A CUBIC IDENTITY FOR THE INFELD-VAN DER WAERDEN FIELD AND ITS APPLICATION.

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Abstract. A cubic identity for the Infeld-van der Waerden field is found and its application to verifying an explicit formula for the spinor components of the metric connection is demonstrated.

1. Introduction.

The Infeld-van der Waerden field $\mathbf{G}$ is a special spin-tensorial field associated with the bundle of Weyl spinors SM over the four-dimensional space-time manifold M in general relativity. Along with the spinor metric $\mathbf{d}$, the Infeld-van der Waerden field $\mathbf{G}$ forms the basic equipment of the spinor bundle SM. In this role $\mathbf{d}$ and $\mathbf{G}$ are similar to the metric tensor $\mathbf{g}$ of M. The fields $\mathbf{d}$, $\mathbf{G}$, and $\mathbf{g}$ are related to each other through a series of identities, e. g. we have

$$\sum_{r=1}^{2}\sum_{\bar r=1}^{2} G^{r\bar r}_{p}\ G^{q}_{r\bar r} = 2\,\delta^{q}_{p}, \qquad\qquad \sum_{q=0}^{3} G^{r\bar r}_{q}\ G^{q}_{s\bar s} = 2\,\delta^{r}_{s}\,\delta^{\bar r}_{\bar s}. \tag{1.1}$$

The identities (1.1) were considered in [1]. They are quadratic with respect to the components of the Infeld-van der Waerden field $\mathbf{G}$. Here we consider the identity

$$\begin{gathered}\sum_{s=1}^{2}\sum_{\bar s=1}^{2} G^{r\bar s}_{p}\,G^{m}_{s\bar s}\,G^{s\bar r}_{q} = G^{r\bar r}_{p}\,\delta^{m}_{q} + G^{r\bar r}_{q}\,\delta^{m}_{p} - \sum_{n=0}^{3} G^{r\bar r}_{n}\,g^{mn}\,g_{pq} +\\ + \sum_{a=0}^{3}\sum_{b=0}^{3}\sum_{n=0}^{3} i\,g_{pa}\,g_{qb}\,\omega^{ambn}\,G^{r\bar r}_{n},\end{gathered} \tag{1.2}$$

which is cubic with respect to the components of $\mathbf{G}$.

The metric tensor $\mathbf{g}$ is canonically associated with its metric connection Γ. The metric connection Γ, in its turn, has an extension $(\Gamma, \mathrm{A}, \bar{\mathrm{A}})$ to the spinor bundle SM. The spinor components A^{i}_{rj} of this extension are given by the formula

$$\begin{gathered}\mathrm{A}^{i}_{rj} = \sum_{\bar s=1}^{2}\sum_{p=0}^{3}\sum_{q=0}^{3} \frac{G^{i\bar s}_{p}\,\Gamma^{p}_{rq}\,G^{q}_{j\bar s}}{4} -\\ - \sum_{\bar s=1}^{2}\sum_{q=0}^{3} \frac{L_{\Upsilon_r}(G^{i\bar s}_{q})\,G^{q}_{j\bar s}}{4} - \sum_{\bar i=1}^{2}\sum_{\bar j=1}^{2} \frac{L_{\Upsilon_r}(\bar d_{\bar j\bar i})\,\bar d^{\bar i\bar j}\,\delta^{i}_{j}}{4}.\end{gathered} \tag{1.3}$$

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The formula (1.3) was derived in [1] by means of direct, but rather huge calculations. In this paper we verify this formula with the use of the identity (1.2).

2. Coordinates and frames.

The components of a vector and the components of a tensor are always relative to some basis. In the case of vectorial and tensorial fields we should have bases at each point of the space-time manifold M. Thus we come to the concept of a frame.

Definition 1.1. A frame of the tangent bundle TM is a quadruple of vector fields $\boldsymbol{\Upsilon}_0$, $\boldsymbol{\Upsilon}_1$, $\boldsymbol{\Upsilon}_2$, $\boldsymbol{\Upsilon}_3$ defined in some open domain of M and linearly independent at each point of this domain.

Let $\boldsymbol{\Upsilon}_0$, $\boldsymbol{\Upsilon}_1$, $\boldsymbol{\Upsilon}_2$, $\boldsymbol{\Upsilon}_3$ be a frame of the tangent bundle TM Then the commutator $[\boldsymbol{\Upsilon}_i, \boldsymbol{\Upsilon}_j]$ is a vector field that can be expressed through the frame fields:

$$[\boldsymbol{\Upsilon}_i, \boldsymbol{\Upsilon}_j] = \sum_{k=0}^{3} c_{ij}^k \, \boldsymbol{\Upsilon}_k. \tag{2.1}$$

The coefficients c_{ij}^k in (2.1) are called the *commutation coefficients* of the frame $\boldsymbol{\Upsilon}_0$, $\boldsymbol{\Upsilon}_1$, $\boldsymbol{\Upsilon}_2$, $\boldsymbol{\Upsilon}_3$. They are uniquely determined for any frame.

Definition 1.2. A frame $\boldsymbol{\Upsilon}_0$, $\boldsymbol{\Upsilon}_1$, $\boldsymbol{\Upsilon}_2$, $\boldsymbol{\Upsilon}_3$ of the tangent bundle TM is a called a *holonomic frame* if all of its commutation coefficients c_{ij}^k are identically zero.

Once we take some local coordinates x^0, x^1, x^2, x^3 in the space-time manifold M, we get the holonomic frame of the coordinate vector fields

$$\boldsymbol{\Upsilon}_0 = \frac{\partial}{\partial x^0}, \qquad \boldsymbol{\Upsilon}_1 = \frac{\partial}{\partial x^1}, \qquad \boldsymbol{\Upsilon}_2 = \frac{\partial}{\partial x^2}, \qquad \boldsymbol{\Upsilon}_3 = \frac{\partial}{\partial x^3} \tag{2.2}$$

defined within the domain of the coordinates x^0, x^1, x^2, x^3 and naturally associated to them. Local coordinates and their coordinate frames (2.2) are sufficient for most calculations in general relativity where no spinors are involved. However, when dealing with spinors we need to use non-holonomic frames.

Definition 1.3. A frame of the spinor bundle SM is a pair of smooth sections $\boldsymbol{\Psi}_1$, $\boldsymbol{\Psi}_2$ of SM defined in some open domain of M and linearly independent at each point of this domain.

Frames of TM are often called *spacial frames*, while frames of SM are called *spinor frames*. Having some spacial frame $\boldsymbol{\Upsilon}_0$, $\boldsymbol{\Upsilon}_1$, $\boldsymbol{\Upsilon}_2$, $\boldsymbol{\Upsilon}_3$ and some spinor frame $\boldsymbol{\Psi}_1$, $\boldsymbol{\Psi}_2$ with common domain, we can express tensorial and spin tensorial fields through their components forming multi-indexed arrays. The indices of such arrays are subdivided into three groups: *spacial indices* ranging from 0 to 3, *spinor indices*, and *conjugate spinor indices* both ranging from 1 to 2. Indices within each group can be either upper or lower. The quantities $g_{p\,q}$ with two lower spacial indices in (1.2) are the components of the metric tensor $\mathbf{g}$, while the quantities g^{mn} are the components of the dual metric tensor. The components of these two tensors form two 4×4 symmetric matrices inverse to each other:

$$\sum_{r=0}^{3} g_{p\,r}\, g^{r\,q} = \delta_p^q. \tag{2.3}$$

Similarly, d_{ij} and d^{ij} are the components of the spinor metric and dual spinor metric respectively. They form 2×2 skew-symmetric matrices inverse to each other:

$$\sum_{r=1}^{2} d_{jr}\,d^{ri}=\delta^i_j. \tag{2.4}$$

The Infeld-van der Waerden field $\mathbf{G}$ in (1.1) is presented by the quantities $G^{r\bar{r}}_p$. It has one upper spinor index r, one upper conjugate spinor index $\bar{r}$, and one spacial index p. The quantities $G^{\,q}_{s\bar{s}}$ are the components of the inverse Infeld-van der Waerden field. They are produced from $G^{r\bar{r}}_p$ by raising the spacial index p and lowering the spinor and conjugate spinor indices r and $\bar{r}$:

$$G^{\,q}_{s\bar{s}}=\sum_{r=1}^{2}\sum_{\bar{r}=1}^{2}\sum_{p=0}^{3} G^{r\bar{r}}_p\,g^{pq}\,d_{rs}\,\bar{d}_{\bar{r}\bar{s}}. \tag{2.5}$$

The quantities $\bar{d}_{\bar{r}\bar{s}}$ in the formula (2.5) are the components of the conjugate spinor metric $\bar{\mathbf{d}}=\tau(\mathbf{d})$. They are related to d_{ij} by means of the complex conjugation:

$$\bar{d}_{\bar{i}\bar{j}}=\overline{d_{\bar{i}\bar{j}}}. \tag{2.6}$$

Similarly, the quantities $\bar{d}^{\bar{i}\bar{j}}$ in the formula (1.3) are defined as follows:

$$\bar{d}^{\bar{i}\bar{j}}=\overline{d^{\bar{i}\bar{j}}}. \tag{2.7}$$

The quantities (2.6) and (2.7) form two mutually inverse matrices:

$$\sum_{r=1}^{2} \bar{d}_{\bar{j}\bar{r}}\,\bar{d}^{\bar{r}\bar{i}}=\delta^i_j. \tag{2.8}$$

The equality (2.8) is similar to the above equalities (2.3) and (2.4).

The quantities $G^{r\bar{r}}_p$ and $G^{\,q}_{s\bar{s}}$ satisfy the following identities:

$$G^{r\bar{r}}_p=\overline{G^{\bar{r}r}_p}, \qquad\qquad G^{\,q}_{s\bar{s}}=\overline{G^{\,q}_{\bar{s}s}}. \tag{2.9}$$

The identities (2.9) mean that for each fixed p and for each fixed q the quantities $G^{r\bar{r}}_p$ and $G^{\,q}_{s\bar{s}}$ form two Hermitian matrices. In the coordinate free form the identities (2.9) are written as $\tau(\mathbf{G})=\mathbf{G}$.

Now let's proceed to the quantities ω^{ambn} in the formula (1.2). They are the components of the *dual volume tensor*. These quantities are produced from the components of the *volume tensor* $\boldsymbol{\omega}$ by means of the index raising procedure:

$$\omega^{ambn}=\sum_{p=0}^{3}\sum_{p=0}^{3}\sum_{r=0}^{3}\sum_{s=0}^{3}\omega_{prqs}\,g^{pa}\,g^{rm}\,g^{qb}\,g^{sn}. \tag{2.10}$$

Both quantities ω^{ambn} and ω_{prqs} in (2.10) can be produced from the components of the metric tensor and the components of the dual metric tensor with the use of the Levi-Civita symbol. Indeed, we have the following formula for ω_{prqs}:

$$\omega_{prqs}=\sqrt{-\det(g_{ij})}\;\varepsilon_{prqs}. \tag{2.11}$$

Similarly, for ω^{ambn} we have the formula

$$\omega^{ambn} = -\sqrt{-\det(g^{ij})}\;\varepsilon^{ambn}. \tag{2.12}$$

Regardless of the upper and lower positions of the indices, the components of the Levi-Civita symbol are given by the formula

$$\varepsilon^{ambn} = \varepsilon_{ambn} = \begin{cases} 0 & \text{if at least two of the four indices } a\,b\,m\,n \text{ do coincide;} \\ 1 & \text{if the indices } a\,m\,b\,n \text{ form an even transposition of the numbers } 0\,1\,2\,3; \\ -1 & \text{if the indices } a\,m\,b\,n \text{ form an odd transposition of the numbers } 0\,1\,2\,3. \end{cases} \tag{2.13}$$

Note that the space-time manifold M is assumed to be equipped with three geometric structures: the **metric**, the **orientation**, and the **polarization** (see [2]). The orientation in M means that we can distinguish right and left frames. The formulas (2.11) and (2.12) are written for right frames. Passing to left frames, we should change the sign in the right hand sides of them.

Let's consider the Lie derivatives $L_{\boldsymbol{\Upsilon}_r}$ in the formula (1.3). In the case of a holonomic frame (2.2) the operators $L_{\boldsymbol{\Upsilon}_r}$ coincide with the corresponding partial derivatives, i.e. $L_{\boldsymbol{\Upsilon}_r} = \partial/\partial x^r$. In the case of a non-holonomic frame we should expand $\boldsymbol{\Upsilon}_0$, $\boldsymbol{\Upsilon}_1$, $\boldsymbol{\Upsilon}_2$, $\boldsymbol{\Upsilon}_3$ in some auxiliary holonomic frame:

$$\boldsymbol{\Upsilon}_r = \sum_{i=0}^{3} \Upsilon^i_r \,\frac{\partial}{\partial x^i}. \tag{2.14}$$

Then the operators $L_{\boldsymbol{\Upsilon}_r}$ in (1.3) act as the differential operators in the right hand sides of the formula (2.14). In other words, for $L_{\boldsymbol{\Upsilon}_r}$ we have the expression

$$L_{\boldsymbol{\Upsilon}_r} = \sum_{i=0}^{3} \Upsilon^i_r \,\frac{\partial}{\partial x^i} \tag{2.15}$$

formally coinciding with (2.14). The Lie derivatives (2.15) satisfy the relationships

$$[L_{\boldsymbol{\Upsilon}_i}, L_{\boldsymbol{\Upsilon}_j}] = \sum_{k=0}^{3} c^k_{ij}\, L_{\boldsymbol{\Upsilon}_k}. \tag{2.16}$$

The commutation coefficients c^k_{ij} in (2.16) are the same as in the formula (2.1).

3. Proof of the identity (1.2).

The proof of the identity (1.2) is similar to that of the identities (1.1) in [1]. It is based on direct calculations. The matter is that the spinor bundle SM is related to the tangent bundle TM through canonically associated frame pairs, while the fields $\mathbf{g}$, $\mathbf{d}$, $\mathbf{G}$, and $\boldsymbol{\omega}$ are given by explicit formulas in such frame pairs.

Definition 3.1. A frame $\boldsymbol{\Upsilon}_0$, $\boldsymbol{\Upsilon}_1$, $\boldsymbol{\Upsilon}_2$, $\boldsymbol{\Upsilon}_3$ of the tangent bundle TM is a called an *orthonormal frame* if the metric tensor $\mathbf{g}$ and its dual metric tensor are given by the standard Minkowski matrix in this frame:

$$g_{ij} = g^{ij} = \begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{Vmatrix}. \tag{3.1}$$

Let's recall again that M is equipped with the **metric**, the **orientation**, and the **polarization**. The polarization is a geometric structure that distinguishes the **future half light cone** from the **past half light cone**.

Definition 3.2. A frame $\mathbf{\Upsilon}_0$, $\mathbf{\Upsilon}_1$, $\mathbf{\Upsilon}_2$, $\mathbf{\Upsilon}_3$ of the tangent bundle TM is a called *positively polarized* if its first vector field $\mathbf{\Upsilon}_0$ belongs to the interior of the future light half cone at each point of its domain.

Definition 3.3. A frame $\mathbf{\Psi}_1$, $\mathbf{\Psi}_2$ of the spinor bundle SM is a called *orthonormal* if the spinor metric $\mathbf{d}$ is given by the following matrix in this frame:

$$d_{ij} = \begin{Vmatrix} 0 & 1 \\ -1 & 0 \end{Vmatrix}. \tag{3.2}$$

Due to the definition 3.3 the dual spinor metric is given by the matrix

$$d^{ij} = \begin{Vmatrix} 0 & -1 \\ 1 & 0 \end{Vmatrix}$$

in any orthonormal frame of the spinor bundle SM.

According to the definition of the spinor bundle SM (see [1]), each positively polarized right orthonormal frame $\mathbf{\Upsilon}_0$, $\mathbf{\Upsilon}_1$, $\mathbf{\Upsilon}_2$, $\mathbf{\Upsilon}_3$ of the tangent bundle TM is associated with some orthonormal frame $\mathbf{\Psi}_1$, $\mathbf{\Psi}_2$ of the spinor bundle SM. Such frames form canonically associated frame pairs. In any canonically associated frame pair the Infeld-van der Waerden field $\mathbf{G}$ is given by the following Pauli matrices:

$$\begin{aligned} G^{i\bar{i}}_0 &= \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix} = \sigma_0, & G^{i\bar{i}}_2 &= \begin{Vmatrix} 0 & -i \\ i & 0 \end{Vmatrix} = \sigma_2, \\ G^{i\bar{i}}_1 &= \begin{Vmatrix} 0 & 1 \\ 1 & 0 \end{Vmatrix} = \sigma_1, & G^{i\bar{i}}_3 &= \begin{Vmatrix} 1 & 0 \\ 0 & -1 \end{Vmatrix} = \sigma_3. \end{aligned} \tag{3.3}$$

Substituting (3.3), (3.2), and (3.1) into the formula (2.5), we calculate the components of the inverse Infeld-van der Waerden field:

$$\begin{aligned} G^0_{i\bar{i}} &= \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix} = \sigma_0, & G^2_{i\bar{i}} &= \begin{Vmatrix} 0 & i \\ -i & 0 \end{Vmatrix} = -\sigma_2, \\ G^1_{i\bar{i}} &= \begin{Vmatrix} 0 & 1 \\ 1 & 0 \end{Vmatrix} = \sigma_1, & G^3_{i\bar{i}} &= \begin{Vmatrix} 1 & 0 \\ 0 & -1 \end{Vmatrix} = \sigma_3. \end{aligned} \tag{3.4}$$

And finally, substituting (3.1) into (2.12), we calculate the components of the inverse volume tensor $\boldsymbol{\omega}$ in the right orthonormal frame $\mathbf{\Upsilon}_0$, $\mathbf{\Upsilon}_1$, $\mathbf{\Upsilon}_2$, $\mathbf{\Upsilon}_3$:

$$\omega^{ambn} = -\varepsilon^{ambn}. \tag{3.5}$$

The quantities ε^{ambn} in (3.5) are defined by the formula (2.13). Now in order to prove the identity (1.2) it is sufficient to substitute (3.1), (3.3), (3.4), and (3.5) into (1.2) and verify that this equality is valid for all particular values of the indices p, q, m, r, and $\bar{r}$. The following computer code does it for us:

```
Verification_List:=[]:
for p from 0 by 1 to 3 do
 for m from 0 by 1 to 3 do
  for q from 0 by 1 to 3 do
   for r from 1 by 1 to 2 do
    for br from 1 by 1 to 2 do
    Equ:=0:
     for s from 1 by 1 to 2 do
      for bs from 1 by 1 to 2 do
       Equ:=Equ+G_ortho[p][r,bs]*Inv_G_ortho[m][s,bs]
               *G_ortho[q][s,br]:
     od od:
     Equ:=Equ-G_ortho[p][r,br]*delta[m,q]
             -G_ortho[q][r,br]*delta[m,p]:
     for n from 0 by 1 to 3 do
      Equ:=Equ+G_ortho[n][r,br]*Inv_g_ortho[m,n]*g_ortho[p,q]:
     od:
     for a from 0 by 1 to 3 do
      for b from 0 by 1 to 3 do
       for n from 0 by 1 to 3 do
        Equ:=Equ-I*g_ortho[p,a]*g_ortho[q,b]*Inv_omega[a,m,b,n]
                 *G_ortho[n][r,br]:
     od od od:
     Verification_List:=[op(Verification_List),evalb(Equ=0)]:
od od od od od:
```

Upon executing the above code it is sufficient to type

```
print(Verification_List):
```

and find that the identity (1.2) is proved for any canonically associated pair of frames. Due to the tensorial nature of this identity, being proved for some particular frame pair, it remains valid for arbitrary pairs of frames.

Note that the above code is designed for the Maple[1] package. However, it can be easily adapted for other symbolic computation packages.

4. Other relationships derived from (1.2).

Let's multiply both sides of the identity (1.2) by $G^{u\bar{u}}_m$ and sum it over the index m. As a result, applying the second identity (1.1), we get

$$\begin{aligned}2\,G^{r\bar{u}}_p\,G^{u\bar{r}}_q &= G^{r\bar{r}}_p\,G^{u\bar{u}}_q + G^{r\bar{r}}_q\,G^{u\bar{u}}_p - 2\,d^{ru}\,\bar{d}^{\bar{r}\bar{u}}\,g_{p\,q}\,+\\ &+\sum^3_{a=0}\sum^3_{b=0}\sum^3_{m=0}\sum^3_{n=0} i\,g_{pa}\,g_{qb}\,\omega^{ambn}\,G^{u\bar{u}}_m\,G^{r\bar{r}}_n.\end{aligned}\tag{4.1}$$

[1] Maple is a trademark of Waterloo Maple Inc.

Symmetrizing the equality (4.1) with respect to p and q, we derive

$$G^{r\bar{u}}_{p}\,G^{u\bar{r}}_{q}+G^{r\bar{u}}_{q}\,G^{u\bar{r}}_{p}=G^{r\bar{r}}_{p}\,G^{u\bar{u}}_{q}+G^{r\bar{r}}_{q}\,G^{u\bar{u}}_{p}-2\,d^{ru}\,\bar{d}^{\bar{r}\bar{u}}\,g_{pq}. \tag{4.2}$$

Alternating the equality (4.1) with respect to the same pair of indices, we obtain

$$G^{r\bar{u}}_{p}\,G^{u\bar{r}}_{q}-G^{r\bar{u}}_{q}\,G^{u\bar{r}}_{p}=\sum^{3}_{a=0}\sum^{3}_{b=0}\sum^{3}_{m=0}\sum^{3}_{n=0}i\,g_{pa}\,g_{qb}\,\omega^{ambn}\,G^{u\bar{u}}_{m}\,G^{r\bar{r}}_{n}. \tag{4.3}$$

The identities (4.2) and (4.3) taken together are equivalent to (4.1).

5. The metric connection and its spinor components.

The metric connection Γ in M associated with the metric tensor $\mathbf{g}$ is a torsion-free connection satisfying the condition

$$\nabla\mathbf{g}=0. \tag{5.1}$$

The equality (5.1) is known as the concordance condition for the metric and connection. In the coordinate form the torsion-free condition is written as

$$\Gamma^{k}_{ij}-\Gamma^{k}_{ji}=c^{k}_{ij}. \tag{5.2}$$

As for the concordance condition (5.1), it expands to

$$L_{\boldsymbol{\Upsilon}_r}(g_{ij})-\sum^{3}_{k=0}\Gamma^{k}_{ri}\,g_{kj}-\sum^{3}_{k=0}\Gamma^{k}_{rj}\,g_{ik}=0. \tag{5.3}$$

The equations (5.2) and (5.3) can be solved with respect to Γ^{k}_{ij}. They yield

$$\begin{gathered}\Gamma^{k}_{ij}=\sum^{3}_{r=0}\frac{g^{kr}}{2}\,\bigl(L_{\boldsymbol{\Upsilon}_i}(g_{rj})+L_{\boldsymbol{\Upsilon}_j}(g_{ir})-L_{\boldsymbol{\Upsilon}_r}(g_{ij})\bigr)\,+\\ +\,\frac{c^{k}_{ij}}{2}-\sum^{3}_{r=0}\sum^{3}_{s=0}\frac{c^{s}_{ir}}{2}\,g^{kr}\,g_{sj}-\sum^{3}_{r=0}\sum^{3}_{s=0}\frac{c^{s}_{jr}}{2}\,g^{kr}\,g_{si}.\end{gathered} \tag{5.4}$$

The only term, which is skew-symmetric with respect to i and j, is $c^{k}_{ij}/2$. Other terms are either symmetric or have their symmetric counterparts. For this reason, substituting (5.4) back into (5.2), we find that this equality is identically fulfilled:

$$\Gamma^{k}_{ij}-\Gamma^{k}_{ji}=\frac{c^{k}_{ij}}{2}-\frac{c^{k}_{ji}}{2}=c^{k}_{ij}.$$

Applying (5.4) to the second term in (5.3), we derive

$$\begin{gathered}-\sum^{3}_{k=0}\Gamma^{k}_{ri}\,g_{kj}=-\frac{1}{2}\,\bigl(L_{\boldsymbol{\Upsilon}_r}(g_{ji})+L_{\boldsymbol{\Upsilon}_i}(g_{rj})-L_{\boldsymbol{\Upsilon}_j}(g_{ri})\bigr)\,-\\ -\frac{1}{2}\sum^{3}_{k=0}c^{k}_{ri}\;g_{kj}+\frac{1}{2}\sum^{3}_{s=0}c^{s}_{rj}\;g_{si}+\frac{1}{2}\sum^{3}_{s=0}c^{s}_{ij}\;g_{sr}.\end{gathered} \tag{5.5}$$

The third term in (5.3) differs from the second term by exchanging i and j:

$$
\begin{gathered}
-\sum_{k=0}^{3}\Gamma^{k}_{rj}\,g_{ki}=-\frac{1}{2}\,\bigl(L_{\boldsymbol\Upsilon_r}(g_{ij})+L_{\boldsymbol\Upsilon_j}(g_{ri})-L_{\boldsymbol\Upsilon_i}(g_{rj})\bigr)\,-\\
-\frac{1}{2}\sum_{k=0}^{3}c^{\,k}_{rj}\,\,g_{ki}+\frac{1}{2}\sum_{s=0}^{3}c^{\,s}_{ri}\,\,g_{sj}+\frac{1}{2}\sum_{s=0}^{3}c^{\,s}_{ji}\,\,g_{sr}.
\end{gathered}
\tag{5.6}
$$

Adding the formulas (5.5) and (5.6), we find that

$$
-\sum_{k=0}^{3}\Gamma^{k}_{ri}\,g_{kj}-\sum_{k=0}^{3}\Gamma^{k}_{rj}\,g_{ki}=-L_{\boldsymbol\Upsilon_r}(g_{ij}).
$$

This relationship is equivalent to (5.3). Thus, for the connection with the components (5.4) we have verified both conditions — the torsion-free condition (5.2) and the concordance condition (5.3).

The metric connection Γ with the components (5.4) has a spinor extension $(\Gamma,\mathrm{A},\bar{\mathrm{A}})$ which is uniquely fixed by the following two concordance conditions:

$$
\nabla\mathbf{d}=0, \qquad\qquad \nabla\mathbf{G}=0.
\tag{5.7}
$$

In the coordinate form the conditions (5.7) are written as follows:

$$
L_{\boldsymbol\Upsilon_r}(d_{ij})-\sum_{k=1}^{2}\mathrm{A}^{k}_{ri}\,\,d_{kj}-\sum_{k=1}^{2}\mathrm{A}^{k}_{rj}\,\,d_{ik}=0,
\tag{5.8}
$$

$$
L_{\boldsymbol\Upsilon_r}(G^{i\bar{i}}_{p})+\sum_{k=1}^{2}\mathrm{A}^{i}_{rk}\,\,G^{k\bar{i}}_{p}+\sum_{\bar{k}=1}^{2}\overline{\mathrm{A}^{\bar{i}}_{r\bar{k}}}\,G^{i\bar{k}}_{p}\,\,-\sum_{k=0}^{3}\Gamma^{k}_{rp}\,G^{i\bar{i}}_{k}=0.
\tag{5.9}
$$

The A-components of the spinor extension of Γ are given by the formula (1.3). Our next goal is to verify the formula (1.3) by substituting it into (5.8) and (5.9).

Let's begin with the second term in the formula (5.8). Applying the formula (1.3) to this term, we derive the following expression for it:

$$
\begin{gathered}
\sum_{k=1}^{2}\mathrm{A}^{k}_{ri}\,\,d_{kj}\,=\frac{1}{4}\sum_{k=1}^{2}\sum_{\bar{s}=1}^{2}\sum_{p=0}^{3}\sum_{q=0}^{3}G^{k\bar{s}}_{p}\,\Gamma^{p}_{rq}\,G^{\,q}_{i\bar{s}}\,d_{kj}\,-\\
-\frac{1}{4}\sum_{k=1}^{2}\sum_{\bar{s}=1}^{2}\sum_{q=0}^{3}L_{\boldsymbol\Upsilon_r}(G^{k\bar{s}}_{q})\,G^{\,q}_{i\bar{s}}\,d_{kj}-\frac{1}{4}\sum_{\bar{i}=1}^{2}\sum_{\bar{j}=1}^{2}L_{\boldsymbol\Upsilon_r}(\bar{d}_{\bar{j}\bar{i}})\,\bar{d}^{\bar{i}\bar{j}}\,d_{ij}.
\end{gathered}
\tag{5.10}
$$

For to transform the first term in the right hand side of (5.10) we use the formulas

$$
\sum_{k=1}^{2}G^{k\bar{s}}_{p}\,d_{kj}=\sum_{m=0}^{3}\sum_{\bar{r}=1}^{2}G^{\,m}_{j\bar{r}}\,\bar{d}^{\bar{r}\bar{s}}\,g_{mp}, \qquad\qquad \Gamma_{rqm}=\sum_{p=0}^{3}g_{mp}\,\Gamma^{p}_{rq}.
\tag{5.11}
$$

The first formula (5.11) is derived from (2.4), (2.5), and (2.8). The second formula (5.11) is a notation defining Γ_{rqm}. In order to transform the second term in the right hand side of (5.10) we use the formula

$$\sum_{\bar{s}=1}^{2}\sum_{q=0}^{3} L_{\Upsilon_r}(G^{k\bar{s}}_{q})\, G^{q}_{i\bar{s}} = -\sum_{\bar{s}=1}^{2}\sum_{q=0}^{3} G^{k\bar{s}}_{q}\, L_{\Upsilon_r}(G^{q}_{i\bar{s}}) \tag{5.12}$$

This formula is derived by applying the differential operator (2.15) to the second identity (1.1). Applying (5.11) and (5.12) to (5.10), we get

$$\begin{gathered}\sum_{k=1}^{2} \mathrm{A}^{k}_{ri}\, d_{kj} = \frac{1}{4}\sum_{\bar{s}=1}^{2}\sum_{q=0}^{3}\sum_{m=0}^{3}\sum_{\bar{r}=1}^{2} G^{m}_{j\bar{r}}\, G^{q}_{i\bar{s}}\, \Gamma_{rqm}\, \bar{d}^{\bar{r}\bar{s}} + \\ +\frac{1}{4}\sum_{k=1}^{2}\sum_{\bar{s}=1}^{2}\sum_{q=0}^{3} G^{k\bar{s}}_{q}\, L_{\Upsilon_r}(G^{q}_{i\bar{s}})\, d_{kj} - \frac{1}{4}\sum_{\bar{i}=1}^{2}\sum_{\bar{j}=1}^{2} L_{\Upsilon_r}(\bar{d}_{\bar{j}\bar{i}})\, \bar{d}^{\bar{i}\bar{j}}\, d_{ij}.\end{gathered} \tag{5.13}$$

Now we replace m by p in the first term and replace q by p in the second term in the right hand side of the formula (5.13). This yields

$$\begin{gathered}\sum_{k=1}^{2} \mathrm{A}^{k}_{ri}\, d_{kj} = \frac{1}{4}\sum_{\bar{s}=1}^{2}\sum_{q=0}^{3}\sum_{p=0}^{3}\sum_{\bar{r}=1}^{2} G^{p}_{j\bar{r}}\, G^{q}_{i\bar{s}}\, \Gamma_{rqp}\, \bar{d}^{\bar{r}\bar{s}} + \\ +\frac{1}{4}\sum_{k=1}^{2}\sum_{\bar{s}=1}^{2}\sum_{p=0}^{3} G^{k\bar{s}}_{p}\, L_{\Upsilon_r}(G^{p}_{i\bar{s}})\, d_{kj} - \frac{1}{4}\sum_{\bar{i}=1}^{2}\sum_{\bar{j}=1}^{2} L_{\Upsilon_r}(\bar{d}_{\bar{j}\bar{i}})\, \bar{d}^{\bar{i}\bar{j}}\, d_{ij}.\end{gathered} \tag{5.14}$$

As a result we can apply the first formula (5.11) to the second term in the right hand side of the formula (5.14). Then we have

$$\begin{gathered}\sum_{k=1}^{2} \mathrm{A}^{k}_{ri}\, d_{kj} = \frac{1}{4}\sum_{\bar{s}=1}^{2}\sum_{q=0}^{3}\sum_{p=0}^{3}\sum_{\bar{r}=1}^{2} G^{p}_{j\bar{r}}\, G^{q}_{i\bar{s}}\, \Gamma_{rqp}\, \bar{d}^{\bar{r}\bar{s}} + \\ +\frac{1}{4}\sum_{\bar{s}=1}^{2}\sum_{\bar{r}=1}^{2}\sum_{m=0}^{3}\sum_{p=0}^{3} G^{m}_{j\bar{r}}\, \bar{d}^{\bar{r}\bar{s}}\, g_{mp}\, L_{\Upsilon_r}(G^{p}_{i\bar{s}}) - \frac{1}{4}\sum_{\bar{i}=1}^{2}\sum_{\bar{j}=1}^{2} L_{\Upsilon_r}(\bar{d}_{\bar{j}\bar{i}})\, \bar{d}^{\bar{i}\bar{j}}\, d_{ij}.\end{gathered} \tag{5.15}$$

For the sake of beauty we replace m by q in the formula (5.15):

$$\begin{gathered}\sum_{k=1}^{2} \mathrm{A}^{k}_{ri}\, d_{kj} = \frac{1}{4}\sum_{\bar{s}=1}^{2}\sum_{q=0}^{3}\sum_{p=0}^{3}\sum_{\bar{r}=1}^{2} G^{p}_{j\bar{r}}\, G^{q}_{i\bar{s}}\, \Gamma_{rqp}\, \bar{d}^{\bar{r}\bar{s}} + \\ +\frac{1}{4}\sum_{\bar{s}=1}^{2}\sum_{\bar{r}=1}^{2}\sum_{p=0}^{3}\sum_{q=0}^{3} G^{q}_{j\bar{r}}\, \bar{d}^{\bar{r}\bar{s}}\, g_{pq}\, L_{\Upsilon_r}(G^{p}_{i\bar{s}}) - \frac{1}{4}\sum_{\bar{i}=1}^{2}\sum_{\bar{j}=1}^{2} L_{\Upsilon_r}(\bar{d}_{\bar{j}\bar{i}})\, \bar{d}^{\bar{i}\bar{j}}\, d_{ij}.\end{gathered} \tag{5.16}$$

Now we proceed to the third term in right hand side of the formula (5.8). In order to transform it we use the skew symmetry $d_{ik} = -d_{ki}$:

$$\sum_{k=1}^{2} \mathrm{A}^{k}_{rj}\, d_{ik} = -\sum_{k=1}^{2} \mathrm{A}^{k}_{rj}\, d_{ki}. \tag{5.17}$$

Then we can apply the formula (5.16) to (5.17). As a result we get

$$\begin{gathered}\sum_{k=1}^{2}\mathrm{A}^{k}_{rj}\ d_{ik}=-\frac{1}{4}\sum_{\bar s=1}^{2}\sum_{q=0}^{3}\sum_{p=0}^{3}\sum_{\bar r=1}^{2}G^{p}_{i\bar r}\,G^{q}_{j\bar s}\,\Gamma_{rqp}\,\bar d^{\bar r\bar s}-\\-\frac{1}{4}\sum_{\bar s=1}^{2}\sum_{\bar r=1}^{2}\sum_{p=0}^{3}\sum_{q=0}^{3}G^{q}_{i\bar r}\,\bar d^{\bar r\bar s}\,g_{pq}\,L_{\boldsymbol\Upsilon_r}(G^{p}_{j\bar s})+\frac{1}{4}\sum_{\bar i=1}^{2}\sum_{\bar j=1}^{2}L_{\boldsymbol\Upsilon_r}(\bar d_{\bar j\bar i})\,\bar d^{\bar i\bar j}\,d_{ji}.\end{gathered}\tag{5.18}$$

Let's exchange p with q and exchange $\bar r$ with $\bar s$ in the first term in right hand side of the above formula (5.18). Moreover, let's do the same in the second term in right hand side of this formula and take into account the symmetry $g_{qp}=g_{pq}$ and the skew symmetry $\bar d^{\bar s\bar r}=-\bar d^{\bar r\bar s}$. As a result of these transformations we obtain

$$\begin{gathered}\sum_{k=1}^{2}\mathrm{A}^{k}_{rj}\ d_{ik}=\frac{1}{4}\sum_{\bar s=1}^{2}\sum_{q=0}^{3}\sum_{p=0}^{3}\sum_{\bar r=1}^{2}G^{q}_{i\bar s}\,G^{p}_{j\bar r}\,\Gamma_{rpq}\,\bar d^{\bar r\bar s}+\\+\frac{1}{4}\sum_{\bar s=1}^{2}\sum_{\bar r=1}^{2}\sum_{p=0}^{3}\sum_{q=0}^{3}L_{\boldsymbol\Upsilon_r}(G^{q}_{j\bar r})\,g_{pq}\,G^{p}_{i\bar s}\,\bar d^{\bar r\bar s}-\frac{1}{4}\sum_{\bar i=1}^{2}\sum_{\bar j=1}^{2}L_{\boldsymbol\Upsilon_r}(\bar d_{\bar j\bar i})\,\bar d^{\bar i\bar j}\,d_{ij}.\end{gathered}\tag{5.19}$$

Now let's add the equalities (5.16) and (5.19). This yields

$$\begin{gathered}\sum_{k=1}^{2}\mathrm{A}^{k}_{ri}\ d_{kj}+\sum_{k=1}^{2}\mathrm{A}^{k}_{rj}\ d_{ik}=\frac{1}{4}\sum_{\bar s=1}^{2}\sum_{q=0}^{3}\sum_{p=0}^{3}\sum_{\bar r=1}^{2}G^{p}_{j\bar r}\,G^{q}_{i\bar s}\bigl(\Gamma_{rqp}+\\+\Gamma_{rpq}\bigr)\,\bar d^{\bar r\bar s}+\frac{1}{4}\sum_{\bar s=1}^{2}\sum_{\bar r=1}^{2}\sum_{p=0}^{3}\sum_{q=0}^{3}\bigl(L_{\boldsymbol\Upsilon_r}(G^{q}_{j\bar r})\,G^{p}_{i\bar s}+G^{q}_{j\bar r}\,L_{\boldsymbol\Upsilon_r}(G^{p}_{i\bar s})\bigr)\times\\\times\,\bar d^{\bar r\bar s}\,g_{pq}-\frac{1}{2}\sum_{\bar i=1}^{2}\sum_{\bar j=1}^{2}L_{\boldsymbol\Upsilon_r}(\bar d_{\bar j\bar i})\,\bar d^{\bar i\bar j}\,d_{ij}.\end{gathered}\tag{5.20}$$

Let's recall that the differential operator (2.15), when applied to a product, obeys the Leibniz rule. Then we can perform the following transformations:

$$\begin{gathered}\sum_{\bar s=1}^{2}\sum_{\bar r=1}^{2}\sum_{p=0}^{3}\sum_{q=0}^{3}\bigl(L_{\boldsymbol\Upsilon_r}(G^{q}_{j\bar r})\,G^{p}_{i\bar s}+G^{q}_{j\bar r}\,L_{\boldsymbol\Upsilon_r}(G^{p}_{i\bar s})\bigr)\,\bar d^{\bar r\bar s}\,g_{pq}=\\=\sum_{\bar s=1}^{2}\sum_{\bar r=1}^{2}\sum_{p=0}^{3}\sum_{q=0}^{3}L_{\boldsymbol\Upsilon_r}(G^{q}_{j\bar r}\,G^{p}_{i\bar s})\,\bar d^{\bar r\bar s}\,g_{pq}=\sum_{\bar s=1}^{2}\sum_{\bar r=1}^{2}\sum_{p=0}^{3}\sum_{q=0}^{3}L_{\boldsymbol\Upsilon_r}(G^{q}_{j\bar r}\times\\\times\,G^{p}_{i\bar s}\ g_{pq})\,\bar d^{\bar r\bar s}-\sum_{\bar s=1}^{2}\sum_{\bar r=1}^{2}\sum_{p=0}^{3}\sum_{q=0}^{3}G^{q}_{j\bar r}\,G^{p}_{i\bar s}\,L_{\boldsymbol\Upsilon_r}(g_{pq})\,\bar d^{\bar r\bar s}.\end{gathered}$$

Due to (2.3) and (2.8) the first formula (5.11) yields

$$\sum_{\bar a=1}^{2}\sum_{k=1}^{2}G^{k\bar a}_{p}\,d_{kj}\,\bar d_{\bar a\bar r}=\sum_{q=0}^{3}G^{q}_{j\bar r}\,g_{pq}.\tag{5.21}$$

Applying the relationship (5.21), we can continue the above transformations

$$\begin{aligned}&\sum_{\bar{s}=1}^{2}\sum_{\bar{r}=1}^{2}\sum_{p=0}^{3}\sum_{q=0}^{3}\bigl(L_{\boldsymbol{\Upsilon}_r}(G^{\,q}_{j\bar{r}})\,G^{\,p}_{i\bar{s}}+G^{\,q}_{j\bar{r}}\,L_{\boldsymbol{\Upsilon}_r}(G^{\,p}_{i\bar{s}})\bigr)\,\bar{d}^{\bar{r}\bar{s}}\,g_{pq}=\\&=\sum_{k=1}^{2}\sum_{\bar{a}=1}^{2}\sum_{\bar{s}=1}^{2}\sum_{\bar{r}=1}^{2}\sum_{p=0}^{3}L_{\boldsymbol{\Upsilon}_r}(G^{k\bar{a}}_{p}\,G^{\,p}_{i\bar{s}}\,d_{kj}\,\bar{d}_{\bar{a}\bar{r}})\,\bar{d}^{\bar{r}\bar{s}}\,-\\&\qquad-\sum_{\bar{s}=1}^{2}\sum_{\bar{r}=1}^{2}\sum_{p=0}^{3}\sum_{q=0}^{3}G^{\,q}_{j\bar{r}}\,G^{\,p}_{i\bar{s}}\,L_{\boldsymbol{\Upsilon}_r}(g_{pq})\,\bar{d}^{\bar{r}\bar{s}}.\end{aligned}$$

Now we can apply the second identity (1.1) to the first term in right hand side of the above equality. As a result we derive

$$\begin{aligned}&\sum_{\bar{s}=1}^{2}\sum_{\bar{r}=1}^{2}\sum_{p=0}^{3}\sum_{q=0}^{3}\bigl(L_{\boldsymbol{\Upsilon}_r}(G^{\,q}_{j\bar{r}})\,G^{\,p}_{i\bar{s}}+G^{\,q}_{j\bar{r}}\,L_{\boldsymbol{\Upsilon}_r}(G^{\,p}_{i\bar{s}})\bigr)\,\bar{d}^{\bar{r}\bar{s}}\,g_{pq}=\\&=\sum_{\bar{s}=1}^{2}\sum_{\bar{r}=1}^{2}L_{\boldsymbol{\Upsilon}_r}(2\,d_{ij}\,\bar{d}_{\bar{s}\bar{r}})\,\bar{d}^{\bar{r}\bar{s}}-\sum_{\bar{s}=1}^{2}\sum_{\bar{r}=1}^{2}\sum_{p=0}^{3}\sum_{q=0}^{3}G^{\,q}_{j\bar{r}}\,G^{\,p}_{i\bar{s}}\,L_{\boldsymbol{\Upsilon}_r}(g_{pq})\,\bar{d}^{\bar{r}\bar{s}}.\end{aligned}\tag{5.22}$$

Applying the Leibniz rule to the first term in the right hand side of (5.22) and then substituting (5.22) back into (5.20), we obtain

$$\begin{aligned}&\sum_{k=1}^{2}\mathrm{A}^{k}_{ri}\,d_{kj}+\sum_{k=1}^{2}\mathrm{A}^{k}_{rj}\,d_{ik}=\frac{1}{4}\sum_{\bar{s}=1}^{2}\sum_{q=0}^{3}\sum_{p=0}^{3}\sum_{\bar{r}=1}^{2}G^{\,p}_{j\bar{r}}\,G^{\,q}_{i\bar{s}}\bigl(\Gamma_{rqp}+\\&+\Gamma_{rpq}\bigr)\,\bar{d}^{\bar{r}\bar{s}}+L_{\boldsymbol{\Upsilon}_r}(d_{ij})-\frac{1}{4}\sum_{\bar{s}=1}^{2}\sum_{\bar{r}=1}^{2}\sum_{p=0}^{3}\sum_{q=0}^{3}G^{\,q}_{j\bar{r}}\,G^{\,p}_{i\bar{s}}\,L_{\boldsymbol{\Upsilon}_r}(g_{pq})\,\bar{d}^{\bar{r}\bar{s}}.\end{aligned}\tag{5.23}$$

The next step is to transform Γ_{rqp} in (5.23). For this purpose we substitute (5.4) into the second formula (5.11). As a result we get

$$\begin{aligned}\Gamma_{rqp}&=\frac{1}{2}\,\bigl(L_{\boldsymbol{\Upsilon}_r}(g_{pq})+L_{\boldsymbol{\Upsilon}_q}(g_{rp})-L_{\boldsymbol{\Upsilon}_p}(g_{rq})\bigr)+\\&+\sum_{s=0}^{3}\frac{c^{\,s}_{rq}}{2}\,g_{sp}-\sum_{s=0}^{3}\frac{c^{\,s}_{rp}}{2}\,g_{sq}-\sum_{s=0}^{3}\frac{c^{\,s}_{qp}}{2}\,g_{sr}.\end{aligned}\tag{5.24}$$

When symmetrizing with respect to the indices p and q most of the terms in (5.24) do cancel each other. The rest of them yield

$$\Gamma_{rqp}+\Gamma_{rpq}=L_{\boldsymbol{\Upsilon}_r}(g_{pq}).\tag{5.25}$$

Substituting (5.25) back into the formula (5.23), we derive

$$\sum_{k=1}^{2}\mathrm{A}^{k}_{ri}\,d_{kj}+\sum_{k=1}^{2}\mathrm{A}^{k}_{rj}\,d_{ik}=L_{\boldsymbol{\Upsilon}_r}(d_{ij}).\tag{5.26}$$

Comparing (5.26) with (5.8), we find that these formulas are equivalent. Thus we conclude that for the spinor connection with the components (1.3) the first concordance condition (5.7) is fulfilled.

Now let's proceed to verifying the second concordance condition (5.7). Its coordinate presentation is given by the formula (5.9). Let's begin with the second term in (5.9). Applying (1.3) to this term, we derive

$$\begin{gathered}\sum_{k=1}^{2}\mathrm{A}^i_{rk}\,G^{k\bar i}_p=\sum_{k=1}^{2}\sum_{\bar s=1}^{2}\sum_{q=0}^{3}\sum_{m=0}^{3}\frac{G^{i\bar s}_q\,\Gamma^q_{rm}\,G^m_{k\bar s}\,G^{k\bar i}_p}{4}\,-\\-\sum_{k=1}^{2}\sum_{\bar s=1}^{2}\sum_{m=0}^{3}\frac{L_{\Upsilon_r}(G^{i\bar s}_m)\,G^m_{k\bar s}\,G^{k\bar i}_p}{4}-\sum_{\bar u=1}^{2}\sum_{\bar s=1}^{2}\frac{L_{\Upsilon_r}(\bar d_{\bar s\bar u})\,\bar d^{\bar u\bar s}\,G^{i\bar i}_p}{4}.\end{gathered}\tag{5.27}$$

In a similar way, applying (1.3) to the third term in (5.9), we obtain

$$\begin{gathered}\sum_{\bar k=1}^{2}\overline{\mathrm{A}^{\bar i}_{r\bar k}}\,G^{i\bar k}_p=\sum_{\bar k=1}^{2}\sum_{s=1}^{2}\sum_{q=0}^{3}\sum_{m=0}^{3}\frac{\overline{G^{\bar i s}_q}\,\Gamma^q_{rm}\,\overline{G^m_{\bar k s}}\,G^{i\bar k}_p}{4}\,-\\-\sum_{\bar k=1}^{2}\sum_{s=1}^{2}\sum_{m=0}^{3}\frac{L_{\Upsilon_r}(\overline{G^{\bar i s}_m})\,\overline{G^m_{\bar k s}}\,G^{i\bar k}_p}{4}-\sum_{u=1}^{2}\sum_{s=1}^{2}\frac{L_{\Upsilon_r}(\overline{\bar d_{su}})\,\overline{\bar d^{us}}\,G^{i\bar i}_p}{4}.\end{gathered}\tag{5.28}$$

In order to transform (5.28) we apply (2.6), (2.7), and (2.9). This yields

$$\begin{gathered}\sum_{\bar k=1}^{2}\overline{\mathrm{A}^{\bar i}_{r\bar k}}\,G^{i\bar k}_p=\sum_{\bar k=1}^{2}\sum_{s=1}^{2}\sum_{q=0}^{3}\sum_{m=0}^{3}\frac{G^{s\bar i}_q\,\Gamma^q_{rm}\,G^m_{s\bar k}\,G^{i\bar k}_p}{4}\,-\\-\sum_{\bar k=1}^{2}\sum_{s=1}^{2}\sum_{m=0}^{3}\frac{L_{\Upsilon_r}(G^{s\bar i}_m)\,G^m_{s\bar k}\,G^{i\bar k}_p}{4}-\sum_{u=1}^{2}\sum_{s=1}^{2}\frac{L_{\Upsilon_r}(d_{su})\,d^{us}\,G^{i\bar i}_p}{4}.\end{gathered}\tag{5.29}$$

Now we replace s by k and replace $\bar k$ by $\bar s$ in the right hand side of (5.29):

$$\begin{gathered}\sum_{\bar k=1}^{2}\overline{\mathrm{A}^{\bar i}_{r\bar k}}\,G^{i\bar k}_p=\sum_{k=1}^{2}\sum_{\bar s=1}^{2}\sum_{q=0}^{3}\sum_{m=0}^{3}\frac{G^{i\bar s}_p\,\Gamma^q_{rm}\,G^m_{k\bar s}\,G^{k\bar i}_q}{4}\,-\\-\sum_{k=1}^{2}\sum_{\bar s=1}^{2}\sum_{m=0}^{3}\frac{G^{i\bar s}_p\,G^m_{k\bar s}\,L_{\Upsilon_r}(G^{k\bar i}_m)}{4}-\sum_{u=1}^{2}\sum_{s=1}^{2}\frac{L_{\Upsilon_r}(d_{su})\,d^{us}\,G^{i\bar i}_p}{4}.\end{gathered}\tag{5.30}$$

Let's return back to the formula (5.12) and rewrite it as follows:

$$\sum_{\bar s=1}^{2}\sum_{m=0}^{3}L_{\Upsilon_r}(G^{i\bar s}_m)\,G^m_{k\bar s}=-\sum_{\bar s=1}^{2}\sum_{m=0}^{3}G^{i\bar s}_m\,L_{\Upsilon_r}(G^m_{k\bar s}).\tag{5.31}$$

There is a formula very similar to (5.31). Here it is:

$$\sum_{k=1}^{2}\sum_{m=0}^{3}G^m_{k\bar s}\,L_{\Upsilon_r}(G^{k\bar i}_m)=-\sum_{k=1}^{2}\sum_{m=0}^{3}L_{\Upsilon_r}(G^m_{k\bar s})\,G^{k\bar i}_m.\tag{5.32}$$

We apply (5.31) to (5.27) and apply (5.32) to (5.30). Then we add these two formulas. As a result we obtain the following formula:

$$
\begin{gathered}
\sum_{k=1}^{2}\mathrm{A}^{i}_{rk}\,G^{k\bar{i}}_{p}+\sum_{\bar{k}=1}^{2}\overline{\mathrm{A}^{\bar{i}}_{r\bar{k}}}\,G^{i\bar{k}}_{p}=\sum_{k=1}^{2}\sum_{\bar{s}=1}^{2}\sum_{q=0}^{3}\sum_{m=0}^{3}\frac{\bigl(G^{i\bar{s}}_{q}\,G^{k\bar{i}}_{p}+G^{i\bar{s}}_{p}\;G^{k\bar{i}}_{q}\bigr)}{4}\,\times\\
\times\,\Gamma^{q}_{rm}\,G^{m}_{k\bar{s}}+\sum_{k=1}^{2}\sum_{\bar{s}=1}^{2}\sum_{m=0}^{3}\frac{\bigl(G^{i\bar{s}}_{m}\,G^{k\bar{i}}_{p}+G^{i\bar{s}}_{p}\,G^{k\bar{i}}_{m}\bigr)}{4}\,L_{\boldsymbol{\Upsilon}_r}(G^{m}_{k\bar{s}})\,-\\
-\sum_{\bar{u}=1}^{2}\sum_{\bar{s}=1}^{2}\frac{L_{\boldsymbol{\Upsilon}_r}(\bar{d}_{\bar{s}\bar{u}})\,\bar{d}^{\bar{u}\bar{s}}\,G^{i\bar{i}}_{p}}{4}-\sum_{u=1}^{2}\sum_{s=1}^{2}\frac{L_{\boldsymbol{\Upsilon}_r}(d_{su})\,d^{us}\,G^{i\bar{i}}_{p}}{4}.
\end{gathered}\tag{5.33}
$$

In order to transform (5.33) we use the formula (4.2) derived from the identity (1.2). We rewrite this formula as follows:

$$
\begin{aligned}
&G^{i\bar{s}}_{q}\,G^{k\bar{i}}_{p}+G^{i\bar{s}}_{p}\,G^{k\bar{i}}_{q}=G^{i\bar{i}}_{q}\,G^{k\bar{s}}_{p}+G^{i\bar{i}}_{p}\,G^{k\bar{s}}_{q}-2\,d^{ik}\,\bar{d}^{\bar{i}\bar{s}}\,g_{pq},\\
&G^{i\bar{s}}_{m}\,G^{k\bar{i}}_{p}+G^{i\bar{s}}_{p}\,G^{k\bar{i}}_{m}=G^{i\bar{i}}_{m}\,G^{k\bar{s}}_{p}+G^{i\bar{i}}_{p}\,G^{k\bar{s}}_{m}-2\,d^{ik}\,\bar{d}^{\bar{i}\bar{s}}\,g_{pm}.
\end{aligned}\tag{5.34}
$$

Substituting (5.34) back into the formula (5.33), we derive

$$
\begin{gathered}
\sum_{k=1}^{2}\mathrm{A}^{i}_{rk}\,G^{k\bar{i}}_{p}+\sum_{\bar{k}=1}^{2}\overline{\mathrm{A}^{\bar{i}}_{r\bar{k}}}\,G^{i\bar{k}}_{p}=\sum_{k=1}^{2}\sum_{\bar{s}=1}^{2}\sum_{q=0}^{3}\sum_{m=0}^{3}\frac{\bigl(G^{i\bar{i}}_{q}\,G^{k\bar{s}}_{p}+G^{i\bar{i}}_{p}\,G^{k\bar{s}}_{q}\bigr)}{4}\,\times\\
\times\,\Gamma^{q}_{rm}\,G^{m}_{k\bar{s}}+\sum_{k=1}^{2}\sum_{\bar{s}=1}^{2}\sum_{m=0}^{3}\frac{\bigl(G^{i\bar{i}}_{m}\,G^{k\bar{s}}_{p}+G^{i\bar{i}}_{p}\,G^{k\bar{s}}_{m}\bigr)}{4}\,L_{\boldsymbol{\Upsilon}_r}(G^{m}_{k\bar{s}})\,-\\
-\sum_{k=1}^{2}\sum_{\bar{s}=1}^{2}\sum_{q=0}^{3}\sum_{m=0}^{3}\frac{d^{ik}\,\bar{d}^{\bar{i}\bar{s}}\,g_{pq}}{2}\,\Gamma^{q}_{rm}\,G^{m}_{k\bar{s}}-\sum_{k=1}^{2}\sum_{\bar{s}=1}^{2}\sum_{m=0}^{3}\frac{d^{ik}\,\bar{d}^{\bar{i}\bar{s}}\,g_{pm}}{2}\,L_{\boldsymbol{\Upsilon}_r}(G^{m}_{k\bar{s}})\,-\\
-\sum_{\bar{u}=1}^{2}\sum_{\bar{s}=1}^{2}\frac{L_{\boldsymbol{\Upsilon}_r}(\bar{d}_{\bar{s}\bar{u}})\,\bar{d}^{\bar{u}\bar{s}}\,G^{i\bar{i}}_{p}}{4}-\sum_{u=1}^{2}\sum_{s=1}^{2}\frac{L_{\boldsymbol{\Upsilon}_r}(d_{su})\,d^{us}\,G^{i\bar{i}}_{p}}{4}.
\end{gathered}
$$

In order to transform this equality we use the first identity (1.1) and the second formula (5.11). As a result we obtain the equality

$$
\begin{gathered}
\sum_{k=1}^{2}\mathrm{A}^{i}_{rk}\,G^{k\bar{i}}_{p}+\sum_{\bar{k}=1}^{2}\overline{\mathrm{A}^{\bar{i}}_{r\bar{k}}}\,G^{i\bar{k}}_{p}=\sum_{q=0}^{3}\frac{\bigl(G^{i\bar{i}}_{q}\,\Gamma^{q}_{rp}+G^{i\bar{i}}_{p}\;\Gamma^{q}_{rq}\bigr)}{2}\,+\\
+\sum_{k=1}^{2}\sum_{\bar{s}=1}^{2}\sum_{m=0}^{3}\frac{G^{i\bar{i}}_{m}\,G^{k\bar{s}}_{p}}{4}\,L_{\boldsymbol{\Upsilon}_r}(G^{m}_{k\bar{s}})+\sum_{k=1}^{2}\sum_{\bar{s}=1}^{2}\sum_{m=0}^{3}\frac{G^{i\bar{i}}_{p}\,G^{k\bar{s}}_{m}}{4}\,L_{\boldsymbol{\Upsilon}_r}(G^{m}_{k\bar{s}})\,-\\
-\sum_{k=1}^{2}\sum_{\bar{s}=1}^{2}\sum_{m=0}^{3}\frac{d^{ik}\,\bar{d}^{\bar{i}\bar{s}}}{2}\,\Gamma_{rmp}\,G^{m}_{k\bar{s}}-\sum_{k=1}^{2}\sum_{\bar{s}=1}^{2}\sum_{m=0}^{3}\frac{d^{ik}\,\bar{d}^{\bar{i}\bar{s}}\,g_{pm}}{2}\,L_{\boldsymbol{\Upsilon}_r}(G^{m}_{k\bar{s}})\,-\\
-\sum_{\bar{u}=1}^{2}\sum_{\bar{s}=1}^{2}\frac{L_{\boldsymbol{\Upsilon}_r}(\bar{d}_{\bar{s}\bar{u}})\,\bar{d}^{\bar{u}\bar{s}}\,G^{i\bar{i}}_{p}}{4}-\sum_{u=1}^{2}\sum_{s=1}^{2}\frac{L_{\boldsymbol{\Upsilon}_r}(d_{su})\,d^{us}\,G^{i\bar{i}}_{p}}{4}.
\end{gathered}\tag{5.35}
$$

Applying the differential operator (2.15) to the second identity (1.1), we derive

$$\sum_{m=0}^{3} G^{i\bar{i}}_{m}\,L_{\boldsymbol{\Upsilon}_r}(G^{m}_{k\bar{s}})=-\sum_{m=0}^{3} L_{\boldsymbol{\Upsilon}_r}(G^{i\bar{i}}_{m})\,G^{m}_{k\bar{s}}. \tag{5.36}$$

The equality (5.36) is similar to (5.31) and (5.32). Now we use the second identity (1.1) once more in order to perform the following calculations:

$$\begin{aligned}
0=\sum_{k=1}^{2}\sum_{\bar{s}=1}^{2}\sum_{m=0}^{3} L_{\boldsymbol{\Upsilon}_r}(G^{k\bar{s}}_{m}\,G^{m}_{k\bar{s}})=\sum_{k=1}^{2}\sum_{\bar{s}=1}^{2}\sum_{m=0}^{3}\sum_{q=1}^{2}\sum_{\bar{r}=1}^{2}\sum_{n=0}^{3} L_{\boldsymbol{\Upsilon}_r}(G^{n}_{q\bar{r}}\,\times\\
\times\, d^{qk}\,\bar{d}^{\bar{r}\bar{s}} g_{nm}\,G^{m}_{k\bar{s}})=\sum_{k=1}^{2}\sum_{\bar{s}=1}^{2}\sum_{m=0}^{3}\sum_{q=1}^{2}\sum_{\bar{r}=1}^{2}\sum_{n=0}^{3} L_{\boldsymbol{\Upsilon}_r}(G^{n}_{q\bar{r}})\,d^{qk}\,\bar{d}^{\bar{r}\bar{s}}\,g_{nm}\,\times\\
\times\, G^{m}_{k\bar{s}}+\sum_{k=1}^{2}\sum_{\bar{s}=1}^{2}\sum_{m=0}^{3}\sum_{q=1}^{2}\sum_{\bar{r}=1}^{2}\sum_{n=0}^{3} G^{n}_{q\bar{r}}\,L_{\boldsymbol{\Upsilon}_r}(d^{qk}\,\bar{d}^{\bar{r}\bar{s}}\,g_{nm})\,G^{m}_{k\bar{s}}\,+\\
+\sum_{k=1}^{2}\sum_{\bar{s}=1}^{2}\sum_{m=0}^{3}\sum_{q=1}^{2}\sum_{\bar{r}=1}^{2}\sum_{n=0}^{3} G^{n}_{q\bar{r}}\,d^{qk}\,\bar{d}^{\bar{r}\bar{s}}\,g_{nm}\,L_{\boldsymbol{\Upsilon}_r}(G^{m}_{k\bar{s}})\,=\\
=\sum_{k=1}^{2}\sum_{\bar{s}=1}^{2}\sum_{m=0}^{3}\sum_{q=1}^{2}\sum_{\bar{r}=1}^{2}\sum_{n=0}^{3} G^{n}_{q\bar{r}}\,L_{\boldsymbol{\Upsilon}_r}(d^{qk}\,\bar{d}^{\bar{r}\bar{s}}\,g_{nm})\,G^{m}_{k\bar{s}}\,+\\
+\sum_{q=1}^{2}\sum_{\bar{r}=1}^{2}\sum_{n=0}^{3} L_{\boldsymbol{\Upsilon}_r}(G^{n}_{q\bar{r}})\,G^{q\bar{r}}_{n}+\sum_{k=1}^{2}\sum_{\bar{s}=1}^{2}\sum_{m=0}^{3} G^{k\bar{s}}_{m}\,L_{\boldsymbol{\Upsilon}_r}(G^{m}_{k\bar{s}}).
\end{aligned}$$

It easy to see that the last two terms in the above formula are equal to each other. For this reason the above formula is equivalent to

$$\sum_{k=1}^{2}\sum_{\bar{s}=1}^{2}\sum_{m=0}^{3} G^{k\bar{s}}_{m}\,L_{\boldsymbol{\Upsilon}_r}(G^{m}_{k\bar{s}})=-\frac{1}{2}\sum_{k=1}^{2}\sum_{\bar{s}=1}^{2}\sum_{m=0}^{3}\sum_{q=1}^{2}\sum_{\bar{r}=1}^{2}\sum_{n=0}^{3} G^{n}_{q\bar{r}}\,L_{\boldsymbol{\Upsilon}_r}(d^{qk}\,\bar{d}^{\bar{r}\bar{s}}\,g_{nm})\,G^{m}_{k\bar{s}}.$$

The operator $L_{\boldsymbol{\Upsilon}_r}$ obeys the Leibniz rule. Therefore, we have

$$\begin{aligned}
\sum_{k=1}^{2}\sum_{\bar{s}=1}^{2}\sum_{m=0}^{3} G^{k\bar{s}}_{m}\,L_{\boldsymbol{\Upsilon}_r}(G^{m}_{k\bar{s}})=-\frac{1}{2}\sum_{k=1}^{2}\sum_{\bar{s}=1}^{2}\sum_{m=0}^{3}\sum_{q=1}^{2}\sum_{\bar{r}=1}^{2}\sum_{n=0}^{3} G^{n}_{q\bar{r}}\,L_{\boldsymbol{\Upsilon}_r}(d^{qk})\,\times\\
\times\,\bar{d}^{\bar{r}\bar{s}}\,g_{nm}\,G^{m}_{k\bar{s}}-\frac{1}{2}\sum_{k=1}^{2}\sum_{\bar{s}=1}^{2}\sum_{m=0}^{3}\sum_{q=1}^{2}\sum_{\bar{r}=1}^{2}\sum_{n=0}^{3} G^{n}_{q\bar{r}}\,d^{qk}\,L_{\boldsymbol{\Upsilon}_r}(\bar{d}^{\bar{r}\bar{s}})\,\times\\
\times\, g_{nm}\,G^{m}_{k\bar{s}}-\frac{1}{2}\sum_{k=1}^{2}\sum_{\bar{s}=1}^{2}\sum_{m=0}^{3}\sum_{q=1}^{2}\sum_{\bar{r}=1}^{2}\sum_{n=0}^{3} G^{n}_{q\bar{r}}\,d^{qk}\,\bar{d}^{\bar{r}\bar{s}}\,L_{\boldsymbol{\Upsilon}_r}(\,g_{nm})\,G^{m}_{k\bar{s}}.
\end{aligned} \tag{5.37}$$

Note that the first formula (5.11), can be rewritten as follows:

$$\sum_{\bar{s}=1}^{2}\sum_{m=0}^{3} \bar{d}^{\bar{r}\bar{s}}\,g_{nm}\,G^{m}_{k\bar{s}}=\sum_{s=1}^{2} G^{s\bar{r}}_{n}\,d_{ks}. \tag{5.38}$$

The equality (5.11) itself was derived from (2.3), (2.4), (2.5), and (2.8). By analogy to it one can derive the following two equalities:

$$\sum_{k=1}^{2}\sum_{m=0}^{3} d^{qk}\,g_{nm}\,G^{\,m}_{k\bar{s}} = \sum_{\bar{k}=1}^{2} G^{q\bar{k}}_{n}\,\bar{d}_{\bar{s}\bar{k}}, \tag{5.39}$$

$$\sum_{k=1}^{2}\sum_{\bar{s}=1}^{2} d^{qk}\bar{d}^{\bar{r}\bar{s}}\,G^{\,m}_{k\bar{s}} = \sum_{p=0}^{3} G^{q\bar{r}}_{p}\,g^{pm}. \tag{5.40}$$

Now we apply the formulas (5.38), (5.39), and (5.40) to the formula (5.37) and take into account the identities (1.1). As a result we obtain

$$\begin{aligned}\sum_{k=1}^{2}\sum_{\bar{s}=1}^{2}\sum_{m=0}^{3} G^{k\bar{s}}_{m}\,L_{\Upsilon_r}(G^{\,m}_{k\bar{s}}) = -2\sum_{k=1}^{2}\sum_{q=1}^{2} L_{\Upsilon_r}(d^{qk})\,d_{kq}\,-\\ -2\sum_{\bar{r}=1}^{2}\sum_{\bar{s}=1}^{2} L_{\Upsilon_r}(\bar{d}^{\bar{r}\bar{s}})\,\bar{d}_{\bar{s}\bar{r}} - \sum_{m=0}^{3}\sum_{n=0}^{3} g^{nm}\,L_{\Upsilon_r}(\,g_{nm}).\end{aligned} \tag{5.41}$$

From (2.4) and (2.8) by applying the differential operator L_{Υ_r} to them we derive

$$\begin{aligned}\sum_{k=1}^{2}\sum_{q=1}^{2} L_{\Upsilon_r}(d^{qk})\,d_{kq} = -\sum_{k=1}^{2}\sum_{q=1}^{2} d^{qk}\,L_{\Upsilon_r}(d_{kq}),\\ \sum_{\bar{r}=1}^{2}\sum_{\bar{s}=1}^{2} L_{\Upsilon_r}(\bar{d}^{\bar{r}\bar{s}})\,\bar{d}_{\bar{s}\bar{r}} = -\sum_{\bar{r}=1}^{2}\sum_{\bar{s}=1}^{2} \bar{d}^{\bar{r}\bar{s}}\,L_{\Upsilon_r}(\bar{d}_{\bar{s}\bar{r}}).\end{aligned} \tag{5.42}$$

Substituting (5.42) into (5.41), we transform (5.41) as follows:

$$\begin{aligned}\sum_{k=1}^{2}\sum_{\bar{s}=1}^{2}\sum_{m=0}^{3} G^{k\bar{s}}_{m}\,L_{\Upsilon_r}(G^{\,m}_{k\bar{s}}) = 2\sum_{u=1}^{2}\sum_{s=1}^{2} L_{\Upsilon_r}(d_{su})\,d^{us}\,+\\ +2\sum_{\bar{u}=1}^{2}\sum_{\bar{u}=1}^{2} L_{\Upsilon_r}(\bar{d}_{\bar{s}\bar{u}})\,\bar{d}^{\bar{u}\bar{s}} - \sum_{m=0}^{3}\sum_{n=0}^{3} g^{nm}\,L_{\Upsilon_r}(\,g_{nm}).\end{aligned} \tag{5.43}$$

Now we substitute (5.36) and (5.43) into (5.35). This yields

$$\begin{aligned}\sum_{k=1}^{2} \mathrm{A}^{i}_{rk}\;G^{k\bar{i}}_{p} + \sum_{\bar{k}=1}^{2}\overline{\mathrm{A}^{\bar{i}}_{r\bar{k}}}\,G^{i\bar{k}}_{p} = \sum_{q=0}^{3}\frac{G^{i\bar{i}}_{q}\,\Gamma^{q}_{rp}}{2}\,+\\ +\sum_{q=0}^{3}\frac{G^{i\bar{i}}_{p}\,\Gamma^{q}_{rq}}{2} - \frac{1}{2}L_{\Upsilon_r}(G^{i\bar{i}}_{p}) - \sum_{m=0}^{3}\sum_{n=0}^{3}\frac{g^{nm}\,L_{\Upsilon_r}(\,g_{nm})}{4}\,G^{i\bar{i}}_{p}\,-\\ -\sum_{k=1}^{2}\sum_{\bar{s}=1}^{2}\sum_{m=0}^{3}\frac{d^{ik}\,\bar{d}^{\bar{i}\bar{s}}}{2}\,\Gamma_{rmp}\,G^{\,m}_{k\bar{s}} - \sum_{k=1}^{2}\sum_{\bar{s}=1}^{2}\sum_{m=0}^{3}\frac{d^{ik}\,\bar{d}^{\bar{i}\bar{s}}\,g_{pm}}{2}\,L_{\Upsilon_r}(G^{\,m}_{k\bar{s}})\,+\\ +\sum_{\bar{u}=1}^{2}\sum_{\bar{s}=1}^{2}\frac{L_{\Upsilon_r}(\bar{d}_{\bar{s}\bar{u}})\,\bar{d}^{\bar{u}\bar{s}}\,G^{i\bar{i}}_{p}}{4} + \sum_{u=1}^{2}\sum_{s=1}^{2}\frac{L_{\Upsilon_r}(d_{su})\,d^{us}\,G^{i\bar{i}}_{p}}{4}.\end{aligned} \tag{5.44}$$

Now let's study the fifth term in the right hand side of the equality (5.44). Applying the identity (5.40) to (5.44), we derive

$$\sum_{k=1}^{2}\sum_{\bar{s}=1}^{2}\sum_{m=0}^{3}\frac{d^{ik}\,\bar{d}^{\bar{i}\bar{s}}}{2}\,\Gamma_{rmp}\,G^{m}_{k\bar{s}}=\sum_{q=0}^{3}\sum_{m=0}^{3}\frac{\Gamma_{rmp}}{2}\,g^{mq}\,G^{i\bar{i}}_{q}. \tag{5.45}$$

In order to transform (5.45) we use the formula (5.25). This yields

$$\begin{gathered}\sum_{k=1}^{2}\sum_{\bar{s}=1}^{2}\sum_{m=0}^{3}\frac{d^{ik}\,\bar{d}^{\bar{i}\bar{s}}}{2}\,\Gamma_{rmp}\,G^{m}_{k\bar{s}}=\\ =\sum_{q=0}^{3}\sum_{m=0}^{3}\frac{g^{qm}\,L_{\Upsilon_r}(g_{pm})}{2}\,G^{i\bar{i}}_{q}-\sum_{q=0}^{3}\frac{\Gamma^{q}_{rp}}{2}\,G^{i\bar{i}}_{q}.\end{gathered} \tag{5.46}$$

Substituting (5.46) back into the formula (5.44), we get the following equality:

$$\begin{gathered}\sum_{k=1}^{2}\mathrm{A}^{i}_{rk}\,G^{k\bar{i}}_{p}+\sum_{\bar{k}=1}^{2}\overline{\mathrm{A}^{\bar{i}}_{r\bar{k}}}\,G^{i\bar{k}}_{p}=\sum_{q=0}^{3}G^{i\bar{i}}_{q}\,\Gamma^{q}_{rp}+\\ +\sum_{q=0}^{3}\frac{G^{i\bar{i}}_{p}\,\Gamma^{q}_{rq}}{2}-\frac{1}{2}L_{\Upsilon_r}(G^{i\bar{i}}_{p})-\sum_{m=0}^{3}\sum_{n=0}^{3}\frac{g^{nm}\,L_{\Upsilon_r}(\,g_{nm})}{4}\,G^{i\bar{i}}_{p}-\\ -\sum_{q=0}^{3}\sum_{m=0}^{3}\frac{g^{qm}\,L_{\Upsilon_r}(g_{pm})}{2}\,G^{i\bar{i}}_{q}-\sum_{k=1}^{2}\sum_{\bar{s}=1}^{2}\sum_{m=0}^{3}\frac{d^{ik}\,\bar{d}^{\bar{i}\bar{s}}\,g_{pm}}{2}\,L_{\Upsilon_r}(G^{m}_{k\bar{s}})+\\ +\sum_{\bar{u}=1}^{2}\sum_{\bar{s}=1}^{2}\frac{L_{\Upsilon_r}(\bar{d}_{\bar{s}\bar{u}})\,\bar{d}^{\bar{u}\bar{s}}\,G^{i\bar{i}}_{p}}{4}+\sum_{u=1}^{2}\sum_{s=1}^{2}\frac{L_{\Upsilon_r}(d_{su})\,d^{us}\,G^{i\bar{i}}_{p}}{4}.\end{gathered} \tag{5.47}$$

Let's proceed to the second term in the right hand side of (5.47). From (5.4) due to the skew symmetry $c^{k}_{ij}=-c^{k}_{ji}$ we derive the formula

$$\sum_{q=0}^{3}\Gamma^{q}_{rq}=\sum_{q=0}^{3}\sum_{m=0}^{3}\frac{g^{qm}\,L_{\Upsilon_r}(g_{qm})}{2}. \tag{5.48}$$

Substituting (5.48) back into (5.47), we find that the second term in the right hand side of this formula cancels the fourth term over there. So we have

$$\begin{gathered}\sum_{k=1}^{2}\mathrm{A}^{i}_{rk}\,G^{k\bar{i}}_{p}+\sum_{\bar{k}=1}^{2}\overline{\mathrm{A}^{\bar{i}}_{r\bar{k}}}\,G^{i\bar{k}}_{p}=\sum_{q=0}^{3}G^{i\bar{i}}_{q}\,\Gamma^{q}_{rp}-\frac{1}{2}L_{\Upsilon_r}(G^{i\bar{i}}_{p})-\\ -\sum_{q=0}^{3}\sum_{m=0}^{3}\frac{g^{qm}\,L_{\Upsilon_r}(g_{pm})}{2}\,G^{i\bar{i}}_{q}-\sum_{k=1}^{2}\sum_{\bar{s}=1}^{2}\sum_{m=0}^{3}\frac{d^{ik}\,\bar{d}^{\bar{i}\bar{s}}\,g_{pm}}{2}\,L_{\Upsilon_r}(G^{m}_{k\bar{s}})+\\ +\sum_{\bar{u}=1}^{2}\sum_{\bar{s}=1}^{2}\frac{L_{\Upsilon_r}(\bar{d}_{\bar{s}\bar{u}})\,\bar{d}^{\bar{u}\bar{s}}\,G^{i\bar{i}}_{p}}{4}+\sum_{u=1}^{2}\sum_{s=1}^{2}\frac{L_{\Upsilon_r}(d_{su})\,d^{us}\,G^{i\bar{i}}_{p}}{4}.\end{gathered} \tag{5.49}$$

The next step is to transform the fourth term in the right hand side of the formula (5.49). For this purpose we recall that $L_{\boldsymbol{\Upsilon}_r}$ is the differential operator (2.15). We apply the Leibniz rule for this operator. As a result we get

$$\begin{aligned}&\sum_{k=1}^{2}\sum_{\bar s=1}^{2}\sum_{m=0}^{3} d^{ik}\,\bar d^{\bar i\bar s}\,g_{pm}\,L_{\boldsymbol{\Upsilon}_r}(G^m_{k\bar s})=\sum_{k=1}^{2}\sum_{\bar s=1}^{2}\sum_{m=0}^{3} L_{\boldsymbol{\Upsilon}_r}(d^{ik}\,\bar d^{\bar i\bar s}\,g_{pm}\,G^m_{k\bar s})-\\&-\sum_{k=1}^{2}\sum_{\bar s=1}^{2}\sum_{m=0}^{3} L_{\boldsymbol{\Upsilon}_r}(d^{ik})\,\bar d^{\bar i\bar s}\,g_{pm}\,G^m_{k\bar s}-\sum_{k=1}^{2}\sum_{\bar s=1}^{2}\sum_{m=0}^{3} d^{ik}\,L_{\boldsymbol{\Upsilon}_r}(\bar d^{\bar i\bar s})\,g_{pm}\,G^m_{k\bar s}-\\&-\sum_{k=1}^{2}\sum_{\bar s=1}^{2}\sum_{m=0}^{3} d^{ik}\,\bar d^{\bar i\bar s}\,L_{\boldsymbol{\Upsilon}_r}(g_{pm})\,G^m_{k\bar s}=L_{\boldsymbol{\Upsilon}_r}(G^{i\bar i}_p)-\sum_{k=1}^{2}\sum_{q=1}^{2} L_{\boldsymbol{\Upsilon}_r}(d^{ik})\,d_{kq}\,\times\\&\times G^{q\bar i}_p-\sum_{\bar s=1}^{2}\sum_{\bar r=1}^{2} L_{\boldsymbol{\Upsilon}_r}(\bar d^{\bar i\bar s})\,\bar d_{\bar s\bar r}\,G^{i\bar r}_p-\sum_{m=0}^{3}\sum_{q=0}^{3} L_{\boldsymbol{\Upsilon}_r}(g_{pm})\,g^{qm}\,G^{i\bar i}_q.\end{aligned}$$

Note that the quantities $L_{\boldsymbol{\Upsilon}_r}(d^{ik})$ and $L_{\boldsymbol{\Upsilon}_r}(\bar d^{\bar i\bar s})$ form two skew-symmetric 2×2 matrices. It is known that any skew-symmetric 2×2 matrix is proportional to any other nonzero skew-symmetric 2×2 matrix. In particular, we can write

$$L_{\boldsymbol{\Upsilon}_r}(d^{ik})=U_r\,d^{ik},\qquad\qquad L_{\boldsymbol{\Upsilon}_r}(\bar d^{\bar i\bar s})=\bar U_r\,\bar d^{\bar i\bar s}.\tag{5.50}$$

The coefficients U_r and $\bar U_r$ in (5.50) can be easily calculated:

$$\begin{aligned}U_r&=\sum_{i=1}^{2}\sum_{k=1}^{2}\frac{L_{\boldsymbol{\Upsilon}_r}(d^{ik})}{2}\,d_{ki}=-\sum_{u=1}^{2}\sum_{s=1}^{2}\frac{L_{\boldsymbol{\Upsilon}_r}(d_{su})\,d^{us}}{2},\\ \bar U_r&=\sum_{\bar i=1}^{2}\sum_{\bar s=1}^{2}\frac{L_{\boldsymbol{\Upsilon}_r}(\bar d^{\bar i\bar s})}{2}\,\bar d_{\bar s\bar i}=-\sum_{\bar u=1}^{2}\sum_{\bar s=1}^{2}\frac{L_{\boldsymbol{\Upsilon}_r}(\bar d_{\bar s\bar u})\,\bar d^{\bar u\bar s}}{2}.\end{aligned}\tag{5.51}$$

Substituting (5.51) back into (5.50) and using (5.50), we obtain

$$\begin{aligned}&\sum_{k=1}^{2}\sum_{\bar s=1}^{2}\sum_{m=0}^{3} d^{ik}\,\bar d^{\bar i\bar s}\,g_{pm}\,L_{\boldsymbol{\Upsilon}_r}(G^m_{k\bar s})=L_{\boldsymbol{\Upsilon}_r}(G^{i\bar i}_p)+\sum_{u=1}^{2}\sum_{s=1}^{2}\frac{L_{\boldsymbol{\Upsilon}_r}(d_{su})\,d^{us}}{2}\times\\&\times G^{i\bar i}_p+\sum_{\bar u=1}^{2}\sum_{\bar s=1}^{2}\frac{L_{\boldsymbol{\Upsilon}_r}(\bar d_{\bar s\bar u})\,\bar d^{\bar u\bar s}}{2}\,G^{i\bar i}_p-\sum_{m=0}^{3}\sum_{q=0}^{3} g^{qm}\,L_{\boldsymbol{\Upsilon}_r}(g_{pm})\,G^{i\bar i}_q.\end{aligned}\tag{5.52}$$

The last step is to substitute (5.52) back into (5.49). As a result we get

$$\sum_{k=1}^{2}\mathrm{A}^i_{rk}\,G^{k\bar i}_p+\sum_{\bar k=1}^{2}\overline{\mathrm{A}^{\bar i}_{r\bar k}}\,G^{i\bar k}_p=\sum_{q=0}^{3}G^{i\bar i}_q\,\Gamma^q_{rp}-L_{\boldsymbol{\Upsilon}_r}(G^{i\bar i}_p).\tag{5.53}$$

Comparing (5.53) with (5.9), we see that these two formulas are equivalent to each other. This means that for the spinor connection with the components (1.3) the second concordance condition (5.7) is also fulfilled. As compared to the paper

[1], in this paper we have verified the formula (1.3) in some other way, though the calculations here are not less huge than those in [1].

References

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