A NOTE ON KOSMANN-LIE
DERIVATIVES OF WEYL SPINORS.

R. A. Sharipov

Abstract. Kosmann-Lie derivatives in the bundle of Weyl spinors are considered. It is shown that the basic spin-tensorial fields of this bundle are constants with respect to these derivatives.

1. Introduction.

Lie derivatives arise in studying continuous symmetries of various geometric structures on manifolds. They are also used in symmetry analysis of ordinary and partial differential equations (see [1]). In general relativity the bundle of Weyl spinors SM is a special geometric structure built over the space-time manifold M. The main goal of this paper is to clarify the procedure of applying Lie derivatives to the basic attributes of this geometric structure, i. e. to the basic spin-tensorial fields associated with the bundle of Weyl spinors.

2. Lie derivatives of spatial structures.

Let M be a space-time manifold of general relativity. This means that it is a four-dimensional orientable manifold equipped with a Minkowski type metric $\mathbf{g}$ and with a polarization. A polarization, which is typically not mentioned, is a geometric structure that marks the future half light cone in the tangent space $T_p(M)$ for each point $p \in M$ (see more details in [2]). A Lie derivative $L_{\mathbf{X}}$ is usually given by some vector field $\mathbf{X}$ in M. Once such a vector field $\mathbf{X}$ is fixed, it produces a one-parametric local group of local diffeomorphisms:

$$\varphi_\varepsilon \colon M \to M. \tag{2.1}$$

The letter t is typically used for the parameter of this local group (see [3]), but here we use the Greek letter ε since t in physics is reserved for the time variable. The local diffeomorphisms (2.1) induce the local diffeomorphisms

$$\varphi_{\varepsilon*} \colon TM \to TM, \qquad \varphi^*_{-\varepsilon} \colon T^*M \to T^*M \tag{2.2}$$

in tangent and cotangent bundles respectively. These induced diffeomorphisms (2.2) act as linear mappings in fibers of TM and T^*M. For this reason they can be extended to local diffeomorphisms of tensor bundles:

$$\varphi_\varepsilon \colon T^r_s M \to T^r_s M. \tag{2.3}$$

Here in (2.3) through $T_s^r M$ we denote the following tensor product of r copies of the tangent bundle TM and s copies of the cotangent bundle T^*M:

$$T_s^r M = \overbrace{TM \otimes \ldots \otimes TM}^{r \text{ times}} \otimes \underbrace{T^*M \otimes \ldots \otimes T^*M}_{s \text{ times}}. \qquad (2.4)$$

Let's study the diffeomorphisms (2.1) and (2.2) in more details. Assume that p and q are two points of the space-time manifold M such that $q = \varphi_\varepsilon(p)$. Then $p = \varphi_{-\varepsilon}(q)$ and we have the following commutative diagram:

$$
\begin{array}{ccc}
T_p(M) & \xrightarrow{\varphi_{\varepsilon *}} & T_q(M) \\
\pi \downarrow & & \downarrow \pi \\
p & \xrightarrow{\varphi_\varepsilon} & q \\
\pi \uparrow & & \uparrow \pi \\
T_p^*(M) & \xrightarrow{\varphi_{-\varepsilon}^*} & T_q^*(M).
\end{array}
\qquad (2.5)
$$

Assume that we have some local chart with the coordinates x^0, x^1, x^2, x^3 in some neighborhood of the point q. Assume also that ε is small enough so that the point $p = \varphi_{-\varepsilon}(q)$ in (2.5) is covered by the same local chart. Then the coordinates of the points p and q in this chart are related to each other as follows:

$$
\begin{cases}
x^0 = u^0(\varepsilon, y^0, y^1, y^2, y^3), \\
x^1 = u^1(\varepsilon, y^0, y^1, y^2, y^3), \\
x^2 = u^2(\varepsilon, y^0, y^1, y^2, y^3), \\
x^3 = u^3(\varepsilon, y^0, y^1, y^2, y^3),
\end{cases}
\quad
\begin{cases}
y^0 = u^0(-\varepsilon, x^0, x^1, x^2, x^3), \\
y^1 = u^1(-\varepsilon, x^0, x^1, x^2, x^3), \\
y^2 = u^2(-\varepsilon, x^0, x^1, x^2, x^3), \\
y^3 = u^3(-\varepsilon, x^0, x^1, x^2, x^3).
\end{cases}
\qquad (2.6)
$$

Using (2.6), we define the matrices $\Phi(\varepsilon)$ and $\Phi(-\varepsilon)$ with the components

$$\Phi_j^i(\varepsilon) = \frac{\partial u^i(\varepsilon, y^0, y^1, y^2, y^3)}{\partial y^j}, \qquad \Phi_j^i(-\varepsilon) = \frac{\partial u^i(-\varepsilon, x^0, x^1, x^2, x^3)}{\partial x^j}. \qquad (2.7)$$

Let $Y_{j_1 \ldots j_s}^{i_1 \ldots i_r}(x^0, x^1, x^2, x^3)$ be the components of some tensorial field $\mathbf{Y}$ of the type (r, s) in the local coordinates x^0, x^1, x^2, x^3 and let $\varphi_\varepsilon(\mathbf{Y})_{j_1 \ldots j_s}^{i_1 \ldots i_r}(x^0, x^1, x^2, x^3)$ be the components of its image $\varphi_\varepsilon(\mathbf{Y})$ under the mapping (2.3). Then we have

$$\varphi_\varepsilon(\mathbf{Y})_{j_1 \ldots j_s}^{i_1 \ldots i_r}(x^0, x^1, x^2, x^3) = \sum_{\substack{h_1, \ldots, h_r \\ k_1, \ldots, k_s}}^{3} \cdots \sum^{3} \Phi_{h_1}^{i_1}(\varepsilon) \, \ldots \, \Phi_{h_r}^{i_r}(\varepsilon) \times$$

$$\times \Phi_{j_1}^{k_1}(-\varepsilon) \, \ldots \, \Phi_{j_s}^{k_s}(-\varepsilon) \, Y_{k_1 \ldots k_s}^{h_1 \ldots h_r}(y^0, y^1, y^2, y^3). \qquad (2.8)$$

According to [3], the Lie derivative $L_\mathbf{X}$ applied to $\mathbf{Y}$ is defined as follows:

$$L_\mathbf{X}(\mathbf{Y}) = \lim_{\varepsilon \to 0} \frac{\mathbf{Y} - \varphi_\varepsilon(\mathbf{Y})}{\varepsilon} = -\left.\frac{d\varphi_\varepsilon(\mathbf{Y})}{d\varepsilon}\right|_{\varepsilon=0}. \qquad (2.9)$$

Let X^0, X^1, X^2, X^3 be the components of the vector field $\mathbf{X}$ in the local coordinates x^0, x^1, x^2, x^3. Then for $\varepsilon \to 0$ we have the following Taylor expansions of the functions (2.6) representing the mappings φ_ε and $\varphi_{-\varepsilon}$:

$$
\begin{cases}
u^0(\varepsilon, y^0, y^1, y^2, y^3) = y^0 + X^0(y^0, y^1, y^2, y^3)\, \varepsilon + \ldots, \\
u^1(\varepsilon, y^0, y^1, y^2, y^3) = y^1 + X^1(y^0, y^1, y^2, y^3)\, \varepsilon + \ldots, \\
u^2(\varepsilon, y^0, y^1, y^2, y^3) = y^2 + X^2(y^0, y^1, y^2, y^3)\, \varepsilon + \ldots, \\
u^3(\varepsilon, y^0, y^1, y^2, y^3) = y^3 + X^3(y^0, y^1, y^2, y^3)\, \varepsilon + \ldots,
\end{cases}
\tag{2.10}
$$

$$
\begin{cases}
u^0(-\varepsilon, x^0, x^1, x^2, x^3) = x^0 - X^0(x^0, x^1, x^2, x^3)\, \varepsilon + \ldots, \\
u^1(-\varepsilon, x^0, x^1, x^2, x^3) = x^1 - X^1(x^0, x^1, x^2, x^3)\, \varepsilon + \ldots, \\
u^2(-\varepsilon, x^0, x^1, x^2, x^3) = x^2 - X^2(x^0, x^1, x^2, x^3)\, \varepsilon + \ldots, \\
u^3(-\varepsilon, x^0, x^1, x^2, x^3) = x^3 - X^3(x^0, x^1, x^2, x^3)\, \varepsilon + \ldots.
\end{cases}
\tag{2.11}
$$

Applying (2.10) and (2.11) to (2.7), we derive

$$
\Phi^i_j(\varepsilon) = \delta^i_j + \frac{\partial X^i(x^0, x^1, x^2, x^3)}{\partial x^j}\, \varepsilon + \ldots,
\tag{2.12}
$$

$$
\Phi^i_j(-\varepsilon) = \delta^i_j - \frac{\partial X^i(x^0, x^1, x^2, x^3)}{\partial x^j}\, \varepsilon + \ldots.
\tag{2.13}
$$

Here $X^i(x^0, x^1, x^2, x^3)$ are the components of the vector field $\mathbf{X}$ in the local coordinates x^0, x^1, x^2, x^3. Now if we denote by $L_{\mathbf{X}}(\mathbf{Y})^{i_1 \ldots i_r}_{j_1 \ldots j_s}$ the components of $L_{\mathbf{X}}(\mathbf{Y})$, then, applying (2.12) and (2.12) to (2.8) and taking into account (2.9), we obtain

$$
L_{\mathbf{X}}(\mathbf{Y})^{i_1 \ldots i_r}_{j_1 \ldots j_s} = \sum_{m=1}^{s} \sum_{k_m=0}^{3} \frac{\partial X^{k_m}}{\partial x^{j_m}}\, Y^{i_1 \ldots \ldots \ldots i_r}_{j_1 \ldots k_m \ldots j_s} -
$$
$$
- \sum_{m=1}^{r} \sum_{k_m=0}^{3} \frac{\partial X^{i_m}}{\partial x^{k_m}}\, Y^{i_1 \ldots k_m \ldots i_r}_{j_1 \ldots \ldots \ldots j_s} + \sum_{k=0}^{3} X^k\, \frac{\partial Y^{i_1 \ldots i_r}_{j_1 \ldots j_s}}{\partial x^k}.
\tag{2.14}
$$

The Lie derivative $L_{\mathbf{X}}$ given by the formula (2.14) possesses the following properties:

(1) the Lie derivative $L_{\mathbf{X}}$ preserves the type of a tensor field, i.e. $\mathbf{Y}$ and $L_{\mathbf{X}}(\mathbf{Y})$ are tensor fields of the same type;

(2) $L_{\mathbf{X}}(\mathbf{Y}_1 \otimes \mathbf{Y}_2) = L_{\mathbf{X}}(\mathbf{Y}_1) \otimes \mathbf{Y}_2 + \mathbf{Y}_1 \otimes L_{\mathbf{X}}(\mathbf{Y}_2)$ for arbitrary two tensorial fields $\mathbf{Y}_1$ and $\mathbf{Y}_2$;

(3) $L_{\mathbf{X}}(C(\mathbf{Y})) = C(L_{\mathbf{X}}(\mathbf{Y}))$, i.e. $L_{\mathbf{X}}$ commute with contractions.

The properties (1)–(3) are easily derived with the use of the formula (2.14) itself.

3. Lie derivatives in frames formalism.

Let x^0, x^1, x^2, x^3 be the local coordinates of some local chart of the space-time manifold M. The coordinates x^0, x^1, x^2, x^3 induce the frame $\mathbf{X}_0$, $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$ of the coordinate vector fields in the domain of these coordinates:

$$
\mathbf{X}_0 = \frac{\partial}{\partial x^0}, \qquad \mathbf{X}_1 = \frac{\partial}{\partial x^1}, \qquad \mathbf{X}_2 = \frac{\partial}{\partial x^2}, \qquad \mathbf{X}_3 = \frac{\partial}{\partial x^3}.
\tag{3.1}
$$

The frame composed by the vector fields (3.1) is a holonomic frame since these vector fields commute with each other:

$$[\mathbf{X}_i, \mathbf{X}_j] = 0.$$

However, one can consider some non-holonomic frame $\boldsymbol{\Upsilon}_0$, $\boldsymbol{\Upsilon}_1$, $\boldsymbol{\Upsilon}_2$, $\boldsymbol{\Upsilon}_3$, i.e. a frame with non-commuting vector fields:

$$[\boldsymbol{\Upsilon}_i, \boldsymbol{\Upsilon}_j] = \sum_{k=0}^{3} c_{ij}^k \, \boldsymbol{\Upsilon}_k. \tag{3.2}$$

The commutation coefficients c_{ij}^k in (3.2) are uniquely determined by the frame vector fields $\boldsymbol{\Upsilon}_0$, $\boldsymbol{\Upsilon}_1$, $\boldsymbol{\Upsilon}_2$, $\boldsymbol{\Upsilon}_3$ since they are linearly independent at each point of their domain. Our nearest goal is to derive the formula analogous to (2.14) for the case where all tensorial fields are represented by their components is some non-holonomic frame $\boldsymbol{\Upsilon}_0$, $\boldsymbol{\Upsilon}_1$, $\boldsymbol{\Upsilon}_2$, $\boldsymbol{\Upsilon}_3$.

Let φ be a scalar field. Then the Lie derivative $L_\mathbf{X}$ of φ is reduced to the differentiation of the function φ along the vector $\mathbf{X}$:

$$L_\mathbf{X}(\varphi) = \sum_{k=0}^{3} X^k \, \frac{\partial \varphi}{\partial x^k}. \tag{3.3}$$

The formula (3.3) is easily derived by substituting $\mathbf{Y} = \varphi$ with $r = 0$ and $s = 0$ into (2.14). Similarly, if $\mathbf{Y}$ is a vector field, from (2.14) we derive

$$L_\mathbf{Y}(\mathbf{Y}) = [\mathbf{X}, \mathbf{Y}]. \tag{3.4}$$

Now assume that both of the vector fields $\mathbf{X}$ and $\mathbf{Y}$ are represented by their expansions in a non-holonomic frame $\boldsymbol{\Upsilon}_0$, $\boldsymbol{\Upsilon}_1$, $\boldsymbol{\Upsilon}_2$, $\boldsymbol{\Upsilon}_3$:

$$\mathbf{X} = \sum_{k=0}^{3} X^k \, \boldsymbol{\Upsilon}_k, \qquad\qquad \mathbf{Y} = \sum_{i=0}^{3} Y^i \, \boldsymbol{\Upsilon}_i. \tag{3.5}$$

Then, substituting (3.5) into (3.4), we derive

$$L_\mathbf{X}(\mathbf{Y})^k = \sum_{i=0}^{3} X^i \, L_{\boldsymbol{\Upsilon}_i}(Y^k) - \sum_{i=0}^{3} Y^i \, L_{\boldsymbol{\Upsilon}_i}(X^k) + \sum_{i=0}^{3} \sum_{j=0}^{3} c_{ij}^k \, X^i Y^j. \tag{3.6}$$

The Lie derivatives $L_{\boldsymbol{\Upsilon}_i}(Y^k)$ and $L_{\boldsymbol{\Upsilon}_i}(X^k)$ in (3.6) are calculated according to the formula (3.3), i.e. we substitute $\varphi = Y^k$ and $\varphi = X^k$ into (3.3). Note that X^k in (3.3) differ from that of (3.5). The components of $\mathbf{X}$ in the formula (3.3) are taken from the expansion of the vector field $\mathbf{X}$ in the holonomic frame (3.1). The non-holonomic version of the formula (3.3) looks like

$$L_\mathbf{X}(\varphi) = \sum_{k=0}^{3} X^k \, L_{\boldsymbol{\Upsilon}_i}(\varphi).$$

Let's denote by $\boldsymbol{\eta}^0$, $\boldsymbol{\eta}^1$, $\boldsymbol{\eta}^2$, $\boldsymbol{\eta}^3$ the dual frame for $\boldsymbol{\Upsilon}_0$, $\boldsymbol{\Upsilon}_1$, $\boldsymbol{\Upsilon}_2$, $\boldsymbol{\Upsilon}_3$. This means that $\boldsymbol{\eta}^0$, $\boldsymbol{\eta}^1$, $\boldsymbol{\eta}^2$, $\boldsymbol{\eta}^3$ are four covectorial fields such that

$$\eta^i(\boldsymbol{\Upsilon}_j) = (\boldsymbol{\eta}^i, \boldsymbol{\Upsilon}_j) = C(\boldsymbol{\eta}^i \otimes \boldsymbol{\Upsilon}_j) = \delta^i_j. \tag{3.7}$$

Using the properties (1)–(3) from Section 2 and using (3.4), from (3.7) we derive

$$L_{\boldsymbol{\Upsilon}_i}(\boldsymbol{\eta}^k) = -\sum_{j=0}^{3} c^k_{ij}\,\boldsymbol{\eta}^j. \tag{3.8}$$

The commutation coefficients c^k_{ij} in (3.8) are the same as in (3.2). Assume that $\mathbf{Y}$ is a covectorial field expanded in the frame $\boldsymbol{\eta}^0$, $\boldsymbol{\eta}^1$, $\boldsymbol{\eta}^2$, $\boldsymbol{\eta}^3$:

$$\mathbf{Y} = \sum_{j=1}^{3} Y_i\,\boldsymbol{\eta}^i. \tag{3.9}$$

From (3.7), (3.8), and (3.9) we derive

$$L_{\mathbf{X}}(\mathbf{Y})_k = \sum_{i=0}^{3} X^i\,L_{\boldsymbol{\Upsilon}_i}(Y_k) + \sum_{i=0}^{3} Y_i\,L_{\boldsymbol{\Upsilon}_k}(X^i) - \sum_{i=0}^{3}\sum_{j=0}^{3} X^i Y_j\,c^j_{ik}. \tag{3.10}$$

Now, using the properties (1)–(3) again and combining (3.6) with (3.10), we can extend the formula (3.6) to the case of an arbitrary tensor field $\mathbf{Y}$ of the type (r, s):

$$L_{\mathbf{X}}(\mathbf{Y})^{i_1 \ldots i_r}_{j_1 \ldots j_s} = \sum_{i=0}^{3} X^i\,L_{\boldsymbol{\Upsilon}_i}(Y^{i_1 \ldots i_r}_{j_1 \ldots j_s}) +$$

$$+ \sum_{m=1}^{s}\sum_{k_m=0}^{3}\left(L_{\boldsymbol{\Upsilon}_{j_m}}(X^{k_m}) - \sum_{i=0}^{3} c^{k_m}_{ij_m} X^i\right) Y^{i_1 \ldots \ldots \ldots i_r}_{j_1 \ldots k_m \ldots j_s} - \tag{3.11}$$

$$- \sum_{m=1}^{r}\sum_{k_m=0}^{3}\left(L_{\boldsymbol{\Upsilon}_{k_m}}(X^{i_m}) - \sum_{i=0}^{3} c^{i_m}_{ik_m} X^i\right) Y^{i_1 \ldots k_m \ldots i_r}_{j_1 \ldots \ldots \ldots j_s}.$$

The formulas (3.6) and (3.10) are special cases of the formula (3.11). If the frame $\boldsymbol{\Upsilon}_0$, $\boldsymbol{\Upsilon}_1$, $\boldsymbol{\Upsilon}_2$, $\boldsymbol{\Upsilon}_3$ coincides with the holonomic frame (3.1), then $c^k_{ij} = 0$ and the formula (3.11) reduces to (2.14).

Now let's return back to the formula (2.8). This formula is valid in frame presentation of tensor fields too. However, the matrices $\Phi^i_j(\varepsilon)$ and $\Phi^i_j(-\varepsilon)$ in this case are not given by the formulas (2.7). Here we use the formulas

$$\Phi^i_j(\varepsilon) = \delta^i_j + \left(L_{\boldsymbol{\Upsilon}_j}(X^i) - \sum_{m=0}^{3} X^m\,c^i_{mj}\right)\varepsilon + \ldots, \tag{3.12}$$

$$\Phi^i_j(-\varepsilon) = \delta^i_j - \left(L_{\boldsymbol{\Upsilon}_j}(X^i) - \sum_{m=0}^{3} X^m\,c^i_{mj}\right)\varepsilon + \ldots. \tag{3.13}$$

The formulas (3.12) and (3.13) are analogous to (2.12) and (2.13).

4. Tangent vector fields on vector bundles.

Let VM be an n-dimensional vector bundle over the space-time manifold M. Any trivialization of VM is given by n sections $\boldsymbol{\Upsilon}_1, \ldots, \boldsymbol{\Upsilon}_n$ linearly independent at each point of their domain U. Let $\mathbf{v}$ be a vector of the fiber $V_q(M)$:

$$\mathbf{v} = v^1\,\boldsymbol{\Upsilon}_1 + \ldots + v^n\,\boldsymbol{\Upsilon}_n. \tag{4.1}$$

Then $\tilde{q} = (q, \mathbf{v})$ is a point of VM. If x^0, x^1, x^2, x^3 are some local coordinates within the domain $U \subset M$ and v^0, v^1, v^2, v^3 are taken from (4.1), then

$$x^0, \ldots, x^3, v^1, \ldots, v^n \tag{4.2}$$

are the coordinates of the point $\tilde{q} = (q, \mathbf{v})$. The coordinates (4.2) are naturally sub-divided into two groups — the base coordinates $x^0, \ldots, x^3$ and the fiber coordinates $v^1, \ldots, v^n$. Let $\ulcorner\mathbf{X}$ be a tangent vector field on VM. In the local coordinates (4.2) it is represented as the following differential operator:

$$\ulcorner\mathbf{X} = \sum_{i=0}^{3} X^i\,\frac{\partial}{\partial x^i} + \sum_{i=1}^{n} V^i\,\frac{\partial}{\partial v^i}. \tag{4.3}$$

Under the canonical projection $\pi\colon VM \to M$ the vector field (4.3) is mapped to

$$\mathbf{X} = \sum_{i=0}^{3} X^i\,\frac{\partial}{\partial x^i}. \tag{4.4}$$

The vector field (4.3) produces the one-parametric local group of diffeomorphisms

$$\varphi_\varepsilon\colon VM \to VM \tag{4.5}$$

that extends the local group (2.1) produced by the vector field (4.4). Due to (4.5) the functions (2.6) are complemented with the functions

$$\begin{cases} v^1 = U^1(\varepsilon, y^0, \ldots, y^3, w^1, \ldots, w^n), \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ v^n = U^n(\varepsilon, y^0, \ldots, y^3, w^1, \ldots, w^n), \end{cases} \tag{4.6}$$

$$\begin{cases} w^1 = U^1(-\varepsilon, x^0, \ldots, x^3, v^1, \ldots, v^n), \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ w^n = U^n(-\varepsilon, x^0, \ldots, x^3, v^1, \ldots, v^n). \end{cases} \tag{4.7}$$

Definition 4.1. The tangent vector field (4.3) on VM is called *concordant with the bundle structure* if the functions (4.6) and (4.7) are linear with respect to their arguments $v^1, \ldots, v^n$ and $w^1, \ldots, w^n$.

In the case of a concordant vector field $\ulcorner\mathbf{X}$ the local diffeomorphisms (4.5) break into the series of linear mappings

$$\varphi_\varepsilon\colon V_p(M) \to V_q(M), \tag{4.8}$$

where $q = \varphi_\varepsilon(p)$. The functions (4.6) present the diffeomorphisms (4.8) in local coordinates. In the case of a concordant vector field they are given by the formulas

$$U^i(\varepsilon, y^0, \ldots, y^3, w^1, \ldots, w^n) = \sum_{j=1}^n U^i_j(\varepsilon, y^0, \ldots, y^3)\, w^j. \tag{4.9}$$

The vertical components of the tangent vector field (4.3) are given by the derivatives

$$V^i = \left.\frac{dU^i(\varepsilon, x^0, \ldots, x^3, v^1, \ldots, v^n)}{d\varepsilon}\right|_{\varepsilon=0}. \tag{4.10}$$

Substituting (4.9) into (4.10), we derive

$$V^i = \sum_{j=1}^n V^i_j(x^0, \ldots, x^3)\, v^j, \quad \text{where } V^i_j = \left.U^i_j\right|_{\varepsilon=0}. \tag{4.11}$$

Theorem 4.1. *The tangent vector field* (4.3) *on a vector bundle* VM *is concordant with the bundle structure if and only if its vertical components are linear functions with respect to* $v^1, \ldots, v^n$ *given by the formula* (4.11)

For the matrices U^i_j in (4.9), which represent the linear mappings (4.8) in the frame $\Upsilon_1, \ldots, \Upsilon_n$, the formulas (4.10) and (4.11) yield

$$U^i_j(\varepsilon) = \delta^i_j + V^i_j(x^0, \ldots, x^3)\,\varepsilon + \ldots,$$

$$U^i_j(-\varepsilon) = \delta^i_j - V^i_j(x^0, \ldots, x^3)\,\varepsilon + \ldots. \tag{4.12}$$

The expansions (4.12) are similar to (2.12), (2.13), (3.12), and (3.13).

Let $\mathbf{Y}$ be a tensor field of the type (r, s) associated with the vector bundle VM, i.e. let $\mathbf{Y}$ be a section of $V^r_s M$, where $V^r_s M$ is the following tensor bundle:

$$V^r_s M = \overbrace{VM \otimes \ldots \otimes VM}^{r \text{ times}} \otimes \underbrace{V^*M \otimes \ldots \otimes V^*M}_{s \text{ times}}. \tag{4.13}$$

Assume that $Y^{i_1 \ldots i_r}_{j_1 \ldots j_s}(x^0, x^1, x^2, x^3)$ are the components of the tensor field $\mathbf{Y}$ in the frame $\Upsilon_1, \ldots, \Upsilon_n$. Then, using the quantities V^i_j from the expansions (4.12), we define the Lie derivative $\mathcal{L}_{\mathbf{X}}(\mathbf{Y})$ of the field $\mathbf{Y}$:

$$\mathcal{L}_{\mathbf{X}}(\mathbf{Y})^{i_1 \ldots i_r}_{j_1 \ldots j_s} = \sum_{i=0}^3 X^i\, L_{\Upsilon_i}(Y^{i_1 \ldots i_r}_{j_1 \ldots j_s}) +$$

$$+ \sum_{m=1}^s \sum_{k_m=1}^n V^{k_m}_{j_m}\, Y^{i_1 \ldots \ldots \ldots i_r}_{j_1 \ldots k_m \ldots j_s} - \sum_{m=1}^r \sum_{k_m=1}^n V^{i_m}_{k_m}\, Y^{i_1 \ldots k_m \ldots i_r}_{j_1 \ldots \ldots \ldots j_s}. \tag{4.14}$$

The Lie derivative (4.14) is called natural if the quantities V^i_j are expressed in some natural way through the components of the vector field $\mathbf{X}$ in (4.4).

5. Natural liftings and Kosmann liftings.

In this section we apply the results of the previous section 4 to the tangent bundle TM, i.e. we set $VM = TM$. Comparing the formula (4.13) with (2.4) and the formula (4.14) with (3.11), we find that

$$V_j^i = L_{\Upsilon_j}(X^i) - \sum_{m=0}^{3} X^m \, c_{mj}^i. \tag{5.1}$$

Now we substitute (5.1) into (4.11) and then we substitute (4.11) into (4.3):

$$\ulcorner\mathbf{X}_N = \sum_{i=0}^{3}\sum_{m=0}^{3} X^m \, \Upsilon_m^i \, \frac{\partial}{\partial x^i} + \sum_{i=0}^{3}\sum_{j=0}^{3}\left(L_{\Upsilon_j}(X^i) - \sum_{m=0}^{3} X^m \, c_{mj}^i\right) v^j \, \frac{\partial}{\partial v^i}. \tag{5.2}$$

The tangent vector field (5.2) on TM is a natural lifting of the vector field

$$\mathbf{X} = \sum_{i=0}^{3}\sum_{m=0}^{3} X^m \, \Upsilon_m^i \, \frac{\partial}{\partial x^i} = \sum_{m=0}^{3} X^m \, \Upsilon_m \tag{5.3}$$

from M to TM. The Lie derivative (4.14) determined by the vector field (5.3) and by its lifting (5.2) is a natural Lie derivative coinciding with (3.11) and (2.14).

Now let's recall that the the space-time manifold M is equipped with the metric $\mathbf{g}$. Its signature is $(+,-,-,-)$. For this reason each fiber $T_p(M)$ of the tangent bundle TM is a pseudo-Euclidean linear vector space.

Definition 5.1. A lifting $\ulcorner\mathbf{X}$ of a vector field $\mathbf{X}$ from M to TM is called a *Kosmann lifting* if the linear mappings (4.8) associated with this lifting are isometries.

Kossman liftings were first introduced by Yvette Kosmann in [4–7].

Theorem 5.1. *The natural lifting* (5.2) *of a vector field* $\mathbf{X}$ *is a Kosmann lifting if and only if* $\mathbf{X}$ *is a Killing vector field.*

The proof is trivial. By definition, Killing vector fields are those whose local diffeomorphisms preserve the metric tensor $\mathbf{g}$. Hence, the mapping $\varphi_{\varepsilon*}$ from (2.2) restricted to any fiber $T_p(M)$ is a linear isometry.

Let's study the isometry condition from the definition 5.1 in more details. Applying the linear mappings (4.8) to the metric tensor, we get the equality

$$g_{ij}(x^0, \ldots, x^3) = \sum_{r=0}^{3}\sum_{s=0}^{3} U_i^r(-\varepsilon) \, U_j^s(-\varepsilon) \, g_{rs}(y^0, \ldots, y^3). \tag{5.4}$$

Differentiating the formula (5.4) with respect to ε, we take into account the formulas (4.12), (2.6), (2.10), (2.11), (4.10), and (4.11). As a result we get

$$0 = \sum_{r=0}^{3} V_i^r \, g_{rj} + \sum_{r=0}^{3} V_j^r \, g_{ir} + \sum_{m=0}^{3} X^m \, L_{\Upsilon_m}(g_{ij}). \tag{5.5}$$

The equality (5.5) can be simplified to

$$\mathcal{L}_{\mathbf{X}}(\mathbf{g}) = 0, \tag{5.6}$$

where the Lie derivative $\mathcal{L}_{\mathbf{X}}$ is calculated according to the formula (4.14). We shall treat the equality (5.6) neither as a condition for $\mathbf{X}$ nor as a condition for $\mathbf{g}$, but as a condition for V_j^i. For this purpose we denote

$$V_{ij} = \sum_{r=0}^{3} V_i^r \, g_{rj}. \tag{5.7}$$

Then the equality (5.5), which is equivalent to (5.6) is written as follows:

$$V_{ij} + V_{ji} = -\sum_{m=0}^{3} X^m \, L_{\boldsymbol{\Upsilon}_m}(g_{ij}). \tag{5.8}$$

The equality (5.8) fixes the symmetric part of V_{ij} for a Kosmann lifting of a vector field. The skew-symmetric part of V_{ij} can be obtained by alternating (5.1). As a result we get the following two formulas:

$$V_{ij}^{\mathrm{sym}} = -\frac{1}{2} \sum_{m=0}^{3} X^m \, L_{\boldsymbol{\Upsilon}_m}(g_{ij}), \tag{5.9}$$

$$V_{ij}^{\mathrm{skew}} = \frac{1}{2} \sum_{r=0}^{3} L_{\boldsymbol{\Upsilon}_i}(X^r) \, g_{rj} - \frac{1}{2} \sum_{r=0}^{3} L_{\boldsymbol{\Upsilon}_j}(X^r) \, g_{ri} - $$
$$- \frac{1}{2} \sum_{r=0}^{3} \sum_{m=0}^{3} X^m \, c_{mi}^r \, g_{rj} + \frac{1}{2} \sum_{r=0}^{3} \sum_{m=0}^{3} X^m \, c_{mj}^r \, g_{ri}. \tag{5.10}$$

Adding the formulas (5.9) and (5.10), we derive

$$V_{ij} = -\frac{1}{2} \sum_{m=0}^{3} X^m \, L_{\boldsymbol{\Upsilon}_m}(g_{ij}) + \frac{1}{2} \sum_{r=0}^{3} L_{\boldsymbol{\Upsilon}_i}(X^r) \, g_{rj} - $$
$$- \frac{1}{2} \sum_{r=0}^{3} L_{\boldsymbol{\Upsilon}_j}(X^r) \, g_{ri} - \frac{1}{2} \sum_{r=0}^{3} \sum_{m=0}^{3} X^m \, c_{mi}^r \, g_{rj} + \frac{1}{2} \sum_{r=0}^{3} \sum_{m=0}^{3} X^m \, c_{mj}^r \, g_{ri}. \tag{5.11}$$

In order to get back to V_i^j we need to raise the index j in (5.11):

$$V_i^j = -\frac{1}{2} \sum_{m=0}^{3} \sum_{r=0}^{3} X^m \, L_{\boldsymbol{\Upsilon}_m}(g_{ir}) \, g^{rj} - \frac{1}{2} \sum_{r=0}^{3} \sum_{s=0}^{3} L_{\boldsymbol{\Upsilon}_s}(X^r) \, g_{ri} \, g^{sj} + $$
$$+ \frac{1}{2} L_{\boldsymbol{\Upsilon}_i}(X^j) - \frac{1}{2} \sum_{m=0}^{3} X^m \, c_{mi}^j + \frac{1}{2} \sum_{r=0}^{3} \sum_{s=0}^{3} \sum_{m=0}^{3} X^m \, c_{ms}^r \, g_{ri} \, g^{sj}. \tag{5.12}$$

Note that in order to fit (4.11) we should exchange the indices i and j in (5.12):

$$V_j^i = -\frac{1}{2} \sum_{m=0}^{3} \sum_{r=0}^{3} g^{ir} \, X^m \, L_{\boldsymbol{\Upsilon}_m}(g_{rj}) - \frac{1}{2} \sum_{r=0}^{3} \sum_{s=0}^{3} g^{is} \, L_{\boldsymbol{\Upsilon}_s}(X^r) \, g_{rj} + $$
$$+ \frac{1}{2} L_{\boldsymbol{\Upsilon}_j}(X^i) - \frac{1}{2} \sum_{m=0}^{3} X^m \, c_{mj}^i + \frac{1}{2} \sum_{r=0}^{3} \sum_{s=0}^{3} \sum_{m=0}^{3} g^{is} \, X^m \, c_{ms}^r \, g_{rj}. \tag{5.13}$$

As a result, substituting (5.13) into (4.11), we derive the formula

$$V^i = -\frac{1}{2}\sum_{m=0}^{3}\sum_{r=0}^{3}\sum_{j=0}^{3} g^{ir}\,X^m\,L_{\Upsilon_m}(g_{rj})\,v^j + \frac{1}{2}\sum_{j=0}^{3} L_{\Upsilon_j}(X^i)\,v^j -$$

$$-\frac{1}{2}\sum_{r=0}^{3}\sum_{s=0}^{3}\sum_{j=0}^{3} g^{is}\,L_{\Upsilon_s}(X^r)\,g_{rj}\,v^j - \frac{1}{2}\sum_{m=0}^{3}\sum_{j=0}^{3} X^m\,c^i_{mj}\,v^j + \qquad (5.14)$$

$$+\frac{1}{2}\sum_{r=0}^{3}\sum_{s=0}^{3}\sum_{m=0}^{3}\sum_{j=0}^{3} g^{is}\,X^m\,c^r_{ms}\,g_{rj}\,v^j.$$

Applying (5.14) to (4.3) and taking into account that we choose $VM = TM$, we get the following tangent vector field on the tangent bundle:

$$\overset{\ulcorner}{\mathbf{X}}_K = \sum_{i=0}^{3}\sum_{m=0}^{3} X^m\,\Upsilon^i_m\,\frac{\partial}{\partial x^i} + \frac{1}{2}\sum_{i=0}^{3}\left(\sum_{j=0}^{3} L_{\Upsilon_j}(X^i)\,v^j -\right.$$

$$-\sum_{m=0}^{3}\sum_{j=0}^{3} X^m\,c^i_{mj}\,v^j + \sum_{r=0}^{3}\sum_{s=0}^{3}\sum_{m=0}^{3}\sum_{j=0}^{3} g^{is}\,X^m\,c^r_{ms}\,g_{rj}\,v^j - \qquad (5.15)$$

$$-\sum_{r=0}^{3}\sum_{s=0}^{3}\sum_{j=0}^{3} g^{is}\,L_{\Upsilon_s}(X^r)\,g_{rj}\,v^j - \sum_{m=0}^{3}\sum_{r=0}^{3}\sum_{j=0}^{3} g^{ir}\,X^m\,L_{\Upsilon_m}(g_{rj})\,v^j\left.\right)\frac{\partial}{\partial v^i}.$$

Definition 5.2. The tangent vector field (5.15) on the tangent bundle TM is called the *standard Kosmann lifting* of the vector field (4.4) from M to TM.

We can calculate the standard Kosmann lifting (5.15) using the holonomic frame (3.1) instead of the non-holonomic frame Υ_0, Υ_1, Υ_2, Υ_3. Then (5.15) reduces to

$$\overset{\ulcorner}{\mathbf{X}}_K = \sum_{i=0}^{3} X^i\,\frac{\partial}{\partial x^i} + \frac{1}{2}\sum_{i=0}^{3}\left(-\sum_{r=0}^{3}\sum_{s=0}^{3}\sum_{j=0}^{3} g^{is}\,\frac{\partial X^r}{\partial x^s}\,g_{rj}\,v^j +\right.$$

$$+\sum_{j=0}^{3}\frac{\partial X^i}{\partial x^j}\,v^j - \sum_{m=0}^{3}\sum_{r=0}^{3}\sum_{j=0}^{3} g^{ir}\,X^m\,\frac{\partial g_{rj}}{\partial x^m}\,v^j\left.\right)\frac{\partial}{\partial v^i}. \qquad (5.16)$$

Now let's remember that the metric $\mathbf{g}$ is associated with the metric connection Γ. It is known as the Levi-Civita connections. The components of the Levi-Civita connection in a holonomic frame (3.1) are given by the formula

$$\Gamma^k_{ij} = \sum_{r=0}^{3}\frac{g^{kr}}{2}\left(\frac{\partial g_{rj}}{\partial x^i} + \frac{\partial g_{ir}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^r}\right). \qquad (5.17)$$

The formula (5.17) is easily derived from the following two conditions (see [8]):

$$\Gamma^k_{ij} = \Gamma^k_{ji}, \qquad\qquad\qquad \nabla_k g_{ij} = 0. \qquad (5.18)$$

Using the connection components Γ_{ij}^k, now we express the partial derivatives in the formula (5.16) through the corresponding covariant derivatives:

$$\begin{aligned}
\frac{\partial X^r}{\partial x^s} &= \nabla_s X^r - \sum_{m=0}^{3} \Gamma_{sm}^r \, X^m, \\
\frac{\partial X^i}{\partial x^j} &= \nabla_j X^i - \sum_{m=0}^{3} \Gamma_{jm}^i \, X^m.
\end{aligned} \tag{5.19}$$

Moreover, from the second equality (5.18) we derive

$$\frac{\partial g_{rj}}{\partial x^m} = \sum_{s=0}^{3} \Gamma_{mj}^s \, g_{rs} + \sum_{s=0}^{3} \Gamma_{mr}^s \, g_{sj}. \tag{5.20}$$

Substituting (5.19) and (5.20) back into (5.16) and taking into account the first equality in (5.17), we get the following formula:

$$\begin{aligned}
\overline{\mathbf{X}}_K = \sum_{i=0}^{3} X^i \frac{\partial}{\partial x^i} + \sum_{i=0}^{3} \sum_{j=0}^{3} \Bigg(&-\frac{1}{2} \sum_{r=0}^{3} \sum_{s=0}^{3} g^{is} \, \nabla_s X^r \, g_{rj} + \\
&+ \frac{1}{2} \nabla_j X^i - \sum_{m=0}^{3} X^m \, \Gamma_{mj}^i \Bigg) v^j \frac{\partial}{\partial v^i}.
\end{aligned} \tag{5.21}$$

The components of the natural lifting (5.2) also can be expressed through the connection components Γ_{ij}^k and covariant derivatives in the holonomic frame (3.1):

$$\overline{\mathbf{X}}_N = \sum_{i=0}^{3} X^i \frac{\partial}{\partial x^i} + \sum_{i=0}^{3} \sum_{j=0}^{3} \Bigg(\nabla_j X^i - \sum_{m=0}^{3} X^m \, \Gamma_{mj}^i \Bigg) v^j \frac{\partial}{\partial v^i}. \tag{5.22}$$

Two different liftings (5.21) and (5.22) are associated with two different Lie derivatives $\mathcal{L}_{\mathbf{X}}$ and $L_{\mathbf{X}}$ respectively. Here $L_{\mathbf{X}}$ is the regular Lie derivative, while $\mathcal{L}_{\mathbf{X}}$ is called the *standard Kosmann-Lie derivative*. Both of them are differentiations of the algebra of tensor fields. Comparing (5.21) and (5.22), we get the relationship

$$\mathcal{L}_{\mathbf{X}} = L_{\mathbf{X}} + S_{\mathbf{X}}. \tag{5.23}$$

Here $S_{\mathbf{X}}$ is a degenerate differentiation in the sense of the proposition 3.3 in Chapter I of [3]. From (5.21) and (5.22) we derive that the degenerate differentiation $S_{\mathbf{X}}$ in (5.23) is given by the tensor field $\mathbf{S}_{\mathbf{X}}$ with the following components:

$$S_j^i(\mathbf{X}) = \frac{\nabla_j X^i + \nabla^i X_j}{2} = \frac{1}{2} \, \nabla_j X^i + \frac{1}{2} \sum_{r=0}^{3} \sum_{s=0}^{3} g^{is} \, \nabla_s X^r \, g_{rj} \tag{5.24}$$

The tensor field $\mathbf{S}_{\mathbf{X}}$ with the components (5.24) is equal to zero if and only if $\mathbf{X}$ is a Killing vector field. This fact is easily derived from the theorem 5.1 or from the equality (5.6), which is fulfilled identically by definition.

Another special case for the formula (5.15) is the case of a non-holonomic, but orthonormal frame Υ_0, Υ_1, Υ_2, Υ_3. In such a frame the components of the metric tensor are constants. They are given by the Minkowski matrix:

$$g_{ij} = g^{ij} = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{Vmatrix}. \tag{5.25}$$

For the components of the Levi-Civita connection in such a frame we have

$$\Gamma_{ij}^k = \frac{c_{ij}^k}{2} - \sum_{r=0}^{3}\sum_{s=0}^{3} \frac{c_{ir}^s}{2}\, g^{kr}\, g_{sj} - \sum_{r=0}^{3}\sum_{s=0}^{3} \frac{c_{jr}^s}{2}\, g^{kr}\, g_{si}. \tag{5.26}$$

The formula (5.26) is derived from the general formula (6.3) in [9]. From the formulas (5.25) and (5.26) we derive the following equalities:

$$L_{\Upsilon_m}(g_{rj}) = 0, \qquad\qquad \Gamma_{ij}^k - \Gamma_{ji}^k = c_{ij}^k. \tag{5.27}$$

Moreover, from $\nabla_m g_{ij} = 0$ in this case we derive

$$\sum_{r=0}^{3}\sum_{s=0}^{3} g^{is}\, \Gamma_{ms}^r\, g_{rj} = -\Gamma_{mj}^i. \tag{5.28}$$

In the non-holonomic frame Υ_0, Υ_1, Υ_2, Υ_3 the formulas (5.19) are replaced by

$$L_{\Upsilon_s}(X^r) = \nabla_s X^r - \sum_{m=0}^{3} \Gamma_{sm}^r\, X^m,$$
$$L_{\Upsilon_j}(X^i) = \nabla_j X^i - \sum_{m=0}^{3} \Gamma_{jm}^i\, X^m. \tag{5.29}$$

Applying (5.27), (5.28), and (5.29) to (5.15), we get the formula

$$\overline{\mathbf{X}}_K = \sum_{i=0}^{3}\sum_{m=0}^{3} X^m\, \Upsilon_m^i\, \frac{\partial}{\partial x^i} + \sum_{i=0}^{3}\sum_{j=0}^{3} \left(\frac{1}{2}\nabla_j X^i - \right.$$
$$\left. - \frac{1}{2}\sum_{r=0}^{3}\sum_{s=0}^{3} g^{is}\, \nabla_s X^r\, g_{rj} - \sum_{m=0}^{3} X^m\, \Gamma_{mj}^i \right) v^j\, \frac{\partial}{\partial v^i}. \tag{5.30}$$

Similarly, applying (5.29) to (5.2) and taking into account (5.27), we get

$$\overline{\mathbf{X}}_N = \sum_{i=0}^{3}\sum_{m=0}^{3} X^m\, \Upsilon_m^i\, \frac{\partial}{\partial x^i} + \sum_{i=0}^{3}\sum_{j=0}^{3} \left(\nabla_j X^i - \sum_{m=0}^{3} X^m\, \Gamma_{mj}^i \right) v^j\, \frac{\partial}{\partial v^i}. \tag{5.31}$$

The formulas (5.30) to (5.31) coincide with (5.21) to (5.22) respectively, though Γ_{mj}^i are not symmetric with respect to m and j in this case. The formulas (5.30)

and (5.31) again lead to the formula (5.23), which is valid irrespective to the choice of a holonomic or non-holonomic frame in M.

Note that the formulas (5.29) are the same as the formula (3.10) in [10]. The formula (5.29) resembles the formulas (3.11) and (3.12) in [10]. However, it doesn't coincide with them. Unlike [10] and [11], for the sake of simplicity in this paper I do not use principal fiber bundles at all.

6. Commutation relationships for Kosmann-Lie derivatives.

Regular Lie derivatives acting upon tensorial fields form a representation of the Lie algebra of vector fields in M. This fact is expressed by the formula

$$[L_{\mathbf{X}}, L_{\mathbf{Y}}] = L_{[\mathbf{X}, \mathbf{Y}]}. \tag{6.1}$$

Commutation relationships for Kosmann-Lie derivatives are different from (6.1):

$$[\mathcal{L}_{\mathbf{X}}, \mathcal{L}_{\mathbf{Y}}] = \mathcal{L}_{[\mathbf{X}, \mathbf{Y}]} + S_{\mathbf{X}, \mathbf{Y}}. \tag{6.2}$$

Here $S_{\mathbf{X}, \mathbf{Y}}$ is a degenerate differentiation given by the tensor field $\mathbf{S}_{\mathbf{X}, \mathbf{Y}}$, where

$$\mathbf{S}_{\mathbf{X}, \mathbf{Y}} = L_{\mathbf{X}}(\mathbf{S}_{\mathbf{Y}}) - L_{\mathbf{Y}}(\mathbf{S}_{\mathbf{X}}) - \mathbf{S}_{[\mathbf{X}, \mathbf{Y}]} + [\mathbf{S}_{\mathbf{X}}, \mathbf{S}_{\mathbf{Y}}]. \tag{6.3}$$

By means of direct calculations the formula (6.3) can be reduced to

$$\mathbf{S}_{\mathbf{X}, \mathbf{Y}} = -[\mathbf{S}_{\mathbf{X}}, \mathbf{S}_{\mathbf{Y}}]. \tag{6.4}$$

Applying (6.4) to (6.2), we get

$$[\mathcal{L}_{\mathbf{X}}, \mathcal{L}_{\mathbf{Y}}] = \mathcal{L}_{[\mathbf{X}, \mathbf{Y}]} - [\mathbf{S}_{\mathbf{X}}, \mathbf{S}_{\mathbf{Y}}]. \tag{6.5}$$

The commutator $[\mathbf{S}_{\mathbf{X}}, \mathbf{S}_{\mathbf{Y}}]$ in the formulas (6.3), (6.4), and (6.5) is understood as a commutator of two operator-valued tensorial fields:

$$[\mathbf{S}_{\mathbf{X}}, \mathbf{S}_{\mathbf{Y}}] = \mathbf{S}_{\mathbf{X}} \circ \mathbf{S}_{\mathbf{Y}} - \mathbf{S}_{\mathbf{X}} \circ \mathbf{S}_{\mathbf{Y}}. \tag{6.6}$$

In the coordinate representation the commutator (6.6) turns to the commutator of two matrices whose components are calculated according to the formula (5.24).

Note that in general case the commutator (6.6) is not zero. For this reason $[\mathcal{L}_{\mathbf{X}}, \mathcal{L}_{\mathbf{Y}}] \neq \mathcal{L}_{[\mathbf{X}, \mathbf{Y}]}$. This fact is pointed out in [10].

7. Kosmann-Lie derivatives for Weyl spinors.

The bundle of Weyl spinors is a two-dimensional complex vector bundle over the space-time manifold M. We denote it SM. The spinor bundle SM is related to the tangent bundle TM in some special way. The relation of SM and TM is formulated in terms of frames. It is based on the well-known group homomorphism

$$\phi: \ \mathrm{SL}(2, \mathbb{C}) \to \mathrm{SO}^+(1, 3, \mathbb{R}). \tag{7.1}$$

Let $\boldsymbol{\Upsilon}_0$, $\boldsymbol{\Upsilon}_1$, $\boldsymbol{\Upsilon}_2$, $\boldsymbol{\Upsilon}_3$ be a positively polarized right orthonormal frame of the tangent bundle TM. By definition (see Section 5 in [12] or Section 1 in [13]) it is canonically associated with some frame $\boldsymbol{\Psi}_1$, $\boldsymbol{\Psi}_2$ of SM in such a way that if two positively polarized right orthonormal frames $\boldsymbol{\Upsilon}_0$, $\boldsymbol{\Upsilon}_1$, $\boldsymbol{\Upsilon}_2$, $\boldsymbol{\Upsilon}_3$ and $\tilde{\boldsymbol{\Upsilon}}_0$, $\tilde{\boldsymbol{\Upsilon}}_1$, $\tilde{\boldsymbol{\Upsilon}}_2$, $\tilde{\boldsymbol{\Upsilon}}_3$ are bound with the transition matrices S and $T = S^{-1}$ in the formulas

$$\tilde{\boldsymbol{\Upsilon}}_i = \sum_{j=0}^{3} S_i^j \, \boldsymbol{\Upsilon}_j, \qquad\qquad \boldsymbol{\Upsilon}_i = \sum_{j=0}^{3} T_i^j \, \tilde{\boldsymbol{\Upsilon}}_j,$$

then their associated spinor frames $\boldsymbol{\Psi}_1$, $\boldsymbol{\Psi}_2$ and $\tilde{\boldsymbol{\Psi}}_1$, $\tilde{\boldsymbol{\Psi}}_2$ are bound with the transition matrices $\mathfrak{S}$ and $\mathfrak{T} = \mathfrak{S}^{-1}$ in the formulas

$$\tilde{\boldsymbol{\Psi}}_i = \sum_{j=1}^{2} \mathfrak{S}_i^j \, \boldsymbol{\Psi}_j, \qquad\qquad \boldsymbol{\Psi}_i = \sum_{j=1}^{2} \mathfrak{T}_i^j \, \tilde{\boldsymbol{\Psi}}_j$$

and the spacial transition matrices S and T are produced from the spinor transition matrices $\mathfrak{S}$ and $\mathfrak{T}$ through the group homomorphism (7.1):

$$S = \phi(\mathfrak{S}), \qquad\qquad T = \phi(\mathfrak{T}).$$

A spinor frame $\boldsymbol{\Psi}_1$, $\boldsymbol{\Psi}_2$ canonically associated with some positively polarized right orthonormal tangent frame $\boldsymbol{\Upsilon}_0$, $\boldsymbol{\Upsilon}_1$, $\boldsymbol{\Upsilon}_2$, $\boldsymbol{\Upsilon}_3$ is called an *orthonormal frame* of the spinor bundle SM. In order to visualize this canonical frame association in SM and TM we use the following diagram:

$$\boxed{\text{Orthonormal frames}} \rightarrow \boxed{\begin{array}{c}\text{Positively polarized}\\ \text{right orthonormal frames}\end{array}}. \qquad (7.2)$$

There are two basic spin-tensorial fields in the bundle of Weyl spinors:

Symbol	Name	Spin-tensorial type
$\mathbf{d}$	Skew-symmetric metric tensor	$(0,2\vert 0,0\vert 0,0)$
$\mathbf{G}$	Infeld-van der Waerden field	$(1,0\vert 1,0\vert 0,1)$

$$(7.3)$$

Their role for SM is similar to that of the metric tensor $\mathbf{g}$ for TM. The spin-tensorial type in the table (3.1) specifies the number of indices in coordinate representation of fields. The first two numbers are the numbers of upper and lower spinor indices, the second two numbers are the numbers of upper and lower conjugate spinor indices, and the last two numbers are the numbers of upper and lower tensorial indices (they are also called spacial indices).

The spin-tensorial fields $\mathbf{d}$ and $\mathbf{G}$ are introduced by means of the explicit formulas for their components in canonically associated frame pairs (7.2). Let $\boldsymbol{\Upsilon}_0$, $\boldsymbol{\Upsilon}_1$, $\boldsymbol{\Upsilon}_2$, $\boldsymbol{\Upsilon}_3$ be some positively polarized right orthonormal frame in TM and let $\boldsymbol{\Psi}_1$, $\boldsymbol{\Psi}_2$ be its associated orthonormal spinor frame in SM. The components of the Infeld-van der Waerden field $\mathbf{G}$ in such a frame pair composed by two canonically associated

frames are given by the following Pauli matrices:

$$G_0^{i\bar{i}} = \left\| \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \right\| = \sigma_0, \qquad\qquad G_2^{i\bar{i}} = \left\| \begin{matrix} 0 & -i \\ i & 0 \end{matrix} \right\| = \sigma_2,$$
$$G_1^{i\bar{i}} = \left\| \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \right\| = \sigma_1, \qquad\qquad G_3^{i\bar{i}} = \left\| \begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix} \right\| = \sigma_3. \tag{7.4}$$

The skew-symmetric metric tensor $\mathbf{d}$ is given by the matrix

$$d_{ij} = \left\| \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} \right\| \tag{7.5}$$

in any orthonormal spinor frame $\boldsymbol{\Psi}_1$, $\boldsymbol{\Psi}_2$. Unlike (7.4), the choice of the associated frame $\boldsymbol{\Upsilon}_0$, $\boldsymbol{\Upsilon}_1$, $\boldsymbol{\Upsilon}_2$, $\boldsymbol{\Upsilon}_3$ is inessential for the components of the matrix (7.5) since $\mathbf{d}$ has no spatial indices at all. The dual metric tensor for $\mathbf{d}$ is given by the matrix

$$d^{ij} = \left\| \begin{matrix} 0 & -1 \\ 1 & 0 \end{matrix} \right\|. \tag{7.6}$$

The matrix (7.6) is inverse to the matrix (7.5).

Let $\mathfrak{S}$ be some matrix from the group $\mathrm{SL}(2, \mathbb{C})$ and let $S = \phi(\mathfrak{S})$ be its image under the homomorphism (7.1). Then we have the relationships

$$\sum_{i=1}^{2} \sum_{\bar{i}=1}^{2} \mathfrak{S}_i^a \, \sigma_m^{i\bar{i}} \, \overline{\mathfrak{S}_{\bar{i}}^{\bar{a}}} = \sum_{k=1}^{3} S_m^k \, \sigma_k^{a\bar{a}}, \tag{7.7}$$

$$\sum_{i=1}^{2} \sum_{j=1}^{2} \mathfrak{S}_a^i \, d_{ij} \, \mathfrak{S}_b^j = d_{ab}, \tag{7.8}$$

where $\sigma_m^{i\bar{i}}$ and $\sigma_k^{a\bar{a}}$ are the components of the Pauli matrices (7.4). In essential, the relationships (7.7) form a definition of the group homomorphism (7.1) (see [13] for more details). The relationships (7.8) are fulfilled due to $\mathfrak{S} \in \mathrm{SL}(2, \mathbb{C})$.

Now let's take some arbitrary vector field $\mathbf{X}$ on M. Then $\ulcorner\mathbf{X}_K$ is its Kosmann lifting to the tangent bundle TM given by the formula (5.30). It induces local one-parametric group of local diffeomorphisms

$$\varphi_\varepsilon \colon TM \to TM \tag{7.9}$$

that extends (2.1), but does not coincide with (2.2). The Kosmann lifting $\ulcorner\mathbf{X}_K$ of the vector field $\mathbf{X}$ is concordant with the bundle structure of the tangent bundle TM in the sense of the definition 4.1. For this reason the local diffeomorphisms (7.9) break into the series of linear mappings

$$\varphi_\varepsilon \colon T_p(M) \to T_q(M), \tag{7.10}$$

where $q = \varphi_\varepsilon(p)$. According to the definition 5.1 the mappings (7.10) are isometries. Therefore, taking some positively polarized right orthonormal frame $\boldsymbol{\Upsilon}_0$, $\boldsymbol{\Upsilon}_1$, $\boldsymbol{\Upsilon}_2$, $\boldsymbol{\Upsilon}_3$ in TM and applying φ_ε to it, we get another orthonormal frame $\varphi_\varepsilon(\boldsymbol{\Upsilon}_0)$,

$\varphi_\varepsilon(\Upsilon_1)$, $\varphi_\varepsilon(\Upsilon_2)$, $\varphi_\varepsilon(\Upsilon_3)$ in TM. Note that φ_ε is homotopic to the identical mapping. For this reason it preserves the discrete properties like polarization and orientation, i.e. $\varphi_\varepsilon(\Upsilon_0)$, $\varphi_\varepsilon(\Upsilon_1)$, $\varphi_\varepsilon(\Upsilon_2)$, $\varphi_\varepsilon(\Upsilon_3)$ a positively polarized right orthonormal frame. Assume that ε is small enough so that both points p and $q = \varphi_\varepsilon(p)$ belong to the domain of the frame Υ_0, Υ_1, Υ_2, Υ_3. Then at the point q we have

$$\varphi_\varepsilon(\Upsilon_j) = \sum_{i=0}^{3} V_j^i(\varepsilon)\,\Upsilon_i. \tag{7.11}$$

The Lorentzian matrix V with the components $V_j^i(\varepsilon) \in \mathrm{SO}^+(1,3,\mathbb{R})$ in (7.11) is a coordinate presentation of the linear mapping (7.10) in the frame Υ_0, Υ_1, Υ_2, Υ_3.

Each of the two positively polarized right orthonormal frames Υ_0, Υ_1, Υ_2, Υ_3 and $\varphi_\varepsilon(\Upsilon_0)$, $\varphi_\varepsilon(\Upsilon_1)$, $\varphi_\varepsilon(\Upsilon_2)$, $\varphi_\varepsilon(\Upsilon_3)$ is associated with some orthonormal frame in SM. Using this fact, we define a linear mapping

$$\varphi_\varepsilon\colon\ S_p(M) \to S_q(M) \tag{7.12}$$

closing the following commutative diagram of frame associations:

$$
\begin{array}{ccc}
\boldsymbol{\Psi}_1,\, \boldsymbol{\Psi}_2 & \longrightarrow & \Upsilon_0,\, \Upsilon_1,\, \Upsilon_2,\, \Upsilon_3 \\[4pt]
{\scriptstyle\varphi_\varepsilon}\Big\downarrow & & \Big\downarrow{\scriptstyle\varphi_\varepsilon} \\[4pt]
\varphi_\varepsilon(\boldsymbol{\Psi}_1),\, \varphi_\varepsilon(\boldsymbol{\Psi}_2) & \longrightarrow & \begin{array}{c}\varphi_\varepsilon(\Upsilon_0),\, \varphi_\varepsilon(\Upsilon_1),\\ \varphi_\varepsilon(\Upsilon_2),\, \varphi_\varepsilon(\Upsilon_3)\end{array}
\end{array}
\tag{7.13}
$$

In the coordinate form the linear mapping (7.12), which closes the diagram (7.13), is presented by some matrix $W \in \mathrm{SL}(2,\mathbb{C})$:

$$\varphi_\varepsilon(\boldsymbol{\Psi}_j) = \sum_{i=1}^{2} W_j^i(\varepsilon)\,\boldsymbol{\Psi}_i. \tag{7.14}$$

The horizontal arrows in the diagram (7.13) are canonical frame associations. For this reason the components of the matrices V and W in (7.11) and (7.14) should satisfy the relationships (7.7) and (7.8):

$$\sum_{i=1}^{2}\sum_{\bar{i}=1}^{2} W_i^a(\varepsilon)\,\sigma_m^{i\bar{i}}\,\overline{W_{\bar{i}}^{\bar{a}}(\varepsilon)} = \sum_{k=1}^{3} V_m^k(\varepsilon)\,\sigma_k^{a\bar{a}}, \tag{7.15}$$

$$\sum_{i=1}^{2}\sum_{j=1}^{2} W_a^i(\varepsilon)\,d_{ij}\,W_b^j(\varepsilon) = d_{ab}. \tag{7.16}$$

According to the general recipe (4.12), now we pass from the matrices $W(\varepsilon)$ and $V(\varepsilon)$ to their expansions as $\varepsilon \to 0$, i.e. we write

$$W_j^i(\varepsilon) = \delta_j^i + W_j^i(x^0, x^1, x^2, x^3)\,\varepsilon + \dots,$$

$$V_j^i(\varepsilon) = \delta_j^i + V_j^i(x^0, x^1, x^2, x^3)\,\varepsilon + \dots. \tag{7.17}$$

Note that the quantities V_j^i in (7.17) are already known. They are taken from (5.13). In our special case, where $\boldsymbol{\Upsilon}_0$, $\boldsymbol{\Upsilon}_1$, $\boldsymbol{\Upsilon}_2$, $\boldsymbol{\Upsilon}_3$ is an orthonormal frame, we can use a more simple formula for V_j^i. It is extracted from (5.30):

$$V_j^i = -\frac{1}{2} \sum_{r=0}^{3} \sum_{s=0}^{3} g^{is} \, \nabla_s X^r \, g_{rj} + \frac{1}{2} \nabla_j X^i - \sum_{m=0}^{3} X^m \, \Gamma_{mj}^i. \tag{7.18}$$

As for the quantities W_j^i, they should be calculated by substituting (7.17) back into (7.15) and (7.16). Like in the case of (5.7), we denote

$$W_{ij} = \sum_{s=1}^{2} W_i^s \, d_{sj}. \tag{7.19}$$

Then, applying (7.17) to (7.16), we derive the following formula:

$$W_{ij} - W_{ji} = 0. \tag{7.20}$$

Due to the formula (7.20) the quantities (7.19) are symmetric with respect to the indices i and j. Our next goal is to resolve the relationships (7.15) with respect to these quantities W_{ij}.

Let's substitute the expansions (7.17) into (7.15) and collect the first order terms with respect to the parameter ε. As a result we get

$$\sum_{i=1}^{2} W_i^a \, \sigma_m^{i\bar{a}} + \sum_{\bar{i}=1}^{2} \sigma_m^{a\bar{i}} \, \overline{W_{\bar{i}}^{\bar{a}}} = \sum_{k=0}^{3} V_m^k \, \sigma_k^{a\bar{a}}. \tag{7.21}$$

Keeping in mind that we use the frames $\boldsymbol{\Upsilon}_0$, $\boldsymbol{\Upsilon}_1$, $\boldsymbol{\Upsilon}_2$, $\boldsymbol{\Upsilon}_3$ and $\boldsymbol{\Psi}_1$, $\boldsymbol{\Psi}_2$ canonically associated to each other in the sense of the diagram (7.2), we replace the components of Pauli matrices by the components of Infeld-van der Waerden field in (7.21):

$$\sum_{i=1}^{2} W_i^a \, G_m^{i\bar{a}} + \sum_{\bar{i}=1}^{2} G_m^{a\bar{i}} \, \overline{W_{\bar{i}}^{\bar{a}}} = \sum_{k=0}^{3} V_m^k \, G_k^{a\bar{a}}. \tag{7.22}$$

In order to transform the formula (7.22) we multiply it by $G_{u\bar{u}}^m$ and sum over the index m. Doing it, we use one of the identities

$$\sum_{m=0}^{3} G_m^{a\bar{a}} \, G_{u\bar{u}}^m = 2 \, \delta_u^a \, \delta_{\bar{u}}^{\bar{a}}, \qquad\qquad \sum_{u=1}^{2} \sum_{\bar{u}=1}^{2} G_m^{u\bar{u}} \, G_{u\bar{u}}^n = 2 \, \delta_m^n, \tag{7.23}$$

where $G_{u\bar{u}}^m$ are the components of the inverse Infeld-van der Waerden field. They are produced from $G_n^{a\bar{a}}$ by lowering upper spinor indices a amd $\bar{a}$ and by raising lower spacial index n according to the following formula:

$$G_{u\bar{u}}^m = \sum_{a=1}^{2} \sum_{\bar{a}=1}^{2} \sum_{n=0}^{3} G_n^{a\bar{a}} \, d_{au} \, \bar{d}_{\bar{a}\bar{u}} \, g^{nm}. \tag{7.24}$$

The components of the conjugate spinor metric $\bar{\mathbf{d}}$ in (7.24) are produced from the components of $\mathbf{d}$ by means of the complex conjugation:

$$\bar{d}_{\bar{i}\bar{j}} = \overline{d_{\bar{i}\bar{j}}}, \qquad\qquad \bar{d}^{\bar{i}\bar{j}} = \overline{d^{\bar{i}\bar{j}}}.$$

The identities (7.23) are taken from [14]. Applying them to (7.22), we get

$$2\,W_u^a\,\delta_{\bar{u}}^{\bar{a}} + 2\,\delta_u^a\,\overline{W_{\bar{u}}^{\bar{a}}} = \sum_{k=0}^{3}\sum_{m=0}^{3} V_m^k\,G_k^{a\bar{a}}\,G_{u\bar{u}}^m. \qquad (7.25)$$

In order to use the symmetry (7.20) we need to lower the indices a and $\bar{a}$ in (7.25):

$$W_{ua}\,\bar{d}_{\bar{u}\bar{a}} + d_{ua}\,\overline{W_{\bar{u}\bar{a}}} = \frac{1}{2}\sum_{k=0}^{3}\sum_{m=0}^{3}\sum_{s=1}^{2}\sum_{\bar{s}=1}^{2} V_m^k\,G_k^{s\bar{s}}\,d_{sa}\,\bar{d}_{\bar{s}\bar{a}}\,G_{u\bar{u}}^m. \qquad (7.26)$$

Note that $\overline{W_{\bar{u}\bar{a}}}$ in (7.26) is symmetric with respect to the indices $\bar{u}$ and $\bar{a}$, while $\bar{d}^{\bar{a}\bar{u}}$ is skew-symmetric with respect to these indices. Therefore, we have

$$\sum_{\bar{u}=1}^{2}\sum_{\bar{a}=1}^{2}\overline{W_{\bar{u}\bar{a}}}\,\bar{d}^{\bar{a}\bar{u}} = 0. \qquad (7.27)$$

Applying (7.27), we multiply (7.26) by $\bar{d}^{\bar{a}\bar{u}}$ and sum up over the indices $\bar{u}$ and $\bar{a}$. As a result we get the following formula for W_{ua}:

$$W_{ua} = \frac{1}{4}\sum_{k=0}^{3}\sum_{m=0}^{3}\sum_{s=1}^{2}\sum_{\bar{s}=1}^{2} G_k^{s\bar{s}}\,V_m^k\,G_{u\bar{s}}^m\,d_{sa}. \qquad (7.28)$$

Now let's return back to the quantities W_j^i by raising the index a in (7.28). As a result of this standard procedure we obtain

$$W_j^i = \frac{1}{4}\sum_{k=0}^{3}\sum_{m=0}^{3}\sum_{\bar{s}=1}^{2} G_k^{i\bar{s}}\,V_m^k\,G_{j\bar{s}}^m. \qquad (7.29)$$

The next step is to substitute (7.18) into (7.29). Doing it, let's recall that the metric connection Γ has the unique extension $(\Gamma, A, \bar{A})$ to the spinor bundle SM. Its spinor components are given by the formula

$$\begin{aligned}
A_{rj}^i &= \sum_{k=0}^{3}\sum_{m=0}^{3}\sum_{\bar{s}=1}^{2} \frac{G_k^{i\bar{s}}\,\Gamma_{rm}^k\,G_{j\bar{s}}^m}{4} - \\
&- \sum_{\bar{s}=1}^{2}\sum_{q=0}^{3} \frac{L_{\Upsilon_r}(G_q^{i\bar{s}})\,G_{j\bar{s}}^q}{4} - \sum_{\bar{i}=1}^{2}\sum_{\bar{j}=1}^{2} \frac{L_{\Upsilon_r}(\bar{d}_{\bar{j}\bar{i}})\,\bar{d}^{\bar{i}\bar{j}}\,\delta_j^i}{4}.
\end{aligned} \qquad (7.30)$$

The formula is derived in [14] and is verified in [15]. In our special case the frames Υ_0, Υ_1, Υ_2, Υ_3 and Ψ_1, Ψ_2 form a canonically associated pair in the sense of the diagram (7.2). In this case the formula (7.30) reduces to

$$A_{rj}^i = \frac{1}{4}\sum_{k=0}^{3}\sum_{m=0}^{3}\sum_{\bar{s}=1}^{2} G_k^{i\bar{s}}\,\Gamma_{rm}^k\,G_{j\bar{s}}^m. \qquad (7.31)$$

The formulas (7.31) and (7.29) are very similar. Therefore, substituting (7.18) into (7.29), due to the presence of Γ^i_{mj} in (7.18) we can write

$$
W^i_j = -\frac{1}{8} \sum_{q=0}^{3} \sum_{p=0}^{3} \sum_{k=0}^{3} \sum_{m=0}^{3} \sum_{\bar{s}=1}^{2} G^{i\bar{s}}_k \, g^{kp} \, \nabla_p X^q \, g_{qm} \, G^m_{j\bar{s}} +
$$
$$
+\frac{1}{8} \sum_{k=0}^{3} \sum_{m=0}^{3} \sum_{\bar{s}=1}^{2} G^{i\bar{s}}_k \, \nabla_m X^k \, G^m_{j\bar{s}} - \sum_{m=0}^{3} X^m \, \mathrm{A}^i_{mj} .
$$

(7.32)

Knowing the quantities (7.18) and (7.32) is sufficient to construct a spinor extension of the Kosmann-Lie derivative $\mathcal{L}_\mathbf{X}$. Let $\mathbf{Y}$ be a spin-tensorial field of the type $(\varepsilon, \eta | \sigma, \zeta | e, f)$. Then for the components of the field $\mathcal{L}_\mathbf{X}(\mathbf{Y})$ we have the formula

$$
\mathcal{L}_\mathbf{X}(\mathbf{Y})^{a_1 \dots a_\varepsilon \bar{a}_1 \dots \bar{a}_\sigma c_1 \dots c_e}_{b_1 \dots b_\eta \bar{b}_1 \dots \bar{b}_\zeta d_1 \dots d_f} = \sum_{m=0}^{3} X^m \, L_{\mathbf{\Upsilon}_m}(Y^{a_1 \dots a_\varepsilon \bar{a}_1 \dots \bar{a}_\sigma c_1 \dots c_e}_{b_1 \dots b_\eta \ \bar{b}_1 \dots \bar{b}_\zeta \ d_1 \dots d_f}) -
$$
$$
-\sum_{\mu=1}^{\varepsilon} \sum_{v_\mu=1}^{2} W^{a_\mu}_{v_\mu} \, Y^{a_1 \dots v_\mu \dots a_\varepsilon \bar{a}_1 \dots \bar{a}_\sigma c_1 \dots c_e}_{b_1 \dots \dots \dots b_\eta \bar{b}_1 \dots \bar{b}_\zeta d_1 \dots d_f} +
$$
$$
+\sum_{\mu=1}^{\eta} \sum_{w_\mu=1}^{2} W^{w_\mu}_{b_\mu} \, Y^{a_1 \dots \dots \dots a_\varepsilon \bar{a}_1 \dots \bar{a}_\sigma c_1 \dots c_e}_{b_1 \dots w_\mu \dots b_\eta \bar{b}_1 \dots \bar{b}_\zeta d_1 \dots d_f} -
$$
$$
-\sum_{\mu=1}^{\sigma} \sum_{v_\mu=1}^{2} \overline{W^{\bar{a}_\mu}_{v_\mu}} \, Y^{a_1 \dots a_\varepsilon \bar{a}_1 \dots v_\mu \dots \bar{a}_\sigma c_1 \dots c_e}_{b_1 \dots b_\eta \bar{b}_1 \dots \dots \dots \bar{b}_\zeta d_1 \dots d_f} +
$$

(7.33)

$$
+\sum_{\mu=1}^{\zeta} \sum_{w_\mu=1}^{2} \overline{W^{w_\mu}_{\bar{b}_\mu}} \, Y^{a_1 \dots a_\varepsilon \bar{a}_1 \dots \dots \dots \bar{a}_\sigma c_1 \dots c_e}_{b_1 \dots b_\eta \bar{b}_1 \dots w_\mu \dots \bar{b}_\zeta d_1 \dots d_f} -
$$
$$
-\sum_{\mu=1}^{e} \sum_{v_\mu=0}^{3} V^{c_\mu}_{v_\mu} \, Y^{a_1 \dots a_\varepsilon \bar{a}_1 \dots \bar{a}_\sigma c_1 \dots v_\mu \dots c_e}_{b_1 \dots b_\eta \bar{b}_1 \dots \bar{b}_\zeta d_1 \dots \dots \dots d_f} +
$$
$$
+\sum_{\mu=1}^{f} \sum_{w_\mu=0}^{3} V^{w_\mu}_{b_\mu} \, Y^{a_1 \dots a_\varepsilon \bar{a}_1 \dots \bar{a}_\sigma c_1 \dots \dots \dots c_e}_{b_1 \dots b_\eta \bar{b}_1 \dots \bar{b}_\zeta d_1 \dots w_\mu \dots d_f} .
$$

It is easy to see that (7.33) is a version of (4.14). In the case of a purely tensorial field $\mathbf{Y}$, i.e. if $\varepsilon = 0$, $\eta = 0$, $\sigma = 0$, and $\zeta = 0$, the Kosmann-Lie derivative (7.33) reduces to (5.23). However, in general case we cannot use this formula (5.23) since the regular Lie derivative $L_\mathbf{X}$ has no spinor extension yet. For this reason, instead of the formula (5.23), in this case we write

$$
\mathcal{L}_\mathbf{X} = \nabla_\mathbf{X} + S_\mathbf{X}.
$$

(7.34)

Like in (5.23), by $S_\mathbf{X}$ in (7.34) we denote a degenerate differentiation. According to the results of [12], each degenerate differentiation extended to spinors is defined by three spin-tensorial field of the types $(1, 1 | 0, 0 | 0, 0)$, $(0, 0 | 1, 1 | 0, 0)$, and $(0, 0 | 0, 0 | 1, 1)$. We denote them $\mathfrak{S}_\mathbf{X}$, $\tilde{\mathfrak{S}}_\mathbf{X}$, and $\mathbf{S}_\mathbf{X}$ respectively. Here are the components of $\mathbf{S}_\mathbf{X}$:

$$
S^i_j(\mathbf{X}) = \frac{\nabla^i X_j - \nabla_j X^i}{2}.
$$

(7.35)

Comparing (7.35) with (5.24), we see that $\mathbf{S_X}$ in (7.34) is different from that of (5.23). The formula (7.35) is extracted from (7.18). Similarly, looking at (7.32), we find the components of the spin-tensorial field $\mathfrak{S_X}$:

$$\mathfrak{S}_j^i(\mathbf{X}) = \sum_{k=0}^3 \sum_{m=0}^3 \sum_{\bar{s}=1}^2 G_k^{i\bar{s}} \, \frac{\nabla^k X_m - \nabla_m X^k}{8} \, G_{j\bar{s}}^m. \tag{7.36}$$

The components of $\bar{\mathfrak{S}}_\mathbf{X}$ are produced from (7.36) by means of the complex conjugation: $\tilde{\mathfrak{S}}_{\bar{j}}^{\bar{i}}(\mathbf{X}) = \overline{\mathfrak{S}_{\bar{j}}^{\bar{i}}(\mathbf{X})}$. For this components we derive the formula

$$\bar{\mathfrak{S}}_{\bar{j}}^{\bar{i}}(\mathbf{X}) = \sum_{k=0}^3 \sum_{m=0}^3 \sum_{s=1}^2 G_k^{s\bar{i}} \, \frac{\nabla^k X_m - \nabla_m X^k}{8} \, G_{s\bar{j}}^m. \tag{7.37}$$

The formula (7.34) complemented with (7.35), (7.36), and (7.37) is equivalent to the formula (7.33).

Let's consider a particular example of applying the formula (7.33). Assume that ψ is a spinor field, i. e. a field with the spin-tensorial type $(1, 0|0, 0|0, 0)$. Then for the components of the spinor field $\mathcal{L}_\mathbf{X}(\psi)$ we have

$$\mathcal{L}_\mathbf{X}(\psi)^i = \sum_{m=0}^3 X^m \, \nabla_m \psi^i + \sum_{k=0}^3 \sum_{m=0}^3 \sum_{s=1}^2 \sum_{j=1}^2 G_k^{s\bar{i}} \, \frac{\nabla^k X_m - \nabla_m X^k}{8} \, G_{s\bar{j}}^m \, \psi^j.$$

This formula resembles the formula (3.19) in [10] and the formula (5.5′) in [11].

Note that the formulas (7.18) and (7.32) were derived under the assumption that $\boldsymbol{\Upsilon}_0$, $\boldsymbol{\Upsilon}_1$, $\boldsymbol{\Upsilon}_2$, $\boldsymbol{\Upsilon}_3$ is a positively polarized right orthonormal frame in TM and $\boldsymbol{\Psi}_1$, $\boldsymbol{\Psi}_2$ its canonically associated orthonormal frame in SM. However, these formulas remain valid for an arbitrary frame pair provided we use the general formula

$$\Gamma_{ij}^k = \sum_{r=0}^3 \frac{g^{kr}}{2} \left(L_{\boldsymbol{\Upsilon}_i}(g_{rj}) + L_{\boldsymbol{\Upsilon}_j}(g_{ir}) - L_{\boldsymbol{\Upsilon}_r}(g_{ij}) \right) +$$
$$+ \frac{c_{ij}^k}{2} - \sum_{r=0}^3 \sum_{s=0}^3 \frac{c_{ir}^s}{2} \, g^{kr} \, g_{sj} - \sum_{r=0}^3 \sum_{s=0}^3 \frac{c_{jr}^s}{2} \, g^{kr} \, g_{si}.$$

for Γ_{mj}^i in (7.18) instead of (5.26) and the general formula (7.30) for A_{mj}^i in (7.32) instead of (7.31). The formula (7.33) is also valid for an arbitrary frame pair under the same provisions.

8. KOSMANN-LIE DERIVATIVES OF THE BASIC FIELDS.

There are three basic field in the theory of Weyl spinors. Two of them $\mathbf{d}$ and $\mathbf{G}$ are listed in the table (7.3). The third is the metric tensor $\mathbf{g}$. Now we shall apply the Kosmann-Lie derivative (7.33) to these basic fields. For this purpose it is convenient to choose some canonically associated pair of frames $\boldsymbol{\Upsilon}_0$, $\boldsymbol{\Upsilon}_1$, $\boldsymbol{\Upsilon}_2$, $\boldsymbol{\Upsilon}_3$ and $\boldsymbol{\Psi}_1$, $\boldsymbol{\Psi}_2$. In such a frame pair the components of all basic fields are constants.

Indeed, they are given by the formulas (5.25), (7.4), and (7.5). Therefore we have

$$\sum_{m=0}^{3} X^m\, L_{\boldsymbol{\Upsilon}_m}(d_{ij}) = 0, \tag{8.1}$$

$$\sum_{m=0}^{3} X^m\, L_{\boldsymbol{\Upsilon}_m}(G_m^{a\bar{a}}) = 0, \tag{8.2}$$

$$\sum_{m=0}^{3} X^m\, L_{\boldsymbol{\Upsilon}_m}(g_{ij}) = 0. \tag{8.3}$$

Applying (7.33) to **d** and taking into account (8.1), (7.19), and (7.20), we obtain

$$\mathcal{L}_{\mathbf{X}}(\mathbf{d})_{ij} = \sum_{s=1}^{2} W_i^s\, d_{sj} + \sum_{s=1}^{2} W_j^s\, d_{is} = W_{ij} - W_{ji} = 0. \tag{8.4}$$

Similarly, applying (7.33) to **G** and taking into account (8.2) and (7.22), we get

$$\mathcal{L}_{\mathbf{X}}(\mathbf{G})_m^{a\bar{a}} = -\sum_{i=1}^{2} W_i^a\, G_m^{i\bar{a}} - \sum_{\bar{i}=1}^{2} G_m^{a\bar{i}}\, \overline{W_{\bar{i}}^{\bar{a}}} + \sum_{k=0}^{3} V_m^k\, G_k^{a\bar{a}} = 0. \tag{8.5}$$

And finally we apply the formula (7.33) to the metric tensor **g**. As a result, taking into account (8.3), (5.7), and (5.8), we derive

$$\mathcal{L}_{\mathbf{X}}(\mathbf{g})_{ij} = \sum_{r=0}^{3} V_i^r\, g_{rj} + \sum_{r=0}^{3} V_j^r\, g_{ir} = V_{ij} + V_{ji} = 0. \tag{8.6}$$

The formulas (8.4), (8.5), and (8.6) are summarized in the following theorem.

Theorem 8.1. *For any vector field* **X** *in M the basic tensorial and spin-tensorial fields* **g**, **d**, *and* **G** *associated with the bundle of Weyl spinors SM are constant with respect to the Kosmann-Lie derivative* $\mathcal{L}_{\mathbf{X}}$.

9. Some concluding remarks.

Note that the quantities V_j^i for (7.33) are taken from (7.18). However, they could be taken from (5.1) either. In the latter case the equality $\mathcal{L}_{\mathbf{X}}(\mathbf{d}) = 0$ would be preserved, but the equality $\mathcal{L}_{\mathbf{X}}(\mathbf{g}) = 0$ would be replaced by

$$\mathcal{L}_{\mathbf{X}}(\mathbf{g}) = L_{\mathbf{X}}(\mathbf{g}).$$

As for the formula (8.5), it would be replaced by the following one:

$$\mathcal{L}_{\mathbf{X}}(\mathbf{G})_m^{a\bar{a}} = \sum_{k=0}^{3} \frac{\nabla_m X^k + \nabla^k X_m}{2}\, G_k^{a\bar{a}}.$$

This choice of V_j^i is preferred in [11]. As for our choice of V_j^i in this paper, in [11] it is referred to as the "metric Lie derivative" introduced by Bourguignon and

Gauduchon in [16]. Since there are various approaches, I should regretfully conclude that there is no canonical definition of the Lie derivative for spinors thus far.

References

1. Ibragimov N. Kh., *Transformation groups in mathematical physics.*, Nauka publishers, Moscow, 1983.
2. Sharipov R. A., *Classical electrodynamics and theory of relativity*, Bashkir State University, Ufa, 1997; see also physics/0311011 in Electronic Archive http://arXiv.org and r-sharipov/r4-b5.htm in GeoCities.
3. Kobayashi Sh., Nomizu K, *Foundations of differential geometry,* Vol. I, Interscience Publishers, New York, London, 1963; Nauka publishers, Moscow, 1981.
4. Kosmann Y., Comptes Rendus Acad. Sc. Paris **262A** (1966), 289–292.
5. Kosmann Y., Comptes Rendus Acad. Sc. Paris **262A** (1966), 394–397.
6. Kosmann Y., Comptes Rendus Acad. Sc. Paris **264A** (1967), 355–358.
7. Kosmann Y., Ann. di Matematica Pura et Appl. **91** (1972), 317–395.
8. Sharipov R. A., *Course of differential geometry*, Bashkir State University, Ufa, 1996; see also math.HO/0412421 in Electronic Archive http://arXiv.org and r-sharipov/r4-b3.htm in GeoCities.
9. Sharipov R. A., *On the spinor structure of the homogeneous and isotropic universe in closed model*, e-print axXiv:0708.1171 in Electronic Archive http://arXiv.org.
10. Fatibene L., Ferraris M., Francaviglia M., Godina M., *A geometric definition of Lie derivative for Spinor Fields*, e-print gr-qc/9608003 in Electronic Archive http://arXiv.org.
11. Godina M., Matteucci P., *The Lie derivative of spinor fields: theory and applications*, e-print math.DG/0504366 in Electronic Archive http://arXiv.org.
12. Sharipov R. A., *Spinor functions of spinors and the concept of extended spinor fields*, e-print math.DG/0511350 in Electronic Archive http://arXiv.org.
13. Sharipov R. A., *A note on Dirac spinors in a non-flat space-time of general relativity*, e-print math.DG/0601262 in Electronic Archive http://arXiv.org.
14. Sharipov R. A., *A note on metric connections for chiral and Dirac spinors*, e-print math.DG/0602359 in Electronic Archive http://arXiv.org.
15. Sharipov R. A., *A cubic identity for the Infeld-van der Waerden field and its application*, e-print arXiv:0801.0008 in Electronic Archive http://arXiv.org.
16. Bourguignon J. P., Gauduchon P., *Spineurs, operateurs de Dirac et variations de metriques*, Comm. Math. Phys. **144** (1992), no. 3, 581–599.

5 Rabochaya street, 450003 Ufa, Russia
Cell Phone: +7(917)476 93 48
E-mail address: r-sharipov@mail.ru
                 R_Sharipov@ic.bashedu.ru

URL: http://www.geocities.com/r-sharipov
       http://www.freetextbooks.boom.ru/index.html