A NOTE ON PAIRS OF METRICS IN A TWO-DIMENSIONAL LINEAR VECTOR SPACE.

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Abstract. Pairs of metrics in a two-dimensional linear vector space are considered, one of which is a Minkowski type metric. Their simultaneous diagonalizability is studied and canonical presentations for them are suggested.

1. Introduction.

Let V be a two-dimensional linear vector space equipped with two metrics $\mathbf{g}$ and $\check{\mathbf{g}}$. It is well known (see [1]) that if $\mathbf{g}$ is positive, then these two metrics can be diagonalized simultaneously in some basis. Here we consider a different case where $\mathbf{g}$ is a metric of the signature $(+,-)$. A two-dimensional space with such a metric is often used as a two-dimensional model of the four-dimensional Minkowski space. For this reason we call $\mathbf{g}$ a Minkowski type metric.

2. Lorentz transformations and diagonalizability.

Each single metric can be diagonalized. This fact means that there is some basis $\mathbf{e}_0$, $\mathbf{e}_1$ in V such that the metric $\mathbf{g}$ is given by the diagonal matrix g_{ij} with the numbers 1 and -1 on its diagonal:

$$g_{ij}=\begin{Vmatrix}1 & 0\\ 0 & -1\end{Vmatrix}, \qquad\qquad \check{g}_{ij}=\begin{Vmatrix}\check{g}_{00} & \check{g}_{01}\\ \check{g}_{01} & \check{g}_{11}\end{Vmatrix}. \tag{2.1}$$

The second metric is not necessarily diagonal in this basis. In order to diagonalize it we perform the following Lorentz transformation of the basis $\mathbf{e}_0$, $\mathbf{e}_1$:

$$\begin{aligned}\tilde{\mathbf{e}}_0&=\cosh(\phi)\,\mathbf{e}_0+\sinh(\phi)\,\mathbf{e}_1,\\ \tilde{\mathbf{e}}_1&=\sinh(\phi)\,\mathbf{e}_0+\cosh(\phi)\,\mathbf{e}_1.\end{aligned}\tag{2.2}$$

Under the basis transformation (2.2) the matrices (2.1) are transformed according to the standard tensorial rule:

$$\tilde{g}_{ij}=\sum_{p=0}^{1}\sum_{q=0}^{1}S^p_i\,S^q_j\,g_{pq}, \qquad\qquad \tilde{\check{g}}_{ij}=\sum_{p=0}^{1}\sum_{q=0}^{1}S^p_i\,S^q_j\,\check{g}_{pq}, \tag{2.3}$$

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where the components of the transition matrix S are determined by (2.2):

$$S=\begin{Vmatrix}\cosh(\phi) & \sinh(\phi)\\ \sinh(\phi) & \cosh(\phi)\end{Vmatrix}. \tag{2.4}$$

Substituting (2.4) into (2.3), one easily finds that the first matrix (2.1) is invariant under the Lorentz transformation (2.2), i. e. $\tilde{g}_{ij}=g_{ij}$. For the non-diagonal matrix element of the second metric $\breve{\mathbf{g}}$ in the new basis $\tilde{\mathbf{e}}_0$, $\tilde{\mathbf{e}}_1$ we have

$$\tilde{\breve{g}}_{01}=\frac{\breve{g}_{00}+\breve{g}_{11}}{2}\,\sinh(2\,\phi)+\breve{g}_{01}\,\cosh(2\,\phi). \tag{2.5}$$

The metric $\breve{\mathbf{g}}$ is diagonalized simultaneously with the metric $\mathbf{g}$ if the equation $\tilde{\breve{g}}_{01}=0$ can be solved with respect to ϕ. Due to the above formula (2.5) the equation $\tilde{\breve{g}}_{01}=0$ is equivalent to

$$(\breve{g}_{00}+\breve{g}_{11})\,\tanh(2\,\phi)=-2\,\breve{g}_{01}. \tag{2.6}$$

Looking at (2.6), we define the following four mutually exclusive cases:

The first case	$\breve{g}_{00}+\breve{g}_{11}=0$ and $\breve{g}_{01}=0$	
The second case	$\breve{g}_{00}+\breve{g}_{11}\neq 0$ and $\breve{g}_{01}=0$	(2.7)
The third case	$\breve{g}_{00}+\breve{g}_{11}=0$ and $\breve{g}_{01}\neq 0$	
The fourth case	$\breve{g}_{00}+\breve{g}_{11}\neq 0$ and $\breve{g}_{01}\neq 0$	

In the **first case** $\breve{g}_{01}=0$. Therefore the second metric $\breve{\mathbf{g}}$ is diagonal in the initial basis $\mathbf{e}_0$, $\mathbf{e}_1$. Moreover, $\breve{g}_{11}=-\breve{g}_{00}$. If we denote $\breve{g}_{00}=a$, then (2.1) is written as

$$g_{ij}=\begin{Vmatrix}1 & 0\\ 0 & -1\end{Vmatrix},\qquad\qquad \breve{g}_{ij}=\begin{Vmatrix}a & 0\\ 0 & -a\end{Vmatrix}. \tag{2.8}$$

As we see in (2.8), the metrics $\mathbf{g}$ and $\breve{\mathbf{g}}$ in the first case do coincide up to the numeric factor a, i. e. $\breve{\mathbf{g}}=a\,\mathbf{g}$. They are always diagonalized simultaneously.

In the **second case** $\breve{g}_{01}=0$ too. Both metrics $\mathbf{g}$ and $\breve{\mathbf{g}}$ are diagonal simultaneously in the initial basis $\mathbf{e}_0$, $\mathbf{e}_1$. If we denote $\breve{g}_{00}=a$ and $\breve{g}_{11}=b$, we get

$$g_{ij}=\begin{Vmatrix}1 & 0\\ 0 & -1\end{Vmatrix},\qquad\qquad \breve{g}_{ij}=\begin{Vmatrix}a & 0\\ 0 & b\end{Vmatrix}. \tag{2.9}$$

The equation (2.6) in this case is solvable and its solution is $\phi=0$. This means that (2.2) is the identical transformation where $\tilde{\mathbf{e}}_0=\mathbf{e}_0$ and $\tilde{\mathbf{e}}_1=\mathbf{e}_1$.

In the **third case** $\breve{g}_{01}\neq 0$, i. e. the second metric $\breve{\mathbf{g}}$ is not diagonal, but we have the relationship $\breve{g}_{11}=-\breve{g}_{00}$. If we denote $\breve{g}_{00}=a$ and $\breve{g}_{01}=b$, then we get

$$g_{ij}=\begin{Vmatrix}1 & 0\\ 0 & -1\end{Vmatrix},\qquad\qquad \breve{g}_{ij}=\begin{Vmatrix}a & b\\ b & -a\end{Vmatrix}. \tag{2.10}$$

The equation (2.6) in this case is not solvable, i. e. the metrics $\mathbf{g}$ and $\check{\mathbf{g}}$ cannot be diagonalized simultaneously. Therefore we postulate (2.10) to be another canonical presentation for the metrics $\mathbf{g}$ and $\check{\mathbf{g}}$.

The fourth case in the table (2.7) is subdivided into three subcases

Case 4, subcase 1	$\lvert 2\,\check{g}_{01}\rvert < \lvert\check{g}_{00}+\check{g}_{11}\rvert$
Case 4, subcase 2	$\lvert 2\,\check{g}_{01}\rvert > \lvert\check{g}_{00}+\check{g}_{11}\rvert$
Case 4, subcase 3	$\lvert 2\,\check{g}_{01}\rvert = \lvert\check{g}_{00}+\check{g}_{11}\rvert$

(2.11)

In the **first subcase** of the case 4 the equation (2.6) is solvable. Indeed, since $\check{g}_{00}+\check{g}_{11}\neq 0$, we can write it as follows:

$$\tanh(2\,\phi)=-\frac{2\,\check{g}_{01}}{\check{g}_{00}+\check{g}_{11}}. \tag{2.12}$$

The function $\tanh(2\,\phi)$ is a growing smooth function on the real axis $\mathbb{R}$, its values range from -1 as $\phi\to-\infty$ to $+1$ as $\phi\to+\infty$. For this reason the equation (2.12) has a unique solution $\phi=\phi_0$. Substituting it into (2.2), we find a new basis $\tilde{\mathbf{e}}_0$, $\tilde{\mathbf{e}}_1$ where both metrics $\mathbf{g}$ and $\check{\mathbf{g}}$ are diagonal. Their matrices take their canonical forms (2.9) in this new basis.

In the **second subcase** of the case 4 the equation (2.6) is not solvable. Therefore we take the sum $\check{\tilde{g}}_{00}+\check{\tilde{g}}_{11}$. The vanishing condition for this sum leads to the following equation for the parameter ϕ:

$$\tanh(2\,\phi)=-\frac{\check{g}_{00}+\check{g}_{11}}{2\,\check{g}_{01}}. \tag{2.13}$$

Looking at the second raw in the table (2.11), we see that the second subcase of the case 4 is that very case where the equation (2.13) is solvable and has a unique solution $\phi=\phi_0$ Substituting this solution into (2.2), we find a new basis $\tilde{\mathbf{e}}_0$, $\tilde{\mathbf{e}}_1$ where the matrices of the metrics $\mathbf{g}$ and $\check{\mathbf{g}}$ take their canonical forms (2.10).

The **third subcase** of the case 4 is a special case. It subdivides into two subcases of the next level. They are listed in the following table:

Case 4, subcase 3A	$2\,\check{g}_{01}=\check{g}_{00}+\check{g}_{11}$
Case 4, subcase 3B	$-2\,\check{g}_{01}=\check{g}_{00}+\check{g}_{11}$

(2.14)

These two subcases (2.14) are studied in the next section.

3. Associated operators.

The first metric $\mathbf{g}$ is a Minkowski type metric with the signature $(+,-)$. It is non-degenerate. For this reason we can define the associated operator $\check{\mathbf{F}}$ for the second metric with respect to the first one. It is introduced by the formula

$$g(\check{\mathbf{F}}(\mathbf{X}),\mathbf{Y})=\check{g}(\mathbf{X},\mathbf{Y}). \tag{3.1}$$

Here $\mathbf{X}$ and $\mathbf{Y}$ are two arbitrary vectors of the space V. Due to the symmetry of the quadratic forms g and $\check{g}$ we can extend (3.1) as follows:

$$g(\check{\mathbf{F}}(\mathbf{X}),\mathbf{Y})=\check{g}(\mathbf{X},\mathbf{Y})=g(\mathbf{X},\check{\mathbf{F}}(\mathbf{Y})),\tag{3.2}$$

The formulas (3.2) mean that $\check{\mathbf{F}}$ is a symmetric operator with respect to the metric $\mathbf{g}$. In the coordinate form the associated operator $\check{\mathbf{F}}$ is represented by a matrix:

$$\check{F}^i_j=\begin{Vmatrix}\check{F}^0_0 & \check{F}^0_1\\ \check{F}^1_0 & \check{F}^1_1\end{Vmatrix}.\tag{3.3}$$

The components of the matrix (3.3) are given by the formula

$$\check{F}^i_j=\sum_{s=0}^{3}g^{is}\,\check{g}_{sj}.$$

Here g^{is} are the components of the matrix inverse to the matrix of the first metric $\mathbf{g}$. Applying this formula to (2.1), for $\check{\mathbf{F}}$ in the basis $\mathbf{e}_0$, $\mathbf{e}_1$ we get

$$\check{F}^i_j=\begin{Vmatrix}\check{g}_{00} & \check{g}_{01}\\ -\check{g}_{01} & -\check{g}_{11}\end{Vmatrix}.\tag{3.4}$$

Using (3.4), we can calculate the invariants for the pair of metrics $\mathbf{g}$ and $\check{\mathbf{g}}$:

$$\operatorname{tr}\check{\mathbf{F}}=\check{g}_{00}-\check{g}_{11},\qquad\qquad \det\check{\mathbf{F}}=(\check{g}_{01})^2-\check{g}_{00}\,\check{g}_{11}.\tag{3.5}$$

Relying on (3.5), we perform the following calculations:

$$\begin{aligned}&(\check{g}_{00}+\check{g}_{11})^2-4\,(\check{g}_{01})^2=(\check{g}_{00}-\check{g}_{11})^2+\\ &+4\,\check{g}_{00}\,\check{g}_{11}-4\,(\check{g}_{01})^2=(\operatorname{tr}\check{\mathbf{F}})^2-4\,\det\check{\mathbf{F}}.\end{aligned}\tag{3.6}$$

Due to the formula (3.6) we can write the conditions in the table (2.11) in the invariant coordinate-free form:

Case 4, subcase 1	$(\operatorname{tr}\check{\mathbf{F}})^2>4\,\det\check{\mathbf{F}}$
Case 4, subcase 2	$(\operatorname{tr}\check{\mathbf{F}})^2<4\,\det\check{\mathbf{F}}$
Case 4, subcase 3	$(\operatorname{tr}\check{\mathbf{F}})^2=4\,\det\check{\mathbf{F}}$

(3.7)

In the subcase 1 of the table (3.7) the associated operator $\check{\mathbf{F}}$ has two real eigenvalues $\lambda_0\neq\lambda_1$. Let $\mathbf{v}_0$ and $\mathbf{v}_1$ be the eigenvectors of the operator $\check{\mathbf{F}}$ corresponding to the eigenvalues λ_0 and λ_1 respectively. Since $\lambda_0\neq\lambda_1$, they are orthogonal to each other with respect to both metrics $\mathbf{g}$ and $\check{\mathbf{g}}$:

$$g(\mathbf{v}_0,\mathbf{v}_1)=0,\qquad\qquad \check{g}(\mathbf{v}_0,\mathbf{v}_1)=0.\tag{3.8}$$

The proof of this fact is derived from (3.2). Indeed, we have

$$\lambda_0\,g(\mathbf{v}_0,\mathbf{v}_1)=g(\check{\mathbf{F}}(\mathbf{v}_0),\mathbf{v}_1)=\check{g}(\mathbf{v}_0,\mathbf{v}_1)=g(\mathbf{v}_0,\check{\mathbf{F}}(\mathbf{v}_1))=\lambda_1\,g(\mathbf{v}_0,\mathbf{v}_1). \tag{3.9}$$

From (3.9) we derive $(\lambda_1-\lambda_0)\,g(\mathbf{v}_0,\mathbf{v}_1)=0$, which yields $g(\mathbf{v}_0,\mathbf{v}_1)=0$. Substituting this equality back to the formulas (3.9), we get $\check{g}(\mathbf{v}_0,\mathbf{v}_1)$. Thus, both equalities (3.8) are proved.

If we choose the vectors $\mathbf{v}_0$, $\mathbf{v}_1$ for a basis, then the equalities (3.9) mean that both metrics $\mathbf{g}$ and $\check{\mathbf{g}}$ are diagonal in this basis. The signature of the metric $\mathbf{g}$ is $(+,-)$. For this reason $g(\mathbf{v}_0,\mathbf{v}_0)$ and $g(\mathbf{v}_1,\mathbf{v}_1)$ are two nonzero numbers of opposite signs. Without loss of generality we can assume that $g(\mathbf{v}_0,\mathbf{v}_0)$ is positive and $g(\mathbf{v}_1,\mathbf{v}_1)$ is negative. We can normalize the eigenvectors $\mathbf{v}_0$ and $\mathbf{v}_1$ so that $g(\mathbf{v}_0,\mathbf{v}_0)=1$ and $g(\mathbf{v}_1,\mathbf{v}_1)=-1$. Then $\mathbf{v}_0$, $\mathbf{v}_1$ is that very basis, where the metrics $\mathbf{g}$ and $\check{\mathbf{g}}$ take their canonical forms (2.9) with $a=\lambda_0$ and $b=-\lambda_1$.

Theorem 3.1. *If* $(\operatorname{tr}\check{\mathbf{F}})^2>4\det\check{\mathbf{F}}$*, then the metrics* $\mathbf{g}$ *and* $\check{\mathbf{g}}$ *are given by the matrices* (2.9) *with* $a+b\neq 0$ *in a basis composed by eigenvectors of the operator* $\check{\mathbf{F}}$*.*

In the subcase 2 of the table (3.7) the associated operator $\check{\mathbf{F}}$ has two complex eigenvalues conjugate to each other: $\lambda_1=\overline{\lambda_0}$. More exactly, $\lambda_0\neq\lambda_1$ are the eigenvalues of the complexified operator $\check{\mathbf{F}}$ in the complexification $\mathbb{C}V=\mathbb{C}\otimes V$ of the vector space V. The complex space $\mathbb{C}V$ is naturally equipped with the involution of complex conjugation:

$$\tau\colon\ \mathbb{C}V\to\mathbb{C}V. \tag{3.10}$$

The space V is embedded into $\mathbb{C}V$ as a $\mathbb{R}$-linear subspace invariant under the involution (3.10). Since $\check{\mathbf{F}}$ is a complexification of an operator acting in V, it commutes with τ. Therefore, if $\mathbf{v}_0$ is an eigenvector corresponding to the eigenvalue λ_0, then $\mathbf{v}_1=\tau(\mathbf{v}_0)$ is an eigenvector corresponding to the eigenvalue $\lambda_1=\overline{\lambda_0}$. Let's define the following two vectors:

$$\mathbf{e}_0=\frac{\mathbf{v}_0+\mathbf{v}_1}{\sqrt{2}} \qquad\qquad \mathbf{e}_1=\frac{\mathbf{v}_0-\mathbf{v}_1}{\sqrt{2}\,i}. \tag{3.11}$$

The vectors (3.11) are invariant under the action of the involution τ. Hence they belong to V. These vectors are nonzero and linearly independent. They form a basis in V. Applying $\check{\mathbf{F}}$ to (3.11), we find

$$\begin{aligned}\check{\mathbf{F}}(\mathbf{e}_0)&=\frac{\lambda_0\,\mathbf{v}_0+\overline{\lambda_0}\,\mathbf{v}_1}{\sqrt{2}}=\operatorname{Re}(\lambda_0)\,\mathbf{e}_0-\operatorname{Im}(\lambda_0)\,\mathbf{e}_1,\\ \check{\mathbf{F}}(\mathbf{e}_1)&=\frac{\lambda_0\,\mathbf{v}_0-\overline{\lambda_0}\,\mathbf{v}_1}{\sqrt{2}\,i}=\operatorname{Im}(\lambda_0)\,\mathbf{e}_0+\operatorname{Re}(\lambda_0)\,\mathbf{e}_1.\end{aligned} \tag{3.12}$$

Note that the vectors $\mathbf{v}_0$ and $\mathbf{v}_1$ are orthogonal to each other with respect to both metrics $\mathbf{g}$ and $\check{\mathbf{g}}$, i.e. the formulas (3.8) are valid. The arguments for that here are the same as in (3.9). Due to (3.8) the metric $\mathbf{g}$ is diagonal in the basis $\mathbf{v}_0$, $\mathbf{v}_1$. It is a non-degenerate metric. Hence, $g(\mathbf{v}_0,\mathbf{v}_0)$ and $g(\mathbf{v}_1,\mathbf{v}_1)$ are nonzero. Due to the complexity of the space $\mathbb{C}V$ the vectors $\mathbf{v}_0$ and $\mathbf{v}_1$ can be normalized to the unity:

$$g(\mathbf{v}_0,\mathbf{v}_0)=1, \qquad\qquad g(\mathbf{v}_1,\mathbf{v}_1)=1. \tag{3.13}$$

From (3.11), (3.8) and (3.13) we easily derive

$$g(\mathbf{e}_0,\mathbf{e}_0)=1,\qquad g(\mathbf{e}_0,\mathbf{e}_1)=0,\qquad g(\mathbf{e}_1,\mathbf{e}_1)=-1.\tag{3.14}$$

Let's denote $\operatorname{Re}(\lambda_0)=a$ and $\operatorname{Im}(\lambda_0)=b$. Then from (3.12) and (3.14), using the formula (3.2), we derive that the metrics $\mathbf{g}$ and $\breve{\mathbf{g}}$ take their canonical forms (2.10).

Theorem 3.2. *If* $(\operatorname{tr}\breve{\mathbf{F}})^2<4\det\breve{\mathbf{F}}$*, then the metrics* $\mathbf{g}$ *and* $\breve{\mathbf{g}}$ *are given by the matrices* (2.10) *in a basis produced from eigenvectors of the complexified associated operator* $\breve{\mathbf{F}}$ *according to the formulas* (3.11)*.*

Now let's proceed to the subcase 3 of the table (3.7). In this case the associated operator $\breve{\mathbf{F}}$ has one real eigenvalue λ_0 of the multiplicity 2. Assume that the metrics $\mathbf{g}$ and $\breve{\mathbf{g}}$ are brought to the form (2.1) in some basis $\mathbf{e}_0,\mathbf{e}_1$. Then this subcase 3 is subdivided into two subcases 3A and 3B of the next level (see the table (2.14)). Actually, the subcase 3B is equivalent to the subcase 3A. Indeed, assume that the condition $-2\,\breve{g}_{01}=\breve{g}_{00}+\breve{g}_{11}$ is fulfilled in the basis $\mathbf{e}_0,\mathbf{e}_1$. Then we perform the following basis transformation:

$$\tilde{\mathbf{e}}_0=\mathbf{e}_0,\qquad \tilde{\mathbf{e}}_1=-\mathbf{e}_1.\tag{3.15}$$

The transformation (3.15) is characterized by the diagonal transition matrix

$$S=\begin{Vmatrix}1&0\\0&-1\end{Vmatrix}.\tag{3.16}$$

Substituting (3.16) into (2.3), we find that $\tilde{g}_{ij}=g_{ij}$, i.e. the matrix of the metric $\mathbf{g}$ is invariant under the basis transformation (3.15), while for the matrices of the second metric we have the following relationships:

$$\tilde{\breve{g}}_{00}=g_{00},\qquad \tilde{\breve{g}}_{01}=-g_{01},\qquad \tilde{\breve{g}}_{11}=g_{11}.\tag{3.17}$$

Due to the formulas (3.17), from $-2\,\breve{g}_{01}=\breve{g}_{00}+\breve{g}_{11}$ we derive $2\,\tilde{\breve{g}}_{01}=\tilde{\breve{g}}_{00}+\tilde{\breve{g}}_{11}$. Thus, the subcase 3B occurring in some basis $\mathbf{e}_0,\mathbf{e}_1$ can be transformed to the subcase 3A in some other basis.

Continuing the study of the subcase 3 in (3.7), we restrict ourselves to the subcase 3A. Using the equality $2\,\breve{g}_{01}=\breve{g}_{00}+\breve{g}_{11}$ we express $\breve{g}_{01}$ through $\breve{g}_{00}$ and $\breve{g}_{11}$:

$$\breve{g}_{01}=\frac{\breve{g}_{00}+\breve{g}_{11}}{2}.\tag{3.18}$$

Then we substitute the expression (3.18) into the matrix (3.4) and calculate the eigenvalue of the associated operator $\breve{\mathbf{F}}$:

$$\lambda_0=\frac{\breve{g}_{00}-\breve{g}_{11}}{2}.\tag{3.19}$$

As we mentioned above, the operator $\breve{\mathbf{F}}$ in this case has exactly one eigenvalue (3.19) of the multiplicity 2. Let $A=\breve{F}-\lambda_0\,I$, where $\breve{F}$ is the matrix of the operator $\breve{\mathbf{F}}$ and I is the unit matrix. Then, using (3.19) for λ_0, we obtain

$$A^i_j=\frac{\breve{g}_{00}+\breve{g}_{11}}{2}\begin{Vmatrix}1&1\\-1&-1\end{Vmatrix}.\tag{3.20}$$

Now, relying on (3.20) we define the following quantity:

$$\sigma=\operatorname{sign}(\check{g}_{00}+\check{g}_{11})=\begin{cases}+1 & \text{if} \quad \check{g}_{00}+\check{g}_{11}>0;\\ \ \ 0 & \text{if} \quad \check{g}_{00}+\check{g}_{11}=0;\\ -1 & \text{if} \quad \check{g}_{00}+\check{g}_{11}<0.\end{cases} \tag{3.21}$$

We subdivide the subcase 3 in (3.7) into three subcases of the next level regarding the value of σ in (3.21). They are listed in the table

Case 4, subcase 3(1)	$\sigma=1$
Case 4, subcase 3(2)	$\sigma=-1$
Case 4, subcase 3(3)	$\sigma=0$

(3.22)

The subcase 3(3) is the most simple in the table (3.22). In this case the matrix (3.20) is equal to zero, i. e. $g_{00}+g_{11}=0$. Then we denote

$$\check{g}_{00}=-\check{g}_{11}=a. \tag{3.23}$$

Substituting (3.23) into (3.18) and (3.19), we find that

$$\lambda_0=a, \qquad\qquad \check{g}_{01}=0. \tag{3.24}$$

Substituting (3.23) and (3.24) back into (2.1), we see that the subcase 3(3) in the table (3.22) is equivalent to the first subcase in the table (2.7).

Theorem 3.3. *If* $(\operatorname{tr}\check{\mathbf{F}})^2=4\det\check{\mathbf{F}}$ *and* $\sigma=0$*, then the metrics* $\mathbf{g}$ *and* $\check{\mathbf{g}}$ *differ only by a scalar factor. They can be brought to the canonical form* (2.8) *in some basis.*

Let's proceed to the subcase 3(1) in the table (3.22). In this case $\check{g}_{00}+\check{g}_{11}>0$. Therefore we denote $\check{g}_{00}+\check{g}_{11}=2\,\beta^2$ and $\lambda_0=a$. Then (3.18) and (3.19) yield

$$\check{g}_{00}=\beta^2+a, \qquad \check{g}_{01}=\beta^2, \qquad \check{g}_{11}=\beta^2-a. \tag{3.25}$$

Substituting (3.25) into (2.1), we find

$$g_{ij}=\begin{Vmatrix}1 & 0\\ 0 & -1\end{Vmatrix}, \qquad\qquad \check{g}_{ij}=\begin{Vmatrix}\beta^2+a & \beta^2\\ \beta^2 & \beta^2-a\end{Vmatrix}. \tag{3.26}$$

The matrices (3.26) present the metrics $\mathbf{g}$ and $\check{\mathbf{g}}$ in some basis $\mathbf{e}_0$, $\mathbf{e}_1$. Now we perform the following basis transformation:

$$\tilde{\mathbf{e}}_1=\frac{1}{2\,\beta}\,\mathbf{e}_0+\frac{1}{2\,\beta}\,\mathbf{e}_1, \qquad\qquad \tilde{\mathbf{e}}_0=\beta\,\mathbf{e}_0-\beta\,\mathbf{e}_1. \tag{3.27}$$

Upon performing the basis transformation (3.27) we find that the metrics $\mathbf{g}$ and $\check{\mathbf{g}}$ are presented by the matrices

$$g_{ij}=\begin{Vmatrix}0 & 1\\ 1 & 0\end{Vmatrix}, \qquad\qquad \check{g}_{ij}=\begin{Vmatrix}1 & a\\ a & 0\end{Vmatrix} \tag{3.28}$$

in the new basis. The presentation (3.28) is a canonical presentation for the metric pair $\mathbf{g}$, $\breve{\mathbf{g}}$ in the subcase 3(1).

Theorem 3.4. *If* $(\operatorname{tr}\breve{\mathbf{F}})^2 = 4\det\breve{\mathbf{F}}$ *and* $\sigma = 1$*, then the metrics* $\mathbf{g}$ *and* $\breve{\mathbf{g}}$ *are presented by the matrices* (3.28) *in some basis.*

The subcase 3(2) is similar to the subcase 3(1). In this case $\breve{g}_{00} + \breve{g}_{11} < 0$ Therefore we denote $\breve{g}_{00} + \breve{g}_{11} = -2\,\beta^2$ and $\lambda_0 = a$. Then (3.18) and (3.19) yield

$$\breve{g}_{00} = a - \beta^2, \qquad \breve{g}_{01} = -\beta^2, \qquad \breve{g}_{11} = -a - \beta^2. \tag{3.29}$$

Due to (3.29) the formulas (2.1) specialize to the following ones:

$$g_{ij} = \begin{Vmatrix} 1 & 0 \\ 0 & -1 \end{Vmatrix}, \qquad \breve{g}_{ij} = \begin{Vmatrix} a - \beta^2 & -\beta^2 \\ -\beta^2 & -a - \beta^2 \end{Vmatrix}. \tag{3.30}$$

Now we perform the following basis transformation:

$$\tilde{\mathbf{e}}_0 = \beta\,\mathbf{e}_0 - \beta\,\mathbf{e}_1, \qquad \tilde{\mathbf{e}}_1 = \frac{1}{2\,\beta}\,\mathbf{e}_0 + \frac{1}{2\,\beta}\,\mathbf{e}_1. \tag{3.31}$$

By means of (3.31) we bring the matrices (3.30) to their canonical forms:

$$g_{ij} = \begin{Vmatrix} 0 & 1 \\ 1 & 0 \end{Vmatrix}, \qquad \breve{g}_{ij} = \begin{Vmatrix} 0 & a \\ a & -1 \end{Vmatrix}. \tag{3.32}$$

Theorem 3.5. *If* $(\operatorname{tr}\breve{\mathbf{F}})^2 = 4\det\breve{\mathbf{F}}$ *and* $\sigma = -1$*, then the metrics* $\mathbf{g}$ *and* $\breve{\mathbf{g}}$ *are presented by the matrices* (3.32) *in some basis.*

4. Classification.

The cases and subcases considered in the previous two sections are excessive. Some of them are equivalent to others and some of them are particular cases of others. The actual classification of metric pairs, one of which is a Minkowski type metric, is given by the theorems 3.1, 3.2, 3.3, 3.4, and 3.5. We gather the results of these theorems into the following table:

Condition	Canonical presentation
$(\operatorname{tr}\breve{\mathbf{F}})^2 > 4\det\breve{\mathbf{F}}$	$g_{ij} = \begin{Vmatrix} 1 & 0 \\ 0 & -1 \end{Vmatrix}$, $\breve{g}_{ij} = \begin{Vmatrix} a & 0 \\ 0 & b \end{Vmatrix}$ with $b \neq -a$
$(\operatorname{tr}\breve{\mathbf{F}})^2 < 4\det\breve{\mathbf{F}}$	$g_{ij} = \begin{Vmatrix} 1 & 0 \\ 0 & -1 \end{Vmatrix}$, $\breve{g}_{ij} = \begin{Vmatrix} a & b \\ b & -a \end{Vmatrix}$ with $b \neq 0$
$(\operatorname{tr}\breve{\mathbf{F}})^2 = 4\det\breve{\mathbf{F}}$ and $\sigma = 0$	$g_{ij} = \begin{Vmatrix} 1 & 0 \\ 0 & -1 \end{Vmatrix}$, $\breve{g}_{ij} = \begin{Vmatrix} a & 0 \\ 0 & -a \end{Vmatrix}$
$(\operatorname{tr}\breve{\mathbf{F}})^2 = 4\det\breve{\mathbf{F}}$ and $\sigma = 1$	$g_{ij} = \begin{Vmatrix} 0 & 1 \\ 1 & 0 \end{Vmatrix}$, $\breve{g}_{ij} = \begin{Vmatrix} 1 & a \\ a & 0 \end{Vmatrix}$
$(\operatorname{tr}\breve{\mathbf{F}})^2 = 4\det\breve{\mathbf{F}}$ and $\sigma = -1$	$g_{ij} = \begin{Vmatrix} 0 & 1 \\ 1 & 0 \end{Vmatrix}$, $\breve{g}_{ij} = \begin{Vmatrix} 0 & a \\ a & -1 \end{Vmatrix}$

(4.1)

The quantity σ in the table (4.1) is an invariant of a pair of metrics. It is very important to note that this invariant cannot be expressed through the invariants of the associated operator ($\operatorname{tr}\check{\mathbf{F}}$ and $\det\check{\mathbf{F}}$). The formula (3.21) defines this invariant in a special basis, where the first metric $\mathbf{g}$ is diagonalized:

$$g_{ij} = \left\|\begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix}\right\|.$$

However, there must be a formula or an algorithm for calculating the invariant σ in an arbitrary basis without diagonalizing the metric $\mathbf{g}$.

5. Dedicatory.

This paper is dedicated to my uncle Amir Minivalievich Nagaev.

References

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