)$ is the graph over the coordinate plane $\mathbb{C}^k_x = \mathbb{C}^k_{z_1 \dots z_k}$. Let $X_j(\bar\theta, \bar\tau)$ be holomorphic vector fields on $Q(\theta, \tau)$ being the natural liftings to $Q(\theta, \tau)$ of the coordinate vector fields $\partial/\partial z_j$, $j = 1, \dots, k$ in $\mathbb{C}^k_{z_1 \dots z_k}$. It follows by lemma 4.2 that for any point $(x, y) \in U$ there exists the unique surface $Q(\theta, \tau)$, $\tau = S(z, \bar\theta)$ passing through (x, y). Hence, the fields $Y_j(\bar\theta) = X_j(\bar\theta, S(z, \bar\theta))$ are holomorphic vector fields in U. Their integral curves evidently are linear sections of the Segre surfaces by parallel planes and, therefore, form the families of complex algebraic curves in $\mathbb{C}^n$ algebraically depending on parameters.

Lemma 4.3. *The set of the vectors $Y_j(\bar\theta)(0)$, $j = 1, \dots, k$, θ runs over a neighborhood of the origin in $\mathbb{C}^k$, spans $\mathbb{C}^n$.*

Proof. If $(\theta, \tau) \in Q(0)$, i.e. $\rho_j(\theta, \tau, 0, 0) = 0$, $j = 1, \dots, d$, then it follows by (3.1) and the implicit function theorem that $\tau = o(|\theta|)$. Now let $0 \in Q(\theta, \tau)$ (recall that this is equivalent to $(\theta, \tau) \in Q(0)$). By the implicit function theorem we get

$$Q(\theta, \tau) = \{(x, y) : \; y + \bar\tau = \phi(x, \bar\theta, y, \bar\tau)\} =$$
$$\{(x, y) : \; y + \bar\tau = \psi(x, \bar\theta, \bar\tau)\}.$$

Here

$$\psi = o(|\theta|) + \big\langle L(x), \; \theta + o(|\theta|)\big\rangle + o(|x|),$$

where

$$\big\langle L(\xi), \; \eta\big\rangle = (\big\langle L_1(\xi), \; \eta\big\rangle, \dots, \big\langle L_d(\xi), \; \eta\big\rangle)$$

is the Levi form of M. Hence

$$Y_j(\theta)(0) = (e_j, \big\langle L(e_j), \; \theta\big\rangle + o(|\theta|)) = (0, \dots, 1, \dots, 0, \big\langle L(e_j), \; \theta\big\rangle + o(|\theta|)),$$

where 1 is on the j-th position and e_j, $j = 1, \dots, k$ is the standard basis of $\mathbb{C}^k$.

Assume there exists $\alpha \in \mathbb{C}^n \backslash \{0\}$ such that

$$\big\langle \alpha, Y_j(\bar\theta)(0)\big\rangle = 0, \; j = 1, \dots, k$$

for any $\theta \in U_x$. Then

$$\big\langle \alpha, \; Y_j(\bar\theta)(0)\big\rangle = \alpha_j + \sum_{\nu=1}^{d} (\big\langle L_\nu(e_j), \; \theta\big\rangle + o(|\theta|))\alpha_{k+\nu} =$$
$$= \alpha_j + \sum_{\nu=1}^{d} \alpha_{k+\nu} \big\langle L_\nu(e_j), \; \theta\big\rangle + o(|\theta|) =$$
$$= \alpha_j + \big\langle \sum_{\nu=1}^{d} \alpha_{k+\nu} L_\nu(e_j), \; \theta\big\rangle + o(|\theta|) \equiv 0$$

as a function in $\theta \in \mathbb{C}^k$. Therefore $\alpha_j = 0$ and $\sum_{\nu=1}^{d} \alpha_{k+\nu} L_\nu(e_j) = 0$ for $j = 1, \dots, k$. This means that the Levi operators of M are linearly dependent. We get a contradiction with the condition of the non-degeneracy of the Levi cone of M. $\square$

Thus, we get n non-singular families of algebraic curves, algebraically depending on the parameters, in general position near the origin and the restriction of F on each curve is algebraic. Now it follows by theorem 2 that F extends to an algebraic mapping on all $\mathbb{C}^n$. This completes the proof of theorem 1 provided theorem 2 holds.

5. Proof of the theorem 2.

First consider only the m-th family of algebraic curves (2.6). Because of non-singularity of this family we can treat $t_m, c_1^{(m)}, \dots, c_{n-1}^{(m)}$ in (2.6) as new local coordinates in the domain D. Let us define the transformation $\varphi_m(\tau)$ as a translation along the t_m-axis in new curvilinear coordinates

$$\varphi^{(m)}(\tau) : t_m, c_1^{(m)}, \dots, c_{n-1}^{(m)} \longrightarrow t_m + \tau, c_1^{(m)}, \dots, c_{n-1}^{(m)} \tag{5.1}$$

The transformations $\varphi^{(m)}(\tau)$ form the local one-parameter group of transformations determined by the vector field of tangent vectors to the curves of m-th family. In the original variables transformations (5.1) are given by n algebraic functions with $n+1$ arguments

$$\begin{aligned} &\tilde{z}_1 = \varphi_1^{(m)}(\tau, z^1, \ldots, z_n) \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ &\tilde{z}_n = \varphi_n(m)(\tau, z^1, \ldots, z_n) \end{aligned} \tag{5.2}$$

The algebraicity of the functions $\varphi_i^{(m)}$ in (5.2) is a consequence of algebraicity of the curves of m-th family and of their algebraic dependence on parameters in (2.6).

Using the transformations $z \longmapsto \tilde{z} = \varphi^{(m)}(\tau)z$ of the form (5.2) we introduce new local holomorphic coordinates $t_1, \ldots, t_n$ in D as follows

$$z = \varphi^{(n)}(t_n) \circ \ldots \circ \varphi^{(1)}(t_1)z^0 \tag{5.3}$$

where z^0 is a fixed point in whose neighborhood these coordinates are defined (recall that our families of curves are in general position). The transformation from $z_1, \ldots, z_n$ to $t_1, \ldots, t_n$ and the inverse are algebraic.

Note that part of coordinate lines in the local coordinates $t_1, \ldots, t_n$ coincide with the curves of the above families. Let $f(t_1, \ldots, t_n)$ be the representation of the function $f(z)$ from theorem 2 in the local coordinates $t_1, \ldots, t_n$. Then the following functions

$$\begin{aligned} &f_1 = f(t, 0, \ldots, 0) \\ &f_2 = f(t_1, t, 0, \ldots, 0) \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ &f_n = f(t_1, \ldots, t_{n-1}, t) \end{aligned} \tag{5.4}$$

are algebraic in t and holomorphic in other arguments. In order to prove theorem 2 it suffices to show the algebraicity of these functions in all their arguments. We shall proceed by induction on i (the number of the function f_i in (5.4)). However, first we need some preliminaries.

Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be an entire multiindex and $|\alpha| = \alpha_1 + \ldots + \alpha_n$. We denote by $f_\alpha(z)$ the following derivative

$$f_\alpha = \frac{\partial^{|\alpha|} f}{\partial z_1^{\alpha_1} \ldots \partial z_n^{\alpha_n}} \tag{5.5}$$

Lemma 5.1. *In the assumptions of theorem 2 one can find the smaller subdomain $D' \subset D$ such that all derivatives $f_\alpha(z)$ are algebraic in t_m after restriction to each curve of any family.*

Proof. The family of curves (2.6) is non-singular, therefore the transformation (2.6) from $z_1, \ldots, z_n$ to $t_m, c_1^{(m)}, \ldots, c_{n-1}^{(m)}$ and its back transformation are implemented by algebraic functions. Hence in place of (5.5) we can consider the following derivatives

$$f_\alpha = \frac{\partial^{\alpha_1 + \ldots \alpha_{n-1}} f}{\partial {c_1^{(m)}}^{\alpha_1} \ldots \partial {c_{n-1}^{(m)}}^{\alpha_{n-1}}} \tag{5.6}$$

and prove their algebraicity in t_m. Differentiation by t_m and transformation to the original variables $z_1, \ldots, z_n$ do not destroy their algebraicity.

In order to prove the algebraicity of the derivatives (2.6) we shall use the algebraicity of the function $f(t_m, c_1^{(m)}, \ldots, c_{n-1}^{(m)})$ in t_m for the fixed values of the other arguments. This means that we have the irreducible polynomial with unit content in the ring $\mathbb{C}[f,t]$ such that $f(t_m)$ satisfies the following equation

$$P(f(t_m), t_m) \equiv 0 \tag{5.7}$$

(see [VW, L]). Note that the coefficients of the polynomial (5.7) and even its degrees in f and t depend on the parameters $c_1^{(m)}, \ldots, c_{n-1}^{(m)})$. Let us define the following sets being the subsets in the range of values of these parameters

$$C_{qk} = \{c_1^{(m)}, \ldots, c_{n-1}^{(m)}) : \deg_f P = q, \deg_t P = k\} \tag{5.8}$$

The union of the countable number of sets C_{qk} coincides with the whole range of values of the parameters $c_1^{(m)}, \ldots, c_{n-1}^{(m)}$. This allows us to use the following well-known Bair theorem.

Bair's Theorem. *A compete metric space cannot be a countable union of nowhere dense subsets.*

The proof can be found in [RS]. We apply this fact to the of range of parameters $c_1^{(m)}, \ldots, c_{n-1}^{(m)}$, and conclude that the closure of at least one of the sets C_{qk} has the non-empty interior. Choose a domain D' whose natural projection lies in the interior of such C_{qk}. Also, let us choose in D' a curve of the family (2.6) with parameters in C_{qk}. Without loss of generality one can assume that this curve corresponds to the parameters $c_i^{(m)} = 0$ and the point $t_m = 0$ on this curve is in D'. For the polynomial (5.7) we have

$$P(f,t) = \sum_{i=0}^{q} \sum_{j=0}^{k} a_{ij}\, f^i\, t^j \tag{5.9}$$

We normalize the polynomial (5.9) by setting some of its nonzero coefficient a_{rs} to be equal to 1. This polynomial vanishes after the substitution $f = f(t_m)$ and $t = t_m$. Let us consider the functions

$$\varphi_{ij} = f(t)^i\, t^j, \text{ where } i = 0, \ldots, q \text{ and } j = 0, \ldots, k \tag{5.10}$$

They are algebraic in t and depend holomorphically on the parameters $c_i^{(m)}$. If these parameters vanish, these functions (as the functions in t) are linearly dependent. But the elimination of the function φ_{rs} with $a_{rs} = 1$ makes the rest functions linearly independent. Otherwise we would have another nonzero polynomial $\tilde{P}(f,t)$ of the form (5.9) for which the equality (5.7) holds. Since P is irreducible, we have $\tilde{P}(f,t) = uP(f,t)$ where $u \in \mathbb{C}[f,t]$. But $\deg_f \tilde{P} \leq deg_f P$ and $\deg_t \tilde{P} \leq deg_t P$, therefore $u \in \mathbb{C} \subset \mathbb{C}[f,t]$. Comparing the coefficients $\tilde{a}_{rs} = 0$ and $a_{rs} = 1$ we find that the equality $\tilde{P}(f,t) = uP(f,t)$ cannot be true for $u \neq 0$.

We denote by X the set of all functions in (5.10), and by X' this set without φ_{rs}. Let us consider the Taylor expansions in t of the functions (5.10). One can treat their coefficients as infinitely-dimensional vectors (columns) of the linear space $\mathbb{C}^\infty$. Such vectors corresponding to the function from X' form the $\infty \times N$-matrix A, where $N = \#X'$ is the number of functions in the set X'. The columns of A are linearly independent if $c_i^{(m)} = 0$. Therefore, there is a $N \times N$-submatrix $\tilde{A}$ of A with non-zero determinant (minor). This minor is holomorphic in $c_i^{(m)}$ and, therefore, does not vanish in a neighborhood of the origin. Hence, the columns of A and the functions from X' are linearly independent for $c_i^{(m)}$ in a neighborhood of the origin.

Let us add the last column B corresponding to the function φ_{rs} to the matrix A, and consider the minors of order $(N+1)$ of the extended matrix $A|B$. They vanish for $c_i^{(m)} = 0$ and for the parameters from the dense set C_{qk}. Therefore, they vanish identically. Thus, the functions from X' are linearly independent and the functions of X are linearly dependent for every $c_i^{(m)}$ in a neighborhood of the origin. Thus, φ_{rs} is a linear combination of the functions from X'. Its coefficients up to the sign coincide with the coefficients of the polynomial (5.9). They are defined uniquely by linear system with the extended matrix $(\tilde{A}|\tilde{B})$, where the N-column $\tilde{B}$ is formed by the elements of B lying on the rows defining $\tilde{A}$. Thus, the coefficients of the polynomial (5.9) are holomorphic in $c_i^{(m)}$ on a neighborhood of the origin. Now one can differentiate the equation (5.7) in $c_i^{(m)}$ and conclude the proof. $\square$

Let us consider the functions (5.4). One can shrink D to $D' \subset D$ following to lemma 5.1. Also, one can assume that the degrees of the polynomials (5.7) in f and t_m depend only on m in D'. Choose the point z^0 from (5.3) in a domain D'. This determines the functions (5.4). For the function f_1 lemma 5.1 gives the algebraicity in t of the derivatives

$$\frac{\partial^s f_1(t, 0, \ldots, 0)}{\partial t_2^s} \tag{5.11}$$

The derivatives (5.11) coincide with the derivatives of the function f_2 from (5.4) for $t = 0$. In fact,

$$\left.\frac{\partial^s f_2(t_1, t, 0, \ldots, 0)}{\partial t^s}\right|_{t=0} = \frac{\partial^s f_1(t_1, 0, \ldots, 0)}{\partial t_2^s} \tag{5.12}$$

We need the following

Lemma 5.2. *An algebraic function $f(t)$ is defined uniquely by its value and the values of a finite number of its derivatives in a regular point. If these values depend algebraically on a parameter τ, then $f = f(t, \tau)$ is an algebraic function function in both variables t and τ.*

Assume the defining irreducible polynomial of the algebraic function $f(t)$ has the form (5.9). Repeating the above arguments, we again consider the functions (5.10) and their Taylor expansions at a regular point (one can assume it to be $t = 0$). The coefficients of these expansions depends linearly on f and its derivatives in $t = 0$. Considering the non-degenerate submatrix $\tilde{A}$, we apply the Cramer rule to the system (5.11) and get the coefficients of the polynomial (5.9). In the second hypothesis of our claim they are algebraic in τ. By the separate algebraicity principle we complete the proof of lemma 5.2.

We apply Lemma 5.2 to the function $f_2(t_1, t, 0, \dots, 0)$, taking into account its algebraicity in t and the algebraicity of derivatives (5.12) in t_1. Therefore, the function $f_2(t_1, t, 0, \dots, 0)$ is algebraic in both variables. This is the base of the induction.

Assume that the functions $f_1, \dots, f_m$ in (5.4) are algebraic. It follows by lemma 5.1 that the derivatives

$$\frac{\partial^s f_m(t_1, \dots, t_m, 0, \dots, 0)}{\partial t_{m+1}^s} = \left. \frac{\partial^s f_{m+1}(t_1, \dots, t_{m+1}, 0, \dots, 0)}{\partial t_{m+1}^s} \right|_{t_{m+1}=0} \tag{5.13}$$

are algebraic as well. Lemma 5.2 and the algebraicity of the derivatives (5.13) in $t_1, \dots, t_m$ give the induction step from m to $m+1$. This completes the proof of Theorem 2.

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