

**A NOTE ON PAIRS OF METRICS
IN A THREE-DIMENSIONAL LINEAR VECTOR SPACE.**

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ABSTRACT. Pairs of metrics in a three-dimensional linear vector space are considered, one of which is a Minkowski type metric with the signature $(+, -, -)$. Such metric pairs are classified and canonical presentations for them in each class are suggested.

1. INTRODUCTION.

In this paper I continue the study of metric pairs initiated in paper [1]. Let V be a three-dimensional linear vector space over the field of real numbers \mathbb{R} . Like in [1] we assume that V is equipped with two metrics \mathbf{g} and $\check{\mathbf{g}}$. The first metric \mathbf{g} is assumed to be a Minkowski type metric, i. e. a metric with the signature $(+, -, -)$. The actual Minkowski metric arises in the four-dimensional space-time of the Special Relativity, while \mathbf{g} is a three-dimensional model of this Minkowski metric.

2. ASSOCIATED LINEAR OPERATORS.

The Minkowski type metric \mathbf{g} with the signature $(+, -, -)$ is non-degenerate. For this reason each other metric $\check{\mathbf{g}}$ in V produces a linear operator $\check{\mathbf{F}}$ associated with it through the metric \mathbf{g} . The operator $\check{\mathbf{F}}$ is defined by the formula

$$g(\check{\mathbf{F}}(\mathbf{X}), \mathbf{Y}) = \check{g}(\mathbf{X}, \mathbf{Y}). \quad (2.1)$$

Due to the symmetry of the form $\check{g}(\mathbf{X}, \mathbf{Y})$ this formula (2.1) is extended to

$$g(\check{\mathbf{F}}(\mathbf{X}), \mathbf{Y}) = \check{g}(\mathbf{X}, \mathbf{Y}) = g(\mathbf{X}, \check{\mathbf{F}}(\mathbf{Y})). \quad (2.2)$$

The formula (2.2) means that the associated operator $\check{\mathbf{F}}$ is symmetric with respect to the metric \mathbf{g} . It is easy to show that $\check{\mathbf{F}}$ is symmetric with respect to $\check{\mathbf{g}}$ as well:

$$\check{g}(\check{\mathbf{F}}(\mathbf{X}), \mathbf{Y}) = g(\check{\mathbf{F}}(\mathbf{X}), \check{\mathbf{F}}(\mathbf{Y})) = \check{g}(\mathbf{X}, \check{\mathbf{F}}(\mathbf{Y})).$$

If the metrics \mathbf{g} and $\check{\mathbf{g}}$ are given by their components g_{ij} and \check{g}_{ij} in some basis $\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2$, then the associated operator $\check{\mathbf{F}}$ is given by the components

$$\check{F}_j^i = \sum_{s=0}^2 g^{is} \check{g}_{sj}. \quad (2.3)$$

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Here g^{ij} are the components of the metric dual for \mathbf{g} . They form the matrix inverse to the matrix formed by the components g_{ij} . Let $P(\lambda)$ be the characteristic polynomial of the operator $\check{\mathbf{F}}$. Then we have

$$P(\lambda) = \det(\check{\mathbf{F}} - \lambda \mathbf{I}) = -\lambda^3 + a_0 \lambda^2 - a_1 \lambda + a_2. \quad (2.4)$$

The coefficients a_0, a_1, a_2 of the characteristic polynomial (2.4) are the invariants of the pair of metrics \mathbf{g} and $\check{\mathbf{g}}$. They are calculated as follows:

$$a_0 = \text{tr } \check{\mathbf{F}}, \quad a_1 = \frac{(\text{tr } \check{\mathbf{F}})^2 - \text{tr } \check{\mathbf{F}}^2}{2}, \quad a_2 = \det \check{\mathbf{F}}. \quad (2.5)$$

A third order polynomial with real coefficients has at least one root in the field of real numbers \mathbb{R} . This real root is an eigenvalue of the associated operator $\check{\mathbf{F}}$. It corresponds to at least one eigenvector $\mathbf{v} \in V$. For this reason we consider the following three mutually exclusive cases:

- (1) there is a simple eigenvalue $\lambda = \lambda_0$ of $\check{\mathbf{F}}$ with a time-like eigenvector \mathbf{v} with respect to the metric \mathbf{g} , i. e. such that $g(\mathbf{v}, \mathbf{v}) > 0$;
- (2) there are no simple eigenvalues with time-like eigenvectors, but there is a simple eigenvalue $\lambda = \lambda_2$ of $\check{\mathbf{F}}$ with a space-like eigenvector \mathbf{v} with respect to the metric \mathbf{g} , i. e. such that $g(\mathbf{v}, \mathbf{v}) < 0$;
- (3) there are no simple eigenvalues of $\check{\mathbf{F}}$ at all.

Saying a **simple eigenvalue** I mean a real root of the polynomial (2.4) whose multiplicity $k = 1$. In the third case, where there no simple eigenvalues, a real root of the polynomial (2.4), which does always exist, is unique and it is a triple eigenvalue with the multiplicity $k = 3$. The subdivision of our consideration into the above three cases is based on the following theorem.

Theorem 2.1. *The associated operator $\check{\mathbf{F}}$ of a metric pair \mathbf{g} and $\check{\mathbf{g}}$ such that \mathbf{g} is a Minkowski type metric with the signature $(+, -, -)$ cannot have a simple eigenvalue with an eigenvector \mathbf{v} such that $g(\mathbf{v}, \mathbf{v}) = 0$.*

Proof. For to prove the theorem 2.1 we consider the complexification $\mathbb{C}V = \mathbb{C} \otimes V$ of the vector space V . It is naturally equipped with the complexifications of the metrics \mathbf{g} and $\check{\mathbf{g}}$. Their associated operator $\check{\mathbf{F}}$ coincides with the complexification of the operator $\check{\mathbf{F}}$ acting in V . The complexified metrics in $\mathbb{C}V$ inherit their signatures from \mathbf{g} and $\check{\mathbf{g}}$ with the only difference — all minuses turn to pluses. In particular, the signature of the complexified metric \mathbf{g} is $(+, +, +)$.

The complex linear vector space $\mathbb{C}V = \mathbb{C} \otimes V$ is naturally equipped with the semilinear involution of complex conjugation:

$$\tau: \mathbb{C}V \rightarrow \mathbb{C}V. \quad (2.6)$$

The space V is embedded into $\mathbb{C}V$ as an \mathbb{R} -linear subspace invariant under the involution (2.6). Assume that the associated operator $\check{\mathbf{F}}$ has a simple eigenvalue $\lambda_0 \in \mathbb{R}$ with an eigenvector $\mathbf{v}_0 \in V \subset \mathbb{C}V$ such that

$$g(\mathbf{v}_0, \mathbf{v}_0) = 0. \quad (2.7)$$

Apart from λ_0 , the operator $\check{\mathbf{F}}$ has two distinct eigenvalues $\lambda_1 \neq \lambda_2$ or one double eigenvalue $\lambda_1 = \lambda_2$. If $\lambda_1 \neq \lambda_2$ we have two extra eigenvectors \mathbf{v}_1 and \mathbf{v}_2 in $\mathbb{C}V$.

The eigenvectors \mathbf{v}_0 , \mathbf{v}_1 , and \mathbf{v}_2 , corresponding to three distinct eigenvalues λ_0 , λ_1 and λ_2 are linearly independent (see [2]). They form a basis in $\mathbb{C}V$. Moreover, the eigenvectors \mathbf{v}_i and \mathbf{v}_j of the associated operator $\check{\mathbf{F}}$ corresponding to $\lambda_i \neq \lambda_j$ are orthogonal with respect to the metric \mathbf{g} . Indeed, from (2.2) we derive

$$\lambda_i g(\mathbf{v}_i, \mathbf{v}_j) = g(\check{\mathbf{F}}(\mathbf{v}_i), \mathbf{v}_j) = g(\mathbf{v}_i, \check{\mathbf{F}}(\mathbf{v}_j)) = \lambda_j g(\mathbf{v}_i, \mathbf{v}_j). \quad (2.8)$$

Since $\lambda_i \neq \lambda_j$, the equality (2.8) yields

$$g(\mathbf{v}_i, \mathbf{v}_j) = 0. \quad (2.9)$$

The formula (2.9) means that the metric \mathbf{g} is diagonalized in the basis formed by the eigenvectors \mathbf{v}_0 , \mathbf{v}_1 , \mathbf{v}_2 , while the formula (2.7) says that it is a degenerate metric with at least one zero in its signature. This result contradicts the initial assumption that the extension of \mathbf{g} to $\mathbb{C}V$ is a metric with the signature $(+, +, +)$.

If $\lambda_1 = \lambda_2$ is a double eigenvalue, we consider the operator $\mathbf{h} = (\check{\mathbf{F}} - \lambda_1 \mathbf{I})^2$. The kernel of this operator $W = \text{Ker } \mathbf{h}$ is the two-dimensional root subspace (see [2]) corresponding to the double eigenvalue $\lambda_1 = \lambda_2$. In this case we have

$$\mathbb{C}V = \langle \mathbf{v}_0 \rangle \oplus W, \quad (2.10)$$

where $\langle \mathbf{v}_0 \rangle$ is the linear span of the eigenvector \mathbf{v}_0 . The subspaces $\langle \mathbf{v}_0 \rangle$ and W in (2.10) are perpendicular to each other. Indeed, let $\mathbf{w} \in W$. Then

$$(\lambda_0 - \lambda_1)^2 g(\mathbf{v}_0, \mathbf{w}) = g(\mathbf{h}(\mathbf{v}_0), \mathbf{w}) = g(\mathbf{v}_0, \mathbf{h}(\mathbf{w})) = 0 \quad (2.11)$$

since $\mathbf{h}(\mathbf{w}) = 0$ by the definition of a root subspace (see [2]). From (2.11), using $\lambda_0 \neq \lambda_1$, we derive $\mathbf{v}_0 \perp W$, while from $\mathbf{v}_0 \perp W$ we conclude that \mathbf{g} should have at least one zero in its signature. Again, this result is a contradiction to the initial assumption that the complexification of \mathbf{g} is a metric with the signature $(+, +, +)$ in $\mathbb{C}V$. The theorem 2.1 is proved. \square

3. THE FIRST CASE.

In the previous section we have divided the study of metric pairs into three mutually exclusive cases. In the **first case** the associated operator $\check{\mathbf{F}}$ has an eigenvalue λ_0 whose eigenvector \mathbf{v}_0 is a time-like vector with respect to the metric \mathbf{g} , i. e. $g(\mathbf{v}_0, \mathbf{v}_0) > 0$. We can normalize this eigenvector so that

$$g(\mathbf{v}_0, \mathbf{v}_0) = 1. \quad (3.1)$$

Let's denote by W the orthogonal complement of \mathbf{v}_0 with respect to the metric \mathbf{g} :

$$W = \{\mathbf{w} \in V : g(\mathbf{v}_0, \mathbf{w}) = 0\}. \quad (3.2)$$

Since \mathbf{g} is non-degenerate, (3.2) is a two-dimensional subspace in V . From (3.1) we derive that W is transversal to the vector \mathbf{v}_0 :

$$V = \langle \mathbf{v}_0 \rangle \oplus W. \quad (3.3)$$

The expansion (3.3) is similar to the expansion (2.10). Using (2.2) and (3.2), we easily prove that the subspace W is perpendicular to \mathbf{v}_0 with respect to the metric $\check{\mathbf{g}}$ as well. Indeed, if $\mathbf{w} \in W$, then we have

$$\check{g}(\mathbf{v}_0, \mathbf{w}) = g(\check{\mathbf{F}}(\mathbf{v}_0), \mathbf{w}) = \lambda_0 g(\mathbf{v}_0, \mathbf{w}) = 0. \quad (3.4)$$

Thus the expansion (3.3) is an expansion of V into a direct sum of two subspaces mutually perpendicular with respect to both metrics \mathbf{g} and $\check{\mathbf{g}}$.

Due to (3.1) the restriction of \mathbf{g} to W is purely negative. Therefore the restrictions of \mathbf{g} and $\check{\mathbf{g}}$ to W can be diagonalized simultaneously. If the vectors \mathbf{v}_1 and \mathbf{v}_2 form a basis of W where both of these restrictions are diagonal, then we can complement them with the vector \mathbf{v}_0 . As a result we get a basis $\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2$ in V such that the metrics \mathbf{g} and $\check{\mathbf{g}}$ are given by the following matrices in this basis:

$$g_{ij} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix}, \quad \check{g}_{ij} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix}. \quad (3.5)$$

Applying the formula (2.3) to the components of the matrices (3.5), we derive that the associated operator $\check{\mathbf{F}}$ is given by the matrix

$$F_j^i = \begin{vmatrix} a & 0 & 0 \\ 0 & -b & 0 \\ 0 & 0 & -c \end{vmatrix} \quad (3.6)$$

in the basis $\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2$. From (3.6) we find that $\lambda_0 = a$. Therefore, we have

$$a \neq -b, \quad a \neq -c. \quad (3.7)$$

Using the matrix (3.6) and taking into account (3.7), we can calculate the characteristic polynomial (2.4) explicitly:

$$P(\lambda) = -(\lambda - a)(\lambda + b)(\lambda + c). \quad (3.8)$$

In general case, apart from (3.7), we have the following inequality:

$$b \neq c. \quad (3.9)$$

In this general case the graph of the polynomial (3.8) intersects the λ -axis at three distinct points as shown on Fig. 3.1. Now we shall derive the condition providing the polynomial (2.4) to have three simple roots. Let's denote by $Q(\lambda)$ the derivative of the

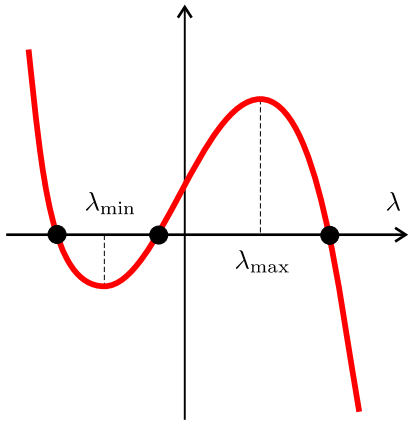


Fig. 3.1

polynomial (2.4): $Q(\lambda) = P'(\lambda)$. Here is the explicit formula for this derivative:

$$Q(\lambda) = -3\lambda^2 + 2a_0\lambda - a_1. \quad (3.10)$$

Let's denote by D_2 the discriminant of the polynomial (3.10). Then we have

$$D_2 = 4(a_0)^2 - 12a_1. \quad (3.11)$$

Using the discriminant (3.11), we calculate λ_{\max} and λ_{\min} on Fig. 3.1:

$$\lambda_{\min} = \frac{a_0}{3} - \frac{\sqrt{D_2}}{6}, \quad \lambda_{\max} = \frac{a_0}{3} + \frac{\sqrt{D_2}}{6}. \quad (3.12)$$

It is clear that the characteristic polynomial (2.4) has three distinct real roots if and only if the following conditions are fulfilled:

$$P(\lambda_{\min}) < 0, \quad P(\lambda_{\max}) > 0. \quad (3.13)$$

Substituting (3.12) into (2.4), by means of direct calculations we find that the inequalities (3.13) are equivalent to the following ones:

$$\begin{aligned} (D_2)^{3/2} &> 8(a_0)^3 - 36a_0a_1 + 108a_2, \\ (D_2)^{3/2} &> -8(a_0)^3 + 36a_0a_1 - 108a_2. \end{aligned} \quad (3.14)$$

The inequalities (3.14) can be united into one inequality:

$$(D_2)^{3/2} > |8(a_0)^3 - 36a_0a_1 + 108a_2|. \quad (3.15)$$

Squaring both sides of (3.15), we get

$$(D_2)^3 > (8(a_0)^3 - 36a_0a_1 + 108a_2)^2. \quad (3.16)$$

In particular, the inequality (3.16) yields $D_2 > 0$. For this reason we did not write $D_2 > 0$ as a separate condition along with (3.13).

Let's substitute (3.11) into (3.16). As a result, the above inequality (3.16) is transformed to the following inequality:

$$-27(a_2)^2 + 18a_0a_1a_2 + (a_1)^2(a_0)^2 - 4(a_0)^3a_2 - 4(a_1)^3 > 0. \quad (3.17)$$

As appears, the left hand side of the inequality (3.17) coincides with the discriminant of the polynomial (2.4) itself, i. e. we have

$$D_3 = -27(a_2)^2 + 18a_0a_1a_2 + (a_1)^2(a_0)^2 - 4(a_0)^3a_2 - 4(a_1)^3. \quad (3.18)$$

Now the inequality (3.17) is written as

$$D_3 > 0. \quad (3.19)$$

Theorem 3.1. *The cubic polynomial (2.4) with the real coefficients a_0, a_1, a_2 has three distinct real roots if and only if its discriminant (3.18) is positive.*

The inequality (3.19) is the only condition that distinguishes the case of three simple roots within the first subcase in the classification scheme suggested in section 2. Indeed, if the associated operator $\tilde{\mathbf{F}}$ has three simple eigenvalues, then we have three linearly independent eigenvectors mutually perpendicular with respect to the metric \mathbf{g} . One of them is certainly a time-like vector, while two others are space-like vectors according to the signature $(+, -, -)$ of the metric \mathbf{g} .

Apart from the general case, there is a special subcase within the first case. In this special subcase the inequality (3.9) is broken and we have the equality

$$b = c. \quad (3.20)$$

Due to (3.20) the graph of the characteristic polynomial (2.4) takes one of two possible shapes shown on Fig. 3.2 and on Fig. 3.3. Due to (3.20) the inequality

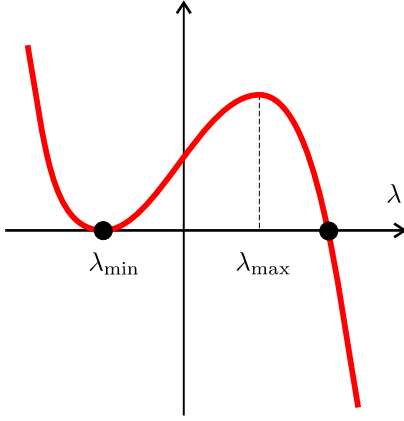


Fig. 3.2

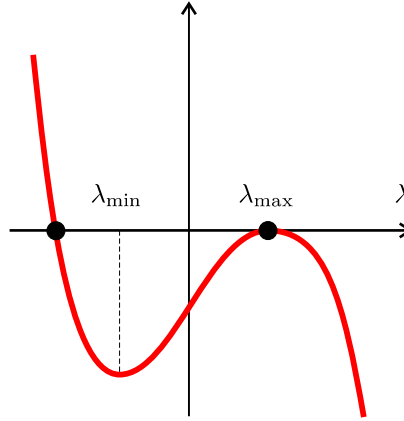


Fig. 3.3

(3.19) for the discriminant D_3 in this special subcase turns to the equality

$$D_3 = 0. \quad (3.21)$$

The inequality (3.15) is replaced by the equality

$$(D_2)^3 = (8(a_0)^3 - 36a_0a_1 + 108a_2)^2, \quad (3.22)$$

which is equivalent to the equality (3.21). The equality (3.22) should be complemented with the inequality for the discriminant D_2 :

$$D_2 > 0. \quad (3.23)$$

The inequality (3.23) follows from the inequalities (3.7).

Note that the equality (3.22) leads to one of the following two equalities resembling the above inequalities (3.14):

$$\begin{aligned} (D_2)^{3/2} &= 8(a_0)^3 - 36a_0a_1 + 108a_2, \\ (D_2)^{3/2} &= -8(a_0)^3 + 36a_0a_1 - 108a_2. \end{aligned} \quad (3.24)$$

The equalities (3.24) are mutually exclusive. If the first equality (3.24) holds, we have $P(\lambda_{\min}) = 0$ and the graph of the polynomial $P(\lambda)$ takes the shape presented on Fig. 3.2. Otherwise, if the second equality (3.24) holds, then $P(\lambda_{\max}) = 0$ and the graph of the polynomial $P(\lambda)$ takes the shape presented on Fig. 3.3. Applying (3.11) to (3.24), we obtain the formulas

$$\begin{aligned}\sqrt{D_2} &= \frac{2(a_0)^3 - 9a_0a_1 + 27a_2}{(a_0)^2 - 3a_1}, \\ \sqrt{D_2} &= -\frac{2(a_0)^3 - 9a_0a_1 + 27a_2}{(a_0)^2 - 3a_1}\end{aligned}\tag{3.25}$$

for these two cases. Substituting (3.25) into (3.12), we derive the formula

$$\lambda_1 = \frac{a_0}{3} - \frac{2(a_0)^3 - 9a_0a_1 + 27a_2}{6(a_0)^2 - 18a_1}.\tag{3.26}$$

Here $\lambda_1 = \lambda_{\min}$ or $\lambda_1 = \lambda_{\max}$ depending on which equality holds in (3.24). The formula (3.26) is a formula for the double root of the polynomial $P(\lambda)$. It is valid provided the conditions (3.21) and (3.23) are fulfilled. Note that

$$a_0 = \text{tr } \check{\mathbf{F}} = \lambda_0 + \lambda_1 + \lambda_2 = \lambda_0 + 2\lambda_1\tag{3.27}$$

since $\lambda_1 = \lambda_2$. From (3.26) and (3.27) we derive

$$\lambda_0 = \frac{a_0}{3} + \frac{2(a_0)^3 - 9a_0a_1 + 27a_2}{3(a_0)^2 - 9a_1}.\tag{3.28}$$

The formula (3.28) is a formula for the simple root of the polynomial $P(\lambda)$. It is also valid provided the conditions (3.21) and (3.23) are fulfilled.

Thus, if $\lambda_1 = \lambda_2 \neq \lambda_0$, the simple eigenvalue λ_0 of the associated operator $\check{\mathbf{F}}$ is expressed through the invariants (2.5) according to the formula (3.28). Let \mathbf{v}_0 be an eigenvector of the operator $\check{\mathbf{F}}$ corresponding to the eigenvalue λ_0 . Let's denote

$$\sigma_0 = \text{sign}(g(\mathbf{v}_0, \mathbf{v}_0)).\tag{3.29}$$

The quantity (3.29) is a special invariant of a pair of metrics \mathbf{g} and $\check{\mathbf{g}}$. According to the theorem 2.1 the invariant σ_0 can take only two possible values $\sigma_0 = -1$ and $\sigma_0 = 1$. In the special subcase of the first case we have

$$\sigma_0 = 1.\tag{3.30}$$

This special subcase in our classification scheme is completely described by the inequality (3.23) and by the equalities (3.21) and (3.30).

4. THE SECOND CASE.

According to our classification scheme in section 2, in the **second case** the associated operator $\check{\mathbf{F}}$ has no simple eigenvalues with time-like eigenvectors, but it has a simple eigenvalue with a space-like eigenvector. Let's denote this eigenvalue

by λ_2 and its space-like eigenvector by \mathbf{v}_2 . We can normalize \mathbf{v}_2 by the condition

$$g(\mathbf{v}_2, \mathbf{v}_2) = -1. \quad (4.1)$$

Let's denote by W the orthogonal complement to \mathbf{v}_2 with respect to the metric \mathbf{g} :

$$W = \{\mathbf{w} \in V : g(\mathbf{v}_2, \mathbf{w}) = 0\}. \quad (4.2)$$

It is clear that W is a two-dimensional subspace in V transversal to the vector \mathbf{v}_2 :

$$V = W \oplus \langle \mathbf{v}_2 \rangle. \quad (4.3)$$

The subspace W in (4.3) is perpendicular to the vector with respect to both metrics \mathbf{g} and $\check{\mathbf{g}}$. The arguments here are the same as in the case of (3.4).

Let's consider the restrictions of the metrics \mathbf{g} and $\check{\mathbf{g}}$ to the subspace (4.2). The restriction of \mathbf{g} to W is a metric with the signature $(+, -)$. For this reason we can apply the results of [1] to the pair of restricted metrics in W . The classification scheme of [1] includes five cases. Only four of them are applicable here. These four cases are those where the restricted metrics \mathbf{g} and $\check{\mathbf{g}}$ cannot be simultaneously diagonalized in W .

The **subcase one** is fixed by the condition that the associated operator of the restricted metrics \mathbf{g} and $\check{\mathbf{g}}$ in W has no real eigenvalues. In terms of the invariants of the associated operator $\check{\mathbf{F}}$ in V this condition is written as the inequality

$$D_3 < 0. \quad (4.4)$$

Theorem 4.1. *The cubic polynomial (2.4) with the real coefficients a_0, a_1, a_2 has three distinct roots one of which is real and two others are complex numbers if and only if its discriminant (3.18) is negative.*

Under the condition (4.4) the matrices of the metrics \mathbf{g} and $\check{\mathbf{g}}$ are brought to

$$g_{ij} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix}, \quad \check{g}_{ij} = \begin{vmatrix} a & b & 0 \\ b & -a & 0 \\ 0 & 0 & c \end{vmatrix}. \quad (4.5)$$

Theorem 4.2. *If the inequality (4.4) is fulfilled, then the matrices of the metrics \mathbf{g} and $\check{\mathbf{g}}$ take their canonical forms (4.5) with $b \neq 0$ in some basis.*

The **subcase two** is fixed by the condition that the associated operator of the restricted metrics \mathbf{g} and $\check{\mathbf{g}}$ in W has a double eigenvalue, which is a real number, and the σ -invariant of the restricted metrics is equal to zero (see [1]). Let's denote by σ_1 the σ -invariant of the restricted metrics in W . Then we have

$$D_3 = 0, \quad D_2 \neq 0, \quad \sigma_0 = -1, \quad \sigma_1 = 0. \quad (4.6)$$

The condition $\sigma_0 = -1$ in (4.6) is derived from (4.1). Under the conditions (4.6) the matrices of the metrics \mathbf{g} and $\check{\mathbf{g}}$ are brought to

$$g_{ij} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix}, \quad \check{g}_{ij} = \begin{vmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & c \end{vmatrix}. \quad (4.7)$$

Theorem 4.3. *If the conditions (4.6) are fulfilled, then the matrices of the metrics \mathbf{g} and $\check{\mathbf{g}}$ take their canonical forms (4.7) with $a \neq -c$ in some basis.*

The **subcase three** differs from the subcase two only by the value of the invariant σ_1 . It is fixed by the following conditions:

$$D_3 = 0, \quad D_2 \neq 0, \quad \sigma_0 = -1, \quad \sigma_1 = 1. \quad (4.8)$$

Under the condition (4.8) the matrices of the metrics \mathbf{g} and $\check{\mathbf{g}}$ are brought to

$$g_{ij} = \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{vmatrix}, \quad \check{g}_{ij} = \begin{vmatrix} 1 & a & 0 \\ a & 0 & 0 \\ 0 & 0 & c \end{vmatrix}. \quad (4.9)$$

Theorem 4.4. *If the conditions (4.8) are fulfilled, then the matrices of the metrics \mathbf{g} and $\check{\mathbf{g}}$ take their canonical forms (4.9) with $a \neq -c$ in some basis.*

The **subcase four** also differs from the subcase two only by the value of the invariant σ_1 . It is fixed by the following conditions:

$$D_3 = 0, \quad D_2 \neq 0, \quad \sigma_0 = -1, \quad \sigma_1 = -1. \quad (4.10)$$

Under the condition (4.10) the matrices of the metrics \mathbf{g} and $\check{\mathbf{g}}$ are brought to

$$g_{ij} = \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{vmatrix}, \quad \check{g}_{ij} = \begin{vmatrix} 0 & a & 0 \\ a & -1 & 0 \\ 0 & 0 & c \end{vmatrix}. \quad (4.11)$$

Theorem 4.5. *If the conditions (4.10) are fulfilled, then the matrices of the metrics \mathbf{g} and $\check{\mathbf{g}}$ take their canonical forms (4.11) with $a \neq -c$ in some basis.*

5. THE THIRD CASE.

In the **third case** of our classification scheme from the section 2 the associated operator $\check{\mathbf{F}}$ has exactly one real eigenvalue λ_0 of the multiplicity $k = 3$. This case is specified by the following equalities:

$$D_3 = 0, \quad D_2 = 0. \quad (5.1)$$

The unique eigenvalue λ_0 in the third case is expressed through the invariant a_0 :

$$\lambda_0 = \frac{a_0}{3}. \quad (5.2)$$

The formula (5.2) is analogous to the formulas (3.26) and (3.28). In the third case we define the following invariant of the associated operator:

$$\sigma_2 = \text{rank}(\check{\mathbf{F}} - \lambda_0 \mathbf{I}). \quad (5.3)$$

The **subcase one** within the third case is determined by the following condition for the value of the integer invariant σ_2 in (5.3):

$$\sigma_2 = 0. \quad (5.4)$$

Combining the conditions (5.1) with the condition (5.4), we get

$$D_3 = 0, \quad D_2 = 0, \quad \sigma_2 = 0. \quad (5.5)$$

Under the condition (5.4) the associated operator $\check{\mathbf{F}}$ is a scalar operator:

$$\check{\mathbf{F}} = \lambda_0 \mathbf{I}. \quad (5.6)$$

Due to (5.6), diagonalizing the metric \mathbf{g} , we simultaneously diagonalize the second metric $\check{\mathbf{g}}$. For this reason the matrices of these metrics are brought to

$$g_{ij} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix}, \quad \check{g}_{ij} = \begin{vmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{vmatrix}, \quad (5.7)$$

where $a = \lambda_0$. This result is formulated as a theorem.

Theorem 5.1. *If the conditions (5.5) are fulfilled, then the matrices of the metrics \mathbf{g} and $\check{\mathbf{g}}$ take their canonical forms (5.7) in some basis.*

Now, keeping the conditions (5.1) unchanged, we increase by one the value of the integer invariant σ_2 in (5.4). As a result we get

$$\sigma_2 = 1. \quad (5.8)$$

If the equality (5.8) is fulfilled, we have the following two subspaces in V :

$$W = \text{Im}(\check{\mathbf{F}} - \lambda_0 I), \quad U = \text{Ker}(\check{\mathbf{F}} - \lambda_0 I). \quad (5.9)$$

For the dimensions of the subspaces (5.9) we have

$$\dim W = 1, \quad \dim U = 2. \quad (5.10)$$

Moreover, in addition to (5.10) we have the following inclusions:

$$W \subset U \subset V. \quad (5.11)$$

Let's choose some nonzero vector $\mathbf{e}_0 \in W$. Due to (5.10) this vector is unique up to some nonzero numeric factor. Since $\mathbf{e}_0 \in W$ and $W \subset U$, from (5.9) we derive

$$\check{\mathbf{F}}(\mathbf{e}_0) = \lambda_0 \mathbf{e}_0. \quad (5.12)$$

The equality (5.12) means that \mathbf{e}_0 is an eigenvector of the associated operator $\check{\mathbf{F}}$ corresponding to its eigenvalue (5.2). By the definition of the subspace W in (5.9) there is another nonzero vector \mathbf{e}_1 such that

$$\mathbf{e}_0 = \check{\mathbf{F}}(\mathbf{e}_1) - \lambda_0 \mathbf{e}_1. \quad (5.13)$$

Apart from \mathbf{e}_0 and \mathbf{e}_1 , we choose some nonzero vector $\mathbf{e}_2 \in U$ such that $\mathbf{e}_0, \mathbf{e}_2$ is a basis of the two-dimensional subspace U . Since $\mathbf{e}_2 \in U$, it is another eigenvector of the operator $\check{\mathbf{F}}$. Indeed, from (5.9) we derive

$$\check{\mathbf{F}}(\mathbf{e}_2) = \lambda_0 \mathbf{e}_2. \quad (5.14)$$

Since $\mathbf{e}_0 \neq 0$, from (5.13) we conclude that $\mathbf{e}_1 \notin U$. Then from (5.11) we derive that the vectors $\mathbf{e}_0, \mathbf{e}_1$, and \mathbf{e}_2 are linearly independent. They form a basis in V . Let's study the components of the metric \mathbf{g} in this basis. Applying the formulas (5.12), (5.13), and (2.2), we derive

$$\begin{aligned} g_{00} &= g(\mathbf{e}_0, \mathbf{e}_0) = g(\mathbf{e}_0, \check{\mathbf{F}}(\mathbf{e}_1)) - \lambda_0 g(\mathbf{e}_0, \mathbf{e}_1) = \\ &= g(\check{\mathbf{F}}(\mathbf{e}_0), \mathbf{e}_1) - \lambda_0 g(\mathbf{e}_0, \mathbf{e}_1) = \lambda_0 g(\mathbf{e}_0, \mathbf{e}_1) - \lambda_0 g(\mathbf{e}_0, \mathbf{e}_1) = 0. \end{aligned} \quad (5.15)$$

Similarly, applying the formulas (5.14), (5.13), and (2.2), we derive

$$\begin{aligned} g_{02} &= g_{20} = g(\mathbf{e}_2, \mathbf{e}_0) = g(\mathbf{e}_2, \check{\mathbf{F}}(\mathbf{e}_1)) - \lambda_0 g(\mathbf{e}_2, \mathbf{e}_1) = \\ &= g(\check{\mathbf{F}}(\mathbf{e}_2), \mathbf{e}_1) - \lambda_0 g(\mathbf{e}_2, \mathbf{e}_1) = \lambda_0 g(\mathbf{e}_2, \mathbf{e}_1) - \lambda_0 g(\mathbf{e}_2, \mathbf{e}_1) = 0. \end{aligned} \quad (5.16)$$

According to the formulas (5.15) and (5.16) the matrix of the metric \mathbf{g} in the basis $\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2$ takes the following form:

$$g_{ij} = \begin{vmatrix} 0 & g_{01} & 0 \\ g_{01} & g_{11} & g_{12} \\ 0 & g_{12} & g_{22} \end{vmatrix}. \quad (5.17)$$

Note that $g_{22} \neq 0$ and $g_{01} \neq 0$ in (5.17). Indeed, otherwise the metric \mathbf{g} would be degenerate. Note also that the choice of the vectors $\mathbf{e}_0, \mathbf{e}_1$, and \mathbf{e}_2 is not unique. We can perform the following transformations of these vectors:

$$\begin{aligned} \mathbf{e}_0 &\rightarrow \alpha \mathbf{e}_0, \\ \mathbf{e}_1 &\rightarrow \beta \mathbf{e}_0 + \alpha \mathbf{e}_1 + \gamma \mathbf{e}_2, \\ \mathbf{e}_2 &\rightarrow \theta \mathbf{e}_0 + \delta \mathbf{e}_2. \end{aligned} \quad (5.18)$$

The basis transformations of the form (5.18) preserve the relationships (5.12), (5.13), and (5.14). Hence, the zero components of the matrix (5.17) remain zero under these transformations. For the beginning we choose the special transformation of the form (5.18) by setting

$$\alpha = 1, \quad \delta = 1, \quad \gamma = 0, \quad \theta = -\frac{g_{12}}{g_{01}}, \quad \beta = -\frac{g_{11}}{2g_{01}} \quad (5.19)$$

in (5.18). Applying the basis transformations (5.18) with the parameters (5.19), we find that the components g_{11} and g_{12} in (5.17) turn to zero. As a result the matrix

of the metric \mathbf{g} takes the following form:

$$g_{ij} = \begin{vmatrix} 0 & g_{01} & 0 \\ g_{01} & 0 & 0 \\ 0 & 0 & g_{22} \end{vmatrix}. \quad (5.20)$$

As we already noted, $g_{22} \neq 0$. If $g_{22} > 0$, then due to (5.20) the signature of the metric \mathbf{g} would be $(+, +, -)$. But, actually, the signature of \mathbf{g} is $(+, -, -)$. For this reason we conclude that $g_{22} < 0$. The component g_{01} is also nonzero. It can be either positive or negative. The sign of the component g_{01} in (5.20) is a new invariant of a pair of forms in the third case. We denote it σ_3 :

$$\sigma_3 = \text{sign}(g_{01}). \quad (5.21)$$

The invariant (5.21) is analogous to other integer invariants σ_0 , σ_1 , and σ_2 considered above. Depending on the value of this invariant we specify the subcase two and the subcase three within the third case.

The **subcase two** within the third case of our classification scheme is fixed by the following values of the invariants D_3 , D_2 , σ_2 and σ_3 :

$$D_3 = 0, \quad D_2 = 0, \quad \sigma_2 = 1, \quad \sigma_3 = 1. \quad (5.22)$$

If the conditions (5.22) hold, the matrix of the metric \mathbf{g} can be brought to

$$g_{ij} = \begin{vmatrix} 0 & b^2 & 0 \\ b^2 & 0 & 0 \\ 0 & 0 & -c^2 \end{vmatrix}. \quad (5.23)$$

The matrix (5.23) is not an ultimate canonical presentation for the metric \mathbf{g} . In order to simplify (5.23) we apply the following basis transformation:

$$\begin{aligned} \mathbf{e}_0 &\rightarrow \frac{1}{b} \mathbf{e}_1, \\ \mathbf{e}_1 &\rightarrow \frac{1}{b} \mathbf{e}_0, \\ \mathbf{e}_2 &\rightarrow \frac{1}{c} \mathbf{e}_2. \end{aligned} \quad (5.24)$$

Upon applying the basis transformation (5.24) we find that the matrices of the metrics \mathbf{g} and $\check{\mathbf{g}}$ take the following canonical forms, where $a = \lambda_0$:

$$g_{ij} = \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{vmatrix}, \quad \check{g}_{ij} = \begin{vmatrix} 1 & a & 0 \\ a & 0 & 0 \\ 0 & 0 & -a \end{vmatrix}. \quad (5.25)$$

Theorem 5.2. *If the conditions (5.22) are fulfilled, then the matrices of the metrics \mathbf{g} and $\check{\mathbf{g}}$ take their canonical forms (5.25) in some basis.*

The **subcase three** differs from the subcase two by the value of the invariant σ_3 . Instead of the equalities (5.22), here we have

$$D_3 = 0, \quad D_2 = 0, \quad \sigma_2 = 1, \quad \sigma_3 = -1. \quad (5.26)$$

Under the conditions (5.26) the matrix (5.20) specifies to

$$g_{ij} = \left\| \begin{array}{ccc} 0 & -b^2 & 0 \\ -b^2 & 0 & 0 \\ 0 & 0 & -c^2 \end{array} \right\|. \quad (5.27)$$

In order to simplify (5.27) we apply the following basis transformation:

$$\begin{aligned} \mathbf{e}_0 &\rightarrow -\frac{1}{b} \mathbf{e}_0, \\ \mathbf{e}_1 &\rightarrow \frac{1}{b} \mathbf{e}_1, \\ \mathbf{e}_2 &\rightarrow \frac{1}{c} \mathbf{e}_2. \end{aligned} \quad (5.28)$$

Upon applying (5.28) we find that the matrices of the metric \mathbf{g} and $\check{\mathbf{g}}$ take the following canonical forms, where $a = \lambda_0$:

$$g_{ij} = \left\| \begin{array}{ccc} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{array} \right\|, \quad \check{g}_{ij} = \left\| \begin{array}{ccc} 0 & a & 0 \\ a & -1 & 0 \\ 0 & 0 & -a \end{array} \right\|. \quad (5.29)$$

Theorem 5.3. *If the conditions (5.26) are fulfilled, then the matrices of the metrics \mathbf{g} and $\check{\mathbf{g}}$ take their canonical forms (5.29) in some basis.*

Now, again keeping the conditions (5.1) unchanged, we increase by one the value of the integer invariant σ_2 in (5.8). As a result we get

$$\sigma_2 = 2. \quad (5.30)$$

Combining (5.1) with (5.30), we write

$$D_3 = 0, \quad D_2 = 0, \quad \sigma_2 = 2. \quad (5.31)$$

The **subcase four** within the third case of our classification scheme is specified by the conditions (5.31). If the equalities (5.31) hold, we have the following subspaces:

$$\begin{aligned} W &= \text{Im}((\check{\mathbf{F}} - \lambda_0 I)^2) = \text{Ker}(\check{\mathbf{F}} - \lambda_0 I), \\ U &= \text{Ker}((\check{\mathbf{F}} - \lambda_0 I)^2) = \text{Im}(\check{\mathbf{F}} - \lambda_0 I). \end{aligned} \quad (5.32)$$

The dimensions of these subspaces (5.32) are also given by the formulas (5.10). Let's choose some vector \mathbf{e}_2 in V such that $\mathbf{e}_2 \notin U$. It is clear that $\mathbf{e}_2 \neq 0$. Then we define the vector \mathbf{e}_1 by means of the formula

$$\mathbf{e}_1 = \check{\mathbf{F}}(\mathbf{e}_2) - \lambda_0 \mathbf{e}_2. \quad (5.33)$$

Due to (5.32) from (5.33) we derive that $\mathbf{e}_1 \in U$, $\mathbf{e}_1 \neq 0$, and $\mathbf{e}_1 \notin W$. Therefore we can define the third vector \mathbf{e}_0 by means of the formula

$$\mathbf{e}_0 = \check{\mathbf{F}}(\mathbf{e}_1) - \lambda_0 \mathbf{e}_1. \quad (5.34)$$

The vector \mathbf{e}_0 belongs to the subspace W . Since $\mathbf{e}_1 \notin W$, the vector \mathbf{e}_0 is nonzero. From $\mathbf{e}_0 \in W$ we derive that \mathbf{e}_0 is an eigenvector of the associated operator $\check{\mathbf{F}}$ corresponding to the eigenvalue λ_0 in (5.2):

$$\check{\mathbf{F}}(\mathbf{e}_0) = \lambda_0 \mathbf{e}_0. \quad (5.35)$$

The vectors \mathbf{e}_0 , \mathbf{e}_1 , and \mathbf{e}_2 are linearly independent. They form a **Jordan normal basis** for the operator $\check{\mathbf{F}}$. Let's study the components of the metric \mathbf{g} in this basis. Applying the formulas (5.34), (5.35), and (2.2), we derive

$$\begin{aligned} g_{00} &= g(\mathbf{e}_0, \mathbf{e}_0) = g(\mathbf{e}_0, \check{\mathbf{F}}(\mathbf{e}_1)) - \lambda_0 g(\mathbf{e}_0, \mathbf{e}_1) = \\ &= g(\check{\mathbf{F}}(\mathbf{e}_0), \mathbf{e}_1) - \lambda_0 g(\mathbf{e}_0, \mathbf{e}_1) = \lambda_0 g(\mathbf{e}_0, \mathbf{e}_1) - \lambda_0 g(\mathbf{e}_0, \mathbf{e}_1) = 0. \end{aligned} \quad (5.36)$$

Similarly, applying the formulas (5.33), (5.35), and (2.2), we derive

$$\begin{aligned} g_{10} &= g_{01} = g(\mathbf{e}_0, \mathbf{e}_1) = g(\mathbf{e}_0, \check{\mathbf{F}}(\mathbf{e}_2)) - \lambda_0 g(\mathbf{e}_0, \mathbf{e}_2) = \\ &= g(\check{\mathbf{F}}(\mathbf{e}_0), \mathbf{e}_2) - \lambda_0 g(\mathbf{e}_0, \mathbf{e}_2) = \lambda_0 g(\mathbf{e}_0, \mathbf{e}_2) - \lambda_0 g(\mathbf{e}_0, \mathbf{e}_2) = 0. \end{aligned} \quad (5.37)$$

And finally, applying the formulas (5.33), (5.34), and (2.2), we derive

$$\begin{aligned} g_{11} &= g(\mathbf{e}_1, \mathbf{e}_1) = g(\mathbf{e}_1, \check{\mathbf{F}}(\mathbf{e}_2)) - \lambda_0 g(\mathbf{e}_1, \mathbf{e}_2) = \\ &= g(\check{\mathbf{F}}(\mathbf{e}_1), \mathbf{e}_2) - \lambda_0 g(\mathbf{e}_1, \mathbf{e}_2) = \lambda_0 g(\mathbf{e}_1, \mathbf{e}_2) + \\ &+ g(\mathbf{e}_0, \mathbf{e}_2) - \lambda_0 g(\mathbf{e}_0, \mathbf{e}_2) = g(\mathbf{e}_0, \mathbf{e}_2) = g_{02} = g_{20}. \end{aligned} \quad (5.38)$$

Due to the formulas (5.36), (5.37), and (5.38), the matrix of the metric \mathbf{g} in the basis $\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2$ takes the following form:

$$g_{ij} = \begin{vmatrix} 0 & 0 & g_{11} \\ 0 & g_{11} & g_{12} \\ g_{11} & g_{12} & g_{22} \end{vmatrix}. \quad (5.39)$$

The component g_{11} in the matrix (5.39) is nonzero since otherwise the metric \mathbf{g} would be degenerate. Note that the Jordan normal basis $\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2$ of the associated

operator $\check{\mathbf{F}}$ is not unique. Applying the transformation

$$\begin{aligned}\mathbf{e}_0 &\rightarrow \alpha \mathbf{e}_0, \\ \mathbf{e}_1 &\rightarrow \beta \mathbf{e}_0 + \alpha \mathbf{e}_1, \\ \mathbf{e}_2 &\rightarrow \gamma \mathbf{e}_0 + \beta \mathbf{e}_1 + \alpha \mathbf{e}_2,\end{aligned}\tag{5.40}$$

we get another Jordan normal basis, where the metric \mathbf{g} is presented by another matrix of the form (5.39). Since $g_{11} \neq 0$ we can set

$$\alpha = 1, \quad \beta = -\frac{g_{12}}{g_{11}}, \quad \gamma = -\frac{g_{22}}{2g_{11}} + \frac{3(g_{12})^2}{8(g_{11})^2}.\tag{5.41}$$

Applying (5.41) to (5.40), we find that the metric \mathbf{g} is given by the following skew-diagonal matrix in the new Jordan normal basis of the operator $\check{\mathbf{F}}$:

$$g_{ij} = \begin{vmatrix} 0 & 0 & g_{11} \\ 0 & g_{11} & 0 \\ g_{11} & 0 & 0 \end{vmatrix}.\tag{5.42}$$

The component g_{11} in (5.42) is negative since otherwise, if $g_{11} > 0$, the signature of the metric \mathbf{g} would be $(+, +, -)$, while actually it is $(+, -, -)$. Therefore, we can substitute $g_{11} = -b^2$ into (5.42). As a result we get the matrix

$$g_{ij} = \begin{vmatrix} 0 & 0 & -b^2 \\ 0 & -b^2 & 0 \\ -b^2 & 0 & 0 \end{vmatrix},\tag{5.43}$$

where $b \neq 0$. The matrix (5.43) is an intermediate result. In order to get the ultimate result we consider the following basis transformation:

$$\begin{aligned}\mathbf{e}_0 &\rightarrow -\frac{1}{b} \mathbf{e}_0, \\ \mathbf{e}_1 &\rightarrow \frac{1}{b} \mathbf{e}_2, \\ \mathbf{e}_2 &\rightarrow -\frac{1}{b} \mathbf{e}_1.\end{aligned}\tag{5.44}$$

Upon applying the basis transformation (5.44) we find that the matrices of the metrics \mathbf{g} and $\check{\mathbf{g}}$ take their canonical forms

$$g_{ij} = \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{vmatrix}, \quad \check{g}_{ij} = \begin{vmatrix} 0 & a & 0 \\ a & 0 & 1 \\ 0 & 1 & -a \end{vmatrix}.\tag{5.45}$$

Theorem 5.4. *If the conditions (5.31) are fulfilled, then the matrices of the metrics \mathbf{g} and $\check{\mathbf{g}}$ take their canonical forms (5.45) in some basis.*

6. CLASSIFICATION.

Thus, all possible cases are exhausted. The total number of them is ten. All of these ten cases are given in the following three tables.

Condition	Canonical presentation
$D_3 > 0$	$g_{ij} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix}, \quad \check{g}_{ij} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix},$ <p>where $a \neq -b$, $a \neq -c$, and $b \neq c$</p>
$D_3 = 0,$ $D_2 > 0,$ $\sigma_0 = 1$	$g_{ij} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix}, \quad \check{g}_{ij} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b \end{vmatrix},$ <p>where $a \neq -b$</p>

(6.1)

Condition	Canonical presentation
$D_3 < 0$	$g_{ij} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix}, \quad \check{g}_{ij} = \begin{vmatrix} a & b & 0 \\ b & -a & 0 \\ 0 & 0 & c \end{vmatrix},$ <p>where $b \neq 0$</p>
$D_3 = 0,$ $D_2 \neq 0,$ $\sigma_0 = -1,$ $\sigma_1 = 0$	$g_{ij} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix}, \quad \check{g}_{ij} = \begin{vmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & c \end{vmatrix},$ <p>where $a \neq -c$</p>
$D_3 = 0,$ $D_2 \neq 0,$ $\sigma_0 = -1,$ $\sigma_1 = 1$	$g_{ij} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix}, \quad \check{g}_{ij} = \begin{vmatrix} 1 & a & 0 \\ a & 0 & 0 \\ 0 & 0 & c \end{vmatrix},$ <p>where $a \neq -c$</p>
$D_3 = 0,$ $D_2 \neq 0,$ $\sigma_0 = -1,$ $\sigma_1 = -1$	$g_{ij} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix}, \quad \check{g}_{ij} = \begin{vmatrix} 0 & a & 0 \\ a & -1 & 0 \\ 0 & 0 & c \end{vmatrix},$ <p>where $a \neq -c$</p>

(6.2)

Condition	Canonical presentation
$D_3 = 0,$ $D_2 = 0,$ $\sigma_2 = 0$	$g_{ij} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix}, \quad \check{g}_{ij} = \begin{vmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{vmatrix}$
$D_3 = 0,$ $D_2 = 0,$ $\sigma_2 = 1,$ $\sigma_3 = 1$	$g_{ij} = \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{vmatrix}, \quad \check{g}_{ij} = \begin{vmatrix} 1 & a & 0 \\ a & 0 & 0 \\ 0 & 0 & -a \end{vmatrix}$
$D_3 = 0,$ $D_2 = 0,$ $\sigma_2 = 1,$ $\sigma_3 = -1$	$g_{ij} = \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{vmatrix}, \quad \check{g}_{ij} = \begin{vmatrix} 0 & a & 0 \\ a & -1 & 0 \\ 0 & 0 & -a \end{vmatrix}$
$D_3 = 0,$ $D_2 = 0,$ $\sigma_2 = 2$	$g_{ij} = \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{vmatrix}, \quad \check{g}_{ij} = \begin{vmatrix} 0 & a & 0 \\ a & 0 & 1 \\ 0 & 1 & -a \end{vmatrix}$

(6.3)

Apart from the discriminants D_3 and D_2 , in the first columns of the tables (6.1), (6.2), and (6.3) we see some special invariants σ_0 , σ_1 , σ_2 , and σ_3 . Currently I don't know any explicit formulas for expressing these invariants through the components of the metrics \mathbf{g} and $\check{\mathbf{g}}$ in an arbitrary basis. However, I expect that such formulas can be derived.

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